A New Three-Term Conjugate Gradient Method for Unconstrained Optimization with Applications in Portfolio Selection and Robotic Motion Control

Maulana Malik*, Auwal Bala Abubakar, Ibrahim Mohammed Sulaiman, Mustafa Mamat, Siti Sabariah Abas, and Sukono

***Abstract*—Three-term conjugate gradient method is one of the efficient method for solving unconstrained optimization models. In this paper, we propose a new three-term conjugate gradient method with a new search direction structure. A remarkable feature of the proposed method is that independent of the line search procedure, the search direction always satisfies the sufficient descent condition. The global convergence properties of the proposed method is established under the strong Wolfe line search by assuming that the objective function is Lipschitz continuous. Numerical results indicate that our proposed method is efficient and robust, thus effective in solving unconstrained optimization models. In addition, the proposed method also considered practical application problem in portfolio selection and robotic motion control.**

***Index Terms*—Three-term conjugate gradient method, unconstrained optimization, sufficient descent condition, global convergence properties, portfolio selection, motion control.**

I. Introduction

THREE-TERM conjugate gradient (TTCG) method is an efficient method for solving unconstrained optimization model as follows:

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}), \tag{1}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable objective function whose gradient is given by $\mathbf{g}(\mathbf{x})=\nabla f(\mathbf{x})$. The TTCG method is an iterative method that generates sequence $\{\mathbf{x}_k\}$ via the following recurrence formula:

$$\mathbf{x}_{k+1}=\mathbf{x}_k+\mathbf{s}_k, \tag{2}$$

where $k\geq 0$, $\mathbf{s}_k=\alpha_k\mathbf{d}_k$, and $\mathbf{x}_0\in\mathbb{R}^n$ is a randomly selected initial point [1]. Note that $\alpha_k>0$ is known as the step length obtained by using some line search technique such as exact or inexact line search [2]. A frequently used line search is the inexact line search, especially Strong Wolfe's line search, which formula is defined as follows:

$$f(\mathbf{x}_k+\alpha_k\mathbf{d}_k)\leq f(\mathbf{x}_k)+\varphi\alpha_k\mathbf{g}_k^T\mathbf{d}_k, \tag{3}$$

$$\mathbf{g}\left(\mathbf{x}_k+\alpha_k\mathbf{d}_k\right)^T\mathbf{d}_k\leq -\sigma\left|\mathbf{g}_k^T\mathbf{d}_k\right|, \tag{4}$$

where $0<\varphi<\sigma<1$ [3]. To get the next iterative point (2) of TTCG method, we need the definition of search direction $\mathbf{d}_k$. The search direction in TTCG method is usually defined as:

$$\mathbf{d}_k:=\begin{cases}-\mathbf{g}_k, & k=0\\ -\mathbf{g}_k+\beta_k\mathbf{d}_{k-1}+\theta_k\mathbf{y}_{k-1}, & k\geq 1\end{cases}, \tag{5}$$

where $\mathbf{g}_k=\mathbf{g}(\mathbf{x}_k)$ is the gradient of f calculated at point $\mathbf{x}_k$, θ_k and β_k are the conjugate gradient parameters, and $\mathbf{y}_{k-1}=\mathbf{g}_k-\mathbf{g}_{k-1}$. Clearly, if the parameter $\theta_k\equiv 0$, TTCG methods reduces to the standard conjugate gradient (CG) methods.

Some of the famous and standard CG methods are the HS method [4], the FR method [5], the PRP method [6], [7], the CD method [8], the LS method [9], the DY method [10], and the RMIL method [11]. The parameters β_k of the above conjugate gradient methods defined as follows:

$$\begin{aligned}
\beta_k^{HS} &= \frac{\mathbf{g}_k^T(\mathbf{g}_k-\mathbf{g}_{k-1})}{\mathbf{d}_{k-1}^T(\mathbf{g}_k-\mathbf{g}_{k-1})},\\
\beta_k^{FR} &= \frac{\|\mathbf{g}_k\|^2}{\|\mathbf{g}_{k-1}\|^2},\\
\beta_k^{PRP} &= \frac{\mathbf{g}_k^T(\mathbf{g}_k-\mathbf{g}_{k-1})}{\|\mathbf{g}_{k-1}\|^2},\\
\beta_k^{CD} &= -\frac{\|\mathbf{g}_k\|^2}{\mathbf{d}_{k-1}^T\mathbf{g}_{k-1}},\\
\beta_k^{LS} &= -\frac{\mathbf{g}_k^T(\mathbf{g}_k-\mathbf{g}_{k-1})}{\mathbf{d}_{k-1}^T\mathbf{g}_{k-1}},\\
\beta_k^{DY} &= \frac{\|\mathbf{g}_k\|^2}{\mathbf{d}_{k-1}^T(\mathbf{g}_k-\mathbf{g}_{k-1})},\\
\beta_k^{RMIL} &= \frac{\mathbf{g}_k^T(\mathbf{g}_k-\mathbf{g}_{k-1})}{\|\mathbf{d}_{k-1}\|^2},
\end{aligned}$$

where $\|.\|$ denotes the Euclidean norm of vectors. Numerous studies have been done on the standard, hybrid, and spectral conjugate gradient methods. For a comprehensive review on new advances, readers should refer to the following articles [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22] and [23].

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*Maulana Malik is an assistant professor at the Department of Mathematics, Universitas Indonesia, Depok 16424, Indonesia and a PhD candidate at the Universiti Sultan Zainal Abidin, Terengganu 22200, Malaysia. (e-mail: m.malik@sci.ui.ac.id)

Auwal Bala Abubakar is a lecturer at the Department of Mathematical Sciences, Faculty of Physical Sciences, Bayero University Kano, Kano 700241, Nigeria and a research associate at the Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa-0204, South Africa. (e-mail: ababubakar.mth@buk.edu.ng)

Ibrahim Mohammed Sulaiman is a post-doctoral researcher at the Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu 22200, Malaysia. (e-mail: sulaimanib@unisza.edu.my)

Mustafa Mamat is a professor of Applied Mathematics at the Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu 22200, Malaysia. (e-mail: must@unisza.edu.my)

Siti Sabariah Abas is a lecturer at the Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu 22200, Malaysia. (e-mail: sabariahabas@unisza.edu.my)

Sukono is an associate professor at the Department of Mathematics, Universitas Padjadjaran, Jatinangor 45363, Indonesia. (e-mail: sukono@unpad.ac.id)

In recent years, several researchers have proposed various TTCG methods. In 2018, Liu et al. [24] proposed a TTCG method of RMIL conjugate gradient method. The given method always satisfies the descent condition

$$\mathbf{g}_k^T \mathbf{d}_k < 0, \text{ for all } k \geq 0, \tag{6}$$

without any line search and also fulfills the global convergence properties

$$\lim_{k\to\infty} \inf \|\mathbf{g}_k\| = 0 \tag{7}$$

under standard Wolfe line search. The proposed method is named as TTRMIL method and search direction of the method defined by

$$\mathbf{d}_k := \begin{cases} -\mathbf{g}_k, & k = 0 \\ -\mathbf{g}_k + \dfrac{\mathbf{g}_k^T \mathbf{y}_{k-1}}{\|\mathbf{d}_{k-1}\|^2}\mathbf{d}_{k-1} - \dfrac{\mathbf{g}_k^T \mathbf{d}_{k-1}}{\|\mathbf{d}_{k-1}\|^2}\mathbf{y}_{k-1}, & k \geq 1 \end{cases}.$$

Also, Baluch et al. [25] extended the approach to propose a TTCG method. The researchers form new search directions with formula

$$\mathbf{d}_k := \begin{cases} -\mathbf{g}_k, & k = 0 \\ -\mathbf{g}_k + \beta_k^{BZAU}\mathbf{d}_{k-1} - \mathbf{y}_{k-1}, & k \geq 1 \end{cases},$$

where

$$\beta_k^{BZAU} = \frac{\mathbf{g}_k^T \mathbf{y}_{k-1}}{-\eta \mathbf{g}_{k-1}^T \mathbf{d}_{k-1} + \mu \left|\mathbf{g}_k^T \mathbf{d}_{k-1}\right|},$$

$$\theta_k^{BZAU} = \frac{\mathbf{g}_k^T \mathbf{d}_{k-1}}{-\eta \mathbf{g}_{k-1}^T \mathbf{d}_{k-1} + \mu \left|\mathbf{g}_k^T \mathbf{d}_{k-1}\right|},$$

for $\eta \in [1, +\infty), \mu \in (\eta, +\infty)$ and named the method TTBZAU (three-term Bakhtawar, Zabidin, Ahmad and Ummu). Under Wolfe Powell line search, the TTBZAU satisfies global convergence properties with convex and non-convex functions and independent of the line search chosen, the method possesses the sufficient descent condition.

Recently, Liu et al. [26] proposed three type of TTCG methods. One of the coefficient of their sttudy is the MTTPRP method, where the search direction is defined as follows:

$$\mathbf{d}_k := \begin{cases} -\mathbf{g}_k, & k = 0 \\ -\mathbf{g}_k + \beta_k^{\#}\mathbf{d}_{k-1} + \theta_k \mathbf{g}_{k-1}, & k \geq 1 \end{cases}, \tag{8}$$

where

$$\beta_k^{\#} = \left(\beta_k^{PRP} - \frac{\mathbf{g}_k^T \mathbf{s}_{k-1}}{\|\mathbf{g}_{k-1}\|^2}\right),$$

$$\theta_k = \frac{\mathbf{g}_k^T \mathbf{d}_{k-1}}{\|\mathbf{g}_{k-1}\|^2},$$

and the method is an extension of the MTTLS method [26]. The convergence analysis of the MTTPRP method is established in a similar way with the MTTLS method, which is to satisfies global convergence properties and possesses the sufficient descent condition.

Inspired the above literature, we develop a TTCG method which satisfies the descent condition and the global convergence properties under strong Wolfe line search. The proposed method aims to possess a better numerical results. The rest of this paper is organized as follows: in section 2, we present our new search direction, algorithm, and proof of sufficient descent condition. Section 3 discusses the proof of global convergence. The numerical results and discussions are recorded in section 4. Application of our new method is presented in section 5. Finally, a conclusion is given in section 6.

II. New Search Direction and Algorithm

Motivated by the structure of MTTPRP method, we make a little change to the MTTPRP method, show that the new method possess descent condition and establish the global convergence proof. Our new method is formed by replacing the $\beta_k^{\#}$ in (8), that is β_k^{PRP} to β_k^{BZAU}, expand the form $\mathbf{g}_k^T \mathbf{s}_{k-1}$ by adding $\|\mathbf{g}_{k-1}\|^2$, the denominator is adjusted to the form of the β_k^{BZAU} denominator, and always has a non negative value. Furthermore, we change θ_k in (8) to θ_k^{BZAU}. Hence, the proposed method has search direction as follows:

$$\mathbf{d}_k := \begin{cases} -\mathbf{g}_k, & k = 0 \\ -\mathbf{g}_k + \beta_k^{\#\#+}\mathbf{d}_{k-1} + \theta_k^{\#\#}\mathbf{g}_{k-1}, & k \geq 1 \end{cases}, \tag{9}$$

where

$$\begin{aligned} \beta_k^{\#\#+} &= \max\left\{0, \beta_k^{\#\#}\right\}, \\ \beta_k^{\#\#} &= \left(\beta_k^{BZAU} - \frac{\|\mathbf{g}_{k-1}\|^2 \mathbf{g}_k^T \mathbf{s}_{k-1}}{\left(-\eta \mathbf{g}_{k-1}^T \mathbf{d}_{k-1} + \mu \left|\mathbf{g}_k^T \mathbf{d}_{k-1}\right|\right)^2}\right), \\ \theta_k^{\#\#} &= \theta_k^{BZAU}, \end{aligned}$$

$\mathbf{s}_{k-1} = \mathbf{x}_k - \mathbf{x}_{k-1}$, and $\eta \in [1, +\infty), \mu \in (\eta, +\infty)$.

The proposed TTCG method is referred to as the MTTBZAU method and the algorithm is described below.

Algorithm 1. *(MTTBZAU method)*

Step 1. *Set* $\mu = 2, \eta = 1, 0 < \delta < \sigma < 1, \mathbf{d}_0 = -\mathbf{g}_0$, $k = 0$, *and given an initial point* $\mathbf{x}_0 \in \mathbb{R}^n$.

Step 2. *If* $\|\mathbf{g}_k\| < \epsilon$, *where* $\epsilon = 10^{-6}$, *then stop; otherwise, continue to Step 3.*

Step 3. *Calculate the search direction* $\mathbf{d}_k$ *by using (9).*

Step 4. *Calculate the step length* $\alpha_k > 0$ *by using strong Wolfe line search (3) and (4).*

Step 5. *Determine* $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$ *by using* $\mathbf{d}_k$ *in Step 3, and* α_k *in Step 4.*

Step 6. *Set* $k = k + 1$, *continue to Step 2.*

III. Global Convergence Analysis

In this section, we establish the descent condition and global convergence properties of MTTBZAU method. We first make standard assumptions for the objective function. These assumptions will be used throughout the paper.

Assumption 1. *(A1) The level set* $\mathcal{Y} = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ *is bounded, i.e. there exist a positive constants* ω *such that* $\|\mathbf{x}\| \leq \omega$, *for all* $\mathbf{x} \in \mathcal{Y}$. *(A2) In a neighborhood* $\mathcal{P}$ *of* $\mathcal{Y}$, *the objective function* f *is continuously differentiable and its gradient is Lipschitz continuous, i.e. there exists a positive constant* L *such that for all* $\mathbf{x}, \mathbf{y} \in \mathcal{P}$, $\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$.

The following lemma will be used to illustrate that the proposed MTTBZAU method satisfy the descent condition.

Lemma 1. *Consider the sequence $\{\mathbf{x}_k\}$ is generated by MTTBZAU method and suppose the function f satisfies Assumption 1, then we have*

$$\mathbf{g}_k^T\mathbf{d}_k \leq -\left(1-\frac{1}{\mu}\right)\|\mathbf{g}_k\|^2, \text{ if } \beta_k^{\#\#+}=\beta_k^{\#\#}. \tag{10}$$

and

$$\mathbf{g}_k^T\mathbf{d}_k \leq -2\omega L\|\mathbf{g}_k\|, \text{ if } \beta_k^{\#\#+}=0. \tag{11}$$

Hence, the search direction (9) satisfies the descent condition (6).

Proof: We prove this lemma by induction. For $k=0$, we obtain $\mathbf{g}_0^T\mathbf{d}_0=-\|\mathbf{g}_0\|^2<0$, so that, (10) holds. Assume that the condition (10) is true for $k=k-1$, that is, $\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}<0$.

Furthermore, multiplying (9) by $\mathbf{g}_k^T$, we get

$$\mathbf{g}_k^T\mathbf{d}_k=-\|\mathbf{g}_k\|^2+\beta_k^{\#\#+}\mathbf{g}_k^T\mathbf{d}_{k-1}+\theta_k^{\#\#}\mathbf{g}_k^T\mathbf{g}_{k-1}. \tag{12}$$

According to value $\beta_k^{\#\#+}$, there are two cases:

Case 1: for $\beta_k^{\#\#+}=\beta_k^{\#\#}$, then from (12), we have

$$\begin{aligned}
\mathbf{g}_k^T\mathbf{d}_k &= -\|\mathbf{g}_k\|^2+\Bigg(\frac{\mathbf{g}_k^T(\mathbf{g}_k-\mathbf{g}_{k-1})}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}- \\
&\quad \frac{\|\mathbf{g}_{k-1}\|^2\mathbf{g}_k^T\mathbf{s}_{k-1}}{\left(-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|\right)^2}\Bigg)\mathbf{g}_k^T\mathbf{d}_{k-1} \\
&\quad +\frac{\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}\mathbf{g}_k^T\mathbf{g}_{k-1} \\
&= -\|\mathbf{g}_k\|^2+\frac{\|\mathbf{g}_k\|^2\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}- \\
&\quad \frac{\mathbf{g}_k^T\mathbf{g}_{k-1}\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}- \\
&\quad \frac{\|\mathbf{g}_{k-1}\|^2\mathbf{g}_k^T\mathbf{s}_{k-1}\mathbf{g}_k^T\mathbf{d}_{k-1}}{\left(-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|\right)^2}+ \\
&\quad \frac{\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}\mathbf{g}_k^T\mathbf{g}_{k-1} \\
&= -\|\mathbf{g}_k\|^2+\frac{\|\mathbf{g}_k\|^2\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}- \\
&\quad \frac{\|\mathbf{g}_{k-1}\|^2\mathbf{g}_k^T\mathbf{s}_{k-1}\mathbf{g}_k^T\mathbf{d}_{k-1}}{\left(-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|\right)^2} \\
&= -\|\mathbf{g}_k\|^2+\frac{\|\mathbf{g}_k\|^2\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}- \\
&\quad \frac{\|\mathbf{g}_{k-1}\|^2\mathbf{g}_k^T\alpha_{k-1}\mathbf{d}_{k-1}\mathbf{g}_k^T\mathbf{d}_{k-1}}{\left(-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|\right)^2} \\
&= -\|\mathbf{g}_k\|^2+\frac{\|\mathbf{g}_k\|^2\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}- \\
&\quad \frac{\|\mathbf{g}_{k-1}\|^2\alpha_{k-1}\left(\mathbf{g}_k^T\mathbf{d}_{k-1}\right)^2}{\left(-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|\right)^2} \\
&\leq -\|\mathbf{g}_k\|^2+\frac{\|\mathbf{g}_k\|^2\mathbf{g}_k^T\mathbf{d}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}.
\end{aligned}$$

Since $\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}<0$, $\eta\in[1,+\infty)$, and $\mu\in(\eta,+\infty)$, that implies

$$\begin{aligned}
\mathbf{g}_k^T\mathbf{d}_k &\leq -\|\mathbf{g}_k\|^2+\frac{\|\mathbf{g}_k\|^2\mathbf{g}_k^T\mathbf{d}_{k-1}}{\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|} \\
&\leq -\|\mathbf{g}_k\|^2+\frac{\|\mathbf{g}_k\|^2\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}{\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|} \\
&= -\left(1-\frac{1}{\mu}\right)\|\mathbf{g}_k\|^2.
\end{aligned}$$

So, relation (10) holds. Furthermore, the descent condition is fulfilled.

Case 2: for $\beta_k^{\#\#+}=0$, then from (12), we get

$$\begin{aligned}
\mathbf{g}_k^T\mathbf{d}_k &= -\|\mathbf{g}_k\|^2+\theta_k^{\#\#}\mathbf{g}_k^T\mathbf{g}_{k-1} \\
&= -\|\mathbf{g}_k\|^2+\frac{\mathbf{g}_k^T\mathbf{d}_{k-1}\mathbf{g}_k^T\mathbf{g}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|} \\
&\leq -\|\mathbf{g}_k\|^2+\frac{|\mathbf{g}_k^T\mathbf{d}_{k-1}||\mathbf{g}_k^T\mathbf{g}_{k-1}|}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}+\mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}.
\end{aligned}$$

Since $\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}<0$, $\eta\in[1,+\infty)$, $\mu\in(\eta,+\infty)$, and from above relation, we obtain

$$\begin{aligned}
\mathbf{g}_k^T\mathbf{d}_k &\leq -\|\mathbf{g}_k\|^2+|\mathbf{g}_k^T\mathbf{g}_{k-1}| \\
&\leq -\|\mathbf{g}_k\|^2+\|\mathbf{g}_k\|\|\mathbf{g}_{k-1}\| \\
&= -\|\mathbf{g}_k\|\left(\|\mathbf{g}_k\|-\|\mathbf{g}_{k-1}\|\right) \\
&\leq -\|\mathbf{g}_k\|\|\mathbf{g}_k-\mathbf{g}_{k-1}\|.
\end{aligned}$$

From Assumption 1 and based on the above inequality, we have

$$\begin{aligned}
\mathbf{g}_k^T\mathbf{d}_k &\leq -L\|\mathbf{g}_k\|\|\mathbf{x}_k-\mathbf{x}_{k-1}\| \\
&\leq -L\|\mathbf{g}_k\|\left(\|\mathbf{x}_k\|+\|\mathbf{x}_{k-1}\|\right) \\
&\leq -2\omega L\|\mathbf{g}_k\|. \qquad (13)
\end{aligned}$$

Hence, this implies that the inequality (11) holds. So, the descent condition is satisfied.
The proof is finished.

■

The following lemma is called the Zoutendijk condition for MTTBZAU method, which will be used in proving the global convergence properties. The proof of the below lemma is similar to the proof in [27]. So, we leave the proof.

Lemma 2. *Suppose that the sequence $\{\mathbf{x}_k\}$ is generated by MTTBZAU method, and the Assumption 1 (A1) holds. If the step length α_k is determined by strong Wolfe line search (3) and (4), and the search direction $\mathbf{d}_k$ satisfies the descent direction, then we have*

$$\sum_{k=0}^{\infty}\frac{\left(\mathbf{g}_k^T\mathbf{d}_k\right)^2}{\|\mathbf{d}_k\|^2}<\infty. \tag{14}$$

The following two lemmas are required to establish the global convergence properties of MTTBZAU method.

Lemma 3. *Suppose that the sequence $\{\mathbf{x}_k\}$ is generated by MTTBZAU method, and the Assumption 1 holds. The step length α_k is determined by strong Wolfe line search (3) and*

(4). If there exists a positive constant ψ such that for any k, $\|\mathbf{g}_k\| \geq \psi$, then $\mathbf{d}_k \neq 0$ for each k and

$$\sum_{k\geq 1} \|\mathbf{u}_k - \mathbf{u}_{k-1}\|^2 < \infty, \tag{15}$$

where $\mathbf{u}_k = \dfrac{\mathbf{d}_k}{\|\mathbf{d}_k\|}$.

Proof: From (10) and the Cauchy-Schwartz inequality, we have $\left(1 - \dfrac{1}{\mu}\right)\|\mathbf{g}_k\| \leq \|\mathbf{d}_k\|$. In addition, since $\|\mathbf{g}_k\| \geq \psi$, then for all k, $\|\mathbf{d}_k\| > 0$. Likewise from (11), we have $\|\mathbf{d}_k\| \geq 2\omega L > 0$, for all k. They imply that $\mathbf{d}_k \neq 0$. Furthermore, $\mathbf{u}_k$ is well defined.

Let

$$r_k = \frac{-\left(1 + \dfrac{\theta_k^{\#\#}\mathbf{g}_{k-1}\mathbf{g}_k^T}{\|\mathbf{g}_k\|^2}\right)\mathbf{g}_k}{\|\mathbf{d}_k\|}, \tag{16}$$

and

$$\delta_k = \beta_k^{\#\#+}\frac{\|\mathbf{d}_{k-1}\|}{\|\mathbf{d}_k\|},$$

then $\mathbf{u}_k = r_k + \delta_k\mathbf{u}_{k-1}$. Because $\mathbf{u}_k$ and $\mathbf{u}_{k-1}$ are unit vectors, then

$$\|r_k\| = \|\mathbf{u}_k - \delta_k\mathbf{u}_{k-1}\| = \|\mathbf{u}_{k-1} - \delta_k\mathbf{u}_k\|.$$

Also as $\delta_k \geq 0$, then

$$\begin{aligned} \|\mathbf{u}_k - \mathbf{u}_{k-1}\| &\leq (1+\delta_k)\|\mathbf{u}_k - \mathbf{u}_{k-1}\| \\ &\leq \|\mathbf{u}_k - \delta_k\mathbf{u}_{k-1}\| + \|\mathbf{u}_{k-1} - \delta_k\mathbf{u}_k\| \\ &= 2\|r_k\|. \end{aligned} \tag{17}$$

By using the definition of $\theta_k^{\#\#}$ and the fact that there exists a positive constant γ such that for all k, $\|\mathbf{g}_k\| \leq \gamma$, we have

$$\begin{aligned} |\theta_k^{\#\#}|\frac{\|\mathbf{g}_{k-1}\|\|\mathbf{g}_k\|}{\|\mathbf{g}_k\|^2} &\leq \left|\frac{\mathbf{g}_k^T\mathbf{d}_{k-1}}{\mu|\mathbf{g}_k^T\mathbf{d}_{k-1}|}\right|\frac{\|\mathbf{g}_{k-1}\|\|\mathbf{g}_k\|}{\|\mathbf{g}_k\|^2} \\ &\leq \frac{\|\mathbf{g}_{k-1}\|}{\mu\|\mathbf{g}_k\|} \\ &\leq \frac{\gamma}{\mu\psi}. \end{aligned} \tag{18}$$

By utilizing (18), we have

$$\begin{aligned} &\left\|-\left(1 + \frac{\theta_k^{\#\#}\mathbf{g}_{k-1}\mathbf{g}_k^T}{\|\mathbf{g}_k\|^2}\right)\mathbf{g}_k\right\| \\ \leq\ & \|\mathbf{g}_k\| + \left(|\theta_k^{\#\#}|\frac{\|\mathbf{g}_{k-1}\|\|\mathbf{g}_k\|}{\|\mathbf{g}_k\|^2}\right)\|\mathbf{g}_k\| \\ \leq\ & \gamma + \frac{\gamma^2}{\mu\psi} \\ =\ & M_1. \end{aligned} \tag{19}$$

Therefore, from Lemma 2, (10), (16), and (19),

$$\begin{aligned} \sum_{k\geq 0}\|r_k\|^2 &= \sum_{k\geq 0}\left(\left\|\frac{-\left(1 + \dfrac{\theta_k^{\#\#}\mathbf{g}_{k-1}\mathbf{g}_k^T}{\|\mathbf{g}_k\|^2}\right)\mathbf{g}_k}{\|\mathbf{d}_k\|}\right\|\right)^2 \\ &= \sum_{k\geq 0}\frac{\left(\left\|-\left(1 + \dfrac{\theta_k^{\#\#}\mathbf{g}_{k-1}\mathbf{g}_k^T}{\|\mathbf{g}_k\|^2}\right)\mathbf{g}_k\right\|\right)^2}{\|\mathbf{d}_k\|^2} \\ &\leq \sum_{k\geq 0}\frac{M_1^2}{\|\mathbf{d}_k\|^2} \\ &\leq \sum_{k\geq 0}\frac{M_1^2}{\left(1 - \frac{1}{\mu}\right)^2\|\mathbf{g}_k\|^4}\frac{\left(1 - \frac{1}{\mu}\right)^2\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} \\ &\leq \frac{M_1^2}{\left(1 - \frac{1}{\mu}\right)^2\psi^4}\sum_{k\geq 0}\frac{\left(1 - \frac{1}{\mu}\right)^2\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} < +\infty. \end{aligned}$$

The above together with (17), we get

$$\sum_{k\geq 0}\|\mathbf{u}_k - \mathbf{u}_{k-1}\|^2 \leq 4\sum_{k\geq 0}\|r_k\|^2 < +\infty.$$

The proof is completed. ■

The following lemma is properties of $\beta_k^{\#\#}$.

Lemma 4. *Suppose that Assumption 1 holds and the sequence $\{\mathbf{x}_k\}$ is generated by MTTBZAU method, then we have*

$$|\beta_k^{\#\#}| \leq Z_3\|\mathbf{s}_{k-1}\|, \tag{20}$$

where $Z_3 = \dfrac{\gamma L}{\eta\left(1 - \frac{1}{\mu}\right)\psi^2} + \dfrac{\gamma^3}{\eta^2\left(1 - \frac{1}{\mu}\right)^2\psi^4}$ *is a constant.*

Proof: From definition of $\beta_k^{\#\#}$ and β_k^{BZAU}, we have

$$\begin{aligned} |\beta_k^{\#\#}| &\leq \left|\frac{\mathbf{g}_k^T\mathbf{y}_{k-1}}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1} + \mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|}\right| + \\ &\quad \left|\frac{\|\mathbf{g}_{k-1}\|^2\mathbf{g}_k^T\mathbf{s}_{k-1}}{\left(-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1} + \mu\left|\mathbf{g}_k^T\mathbf{d}_{k-1}\right|\right)^2}\right| \\ &\leq \frac{\|\mathbf{g}_k\|\|\mathbf{y}_{k-1}\|}{-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}} + \frac{\|\mathbf{g}_{k-1}\|^2\|\mathbf{g}_k\|\|\mathbf{s}_{k-1}\|}{\left(-\eta\mathbf{g}_{k-1}^T\mathbf{d}_{k-1}\right)^2} \\ &\leq \frac{\|\mathbf{g}_k\|L\|\mathbf{x}_k - \mathbf{x}_{k-1}\|}{\eta\left(1 - \frac{1}{\mu}\right)\|\mathbf{g}_k\|^2} + \frac{\|\mathbf{g}_{k-1}\|^2\|\mathbf{g}_k\|\|\mathbf{s}_{k-1}\|}{\eta^2\left(1 - \frac{1}{\mu}\right)^2\|\mathbf{g}_k\|^4} \\ &\leq \frac{\gamma L\|\mathbf{s}_{k-1}\|}{\eta\left(1 - \frac{1}{\mu}\right)\psi^2} + \frac{\gamma^3\|\mathbf{s}_{k-1}\|}{\eta^2\left(1 - \frac{1}{\mu}\right)^2\psi^4} = Z_3\|\mathbf{s}_{k-1}\|, \end{aligned}$$

where $Z_3 = \dfrac{\gamma L}{\eta\left(1 - \frac{1}{\mu}\right)\psi^2} + \dfrac{\gamma^3}{\eta^2\left(1 - \frac{1}{\mu}\right)^2\psi^4}$. ■

We now present the proof of the global convergence properties of MTTBZAU method. The proof of this theorem is similar to [28], and [29], but differs slightly in some forms.

Theorem 1. *Suppose that the sequence $\{\mathbf{x}_k\}$ is generated by MTTBZAU method, and the Assumption 1 hold. The step* *length α_k is calculated by strong Wolfe line search (3) and (4), then*

$$\lim_{k\to\infty} \inf \|\mathbf{g}_k\| = 0. \tag{21}$$

Proof: Suppose by contradiction that (21) is not true. Then there exists $\psi > 0$ such that for all k, $\|\mathbf{g}_k\| \geq \psi$.

We first prove that the $\mathbf{s}_k$ is bounded. Based on Assumption 1, we have

$$\|\mathbf{x}_l - \mathbf{x}_k\| \leq \|\mathbf{x}_k\| + \|\mathbf{x}_l\| \leq 2\omega. \tag{22}$$

For any $l, k \in \mathbb{Z}^+$, $l > k$, and from definition of $\mathbf{u}_k = \dfrac{\mathbf{d}_k}{\|\mathbf{d}_k\|}$, we have

$$\begin{aligned}\mathbf{x}_l - \mathbf{x}_k &= \sum_{i=k}^{l-1}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \sum_{i=k}^{l-1}\mathbf{s}_i = \sum_{i=k}^{l-1}\|\mathbf{s}_i\|\mathbf{u}_i \\ &= \sum_{i=k}^{l-1}\|\mathbf{s}_i\|(\mathbf{u}_k + \mathbf{u}_i - \mathbf{u}_k) \\ &= \sum_{i=k}^{l-1}\|\mathbf{s}_i\|\mathbf{u}_k + \sum_{i=k}^{l-1}\|\mathbf{s}_i\|(\mathbf{u}_i - \mathbf{u}_k). \end{aligned} \tag{23}$$

From (22), (23), and the triangle inequality, we have

$$\begin{aligned}\sum_{i=k}^{l-1}\|\mathbf{s}_i\| &\leq \|\mathbf{x}_l - \mathbf{x}_k\| + \sum_{i=k}^{l-1}\|\mathbf{s}_i\|\|\mathbf{u}_i - \mathbf{u}_k\| \\ &\leq 2\omega + \sum_{i=k}^{l-1}\|\mathbf{s}_i\|\|\mathbf{u}_i - \mathbf{u}_k\|. \end{aligned} \tag{24}$$

Note that from (20) and (22), we have

$$|\beta_k^{\#\#}| \leq Z_3\|\mathbf{s}_{k-1}\| \leq Z_3 L\|\mathbf{x}_k - \mathbf{x}_{k-1}\| \leq 2\omega Z_3 L = \mathcal{D},$$

and let $\triangle$ be a positive integer, chosen large enough such that

$$\triangle \geq \mathcal{D}.$$

Based on the criteria of Lemma 3, we can choose an index k_0 that is large enough, so that we have the following relationship

$$\sum_{i\geq k_0}\|\mathbf{u}_{i+1} - \mathbf{u}_i\|^2 \leq \frac{1}{4\triangle}. \tag{25}$$

Furthermore, if $i > k \geq k_0$ and $i - k \leq \triangle$, then using (25) and the Cauchy-Schwarz inequality, we get:

$$\begin{aligned}\|\mathbf{u}_i - \mathbf{u}_k\| &\leq \sum_{i=k}^{i-1}\|\mathbf{u}_{i+1} - \mathbf{u}_i\| \\ &\leq \sqrt{i-k}\left(\sum_{i=k}^{i-1}\|\mathbf{u}_{i+1} - \mathbf{u}_i\|^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{\triangle}\left(\frac{1}{4\triangle}\right)^{\frac{1}{2}} = \frac{1}{2}. \end{aligned} \tag{26}$$

From (24) together with (26), we obtain

$$\sum_{i=k}^{l-1}\|\mathbf{s}_i\| \leq 2\omega + \sum_{i=k}^{l-1}\|\mathbf{s}_i\|\frac{1}{2}.$$

Hence,

$$\sum_{i=k}^{l-1}\|\mathbf{s}_i\| \leq 4\omega. \tag{27}$$

Next, we will prove that the search direction $\mathbf{d}_k$ is bounded. Based on the $\beta_k^{\#\#+}$ value, there are two cases:

Case 1: if $\beta_k^{\#\#+} = 0$, then from (9) we have

$$\begin{aligned}\|\mathbf{d}_k\| &= \|-\mathbf{g}_k + \theta_k^{\#\#}\mathbf{g}_{k-1}\| \\ &\leq \|\mathbf{g}_k\| + |\theta_k^{\#\#}|\,\|\mathbf{g}_{k-1}\| \\ &\leq \gamma + \frac{\gamma}{\mu} := \mathcal{Q}. \end{aligned} \tag{28}$$

Furthermore, by using (11) and (28), it is clear that

$$\sum_{k=0}^{\infty}\frac{\left(\mathbf{g}_k^T\mathbf{d}_k\right)^2}{\|\mathbf{d}_k\|^2} \geq \sum_{k=0}^{\infty}\frac{4\omega^2 L^2\psi^2}{\mathcal{Q}^2} = +\infty,$$

which contradict (14) in Lemma 2. Hence, the relation (21) holds.

Case 2: if $\beta_k^{\#\#+} = \beta_k^{\#\#}$, then from (9) and (19) we have

$$\begin{aligned}\|\mathbf{d}_k\|^2 &= \left(\left\|-\mathbf{g}_k + \beta_k^{\#\#}\mathbf{d}_{k-1} + \theta_k^{\#\#}\mathbf{g}_{k-1}\right\|\right)^2 \\ &\leq \left(\left\|-\left(1 + \frac{\theta_k^{\#\#}\mathbf{g}_{k-1}\mathbf{g}_k^T}{\|\mathbf{g}_k\|^2}\right)\mathbf{g}_k + |\beta_k^{\#\#}|\|\mathbf{d}_{k-1}\|\right\|\right)^2 \\ &\leq \left(M_1 + |\beta_k^{\#\#}|\|\mathbf{d}_{k-1}\|\right)^2 \\ &\leq 2M_1^2 + 2|\beta_k^{\#\#}|^2\|\mathbf{d}_{k-1}\|^2 \\ &\leq 2M_1^2 + 2Z_3^2\|\mathbf{s}_{k-1}\|^2\|\mathbf{d}_{k-1}\|^2. \end{aligned}$$

Then, from the inequality above, we define $S_j = 2Z_3^2\|\mathbf{s}_j\|^2$, and in the same way as inequality (3.10) in [33], for all $l \geq k_0 + 1$, we get

$$\|\mathbf{d}_l\|^2 \leq 2M_1^2\left(\sum_{i=k_0+1}^{l}\prod_{j=i}^{l-1}S_j\right) + \|\mathbf{d}_{k_0}\|^2\prod_{j=i}^{l-1}S_j. \tag{29}$$

By using (27) and according to Theorem 3.1 of the reference [33], we can deduce that the right-hand of (29) is bounded, and the bound don't depend on l, suppose that $\mathcal{R}$. So, we obtain

$$\|\mathbf{d}_k\|^2 \leq \mathcal{R}. \tag{30}$$

Furthermore, based on (10) and (30), we have

$$\sum_{k=0}^{\infty}\frac{\left(\mathbf{g}_k^T\mathbf{d}_k\right)^2}{\|\mathbf{d}_k\|^2} \geq \sum_{k=0}^{\infty}\frac{\left(1 - \frac{1}{\mu}\right)^2\psi^4}{\mathcal{R}} = +\infty,$$

Therefore,

$$\sum_{k=0}^{\infty}\frac{\left(\mathbf{g}_k^T\mathbf{d}_k\right)^2}{\|\mathbf{d}_k\|^2} > \infty.$$

This contradicts (14). Hence, $\lim_{k\to\infty}\inf\|\mathbf{g}_k\| = 0$. The proof is completed. ■

IV. Computational Experiments

In this part, we present the results of the computational experiments to illustrate the efficacy of the proposed MTTBZAU method for solving unconstrained optimization problems. These was achieved by comparing the experimental results of proposed MTTBZAU method with TTRMIL [24] and MTTPRP [26] methods. Most of the test functions used for these experiment are considered from [30], [31], and [32]. The strong Wolfe line search conditions (3) and (4) were used in the computation for all methods with parameters value:

- MTTPRP: $\sigma = 0.8$, $\varphi = 10^{-4}$.
- TTRMIL and MTTBZAU: $\sigma = 10^{-3}$, $\varphi = 10^{-4}$.

In performing the computational test, the number of iterations (NOI), the number of function evaluations (NOF), and the CPU time are considered as parameters to determine the robustness of the proposed method. The value $\mu = 2$ and $\eta = 1$ was considered for the proposed method based on the article [25]. To stop the calculation, we use the same criteria for all methods $\|\mathbf{g}_k\| \leq 10^{-6}$ and the calculation is considered failed if the number of iterations exceeds 10,000.

MATLAB software with personal laptop; Intel Core i7 processor, 16 GB RAM, 64 bit Windows 10 Pro operating system was used to obtain the numerical results of each method. A total of thirty-seven test functions with ninety-eight problems from different dimensions and initial points are mostly considered from Andrei [30]. These problems are listed in Table I and have also been used in papers [15], [16], and [17].

The result of computational experiments from all problems in Table I are listed in Table II. With regard to the results in Table II, it is not be sufficient to determine which method has good numerical results. Therefore, we present the performance profile introduced by Dolan and Moré [34] in Figs. 1, 2, and 3 to clearly show the difference in numerical effects among the TTRMIL, MTTPRP, and MTTBZAU methods, based on following conditions. Suppose S as a set of solvers on a test set of problems P and $a_{p,s}$ as the NOI or NOF or CPU time needed to solve problem p by solver s. The ratio $r_{p,s}$ is the performance profile ratio, used to compare the performance and its formulated as:

$$r_{p,s} = \frac{a_{p,s}}{\min\{a_{p,s} : p \in P \text{ and } s \in S\}}.$$

Denote n_p as test problems and we let $r_{p,s} = 2\max\{r_{p,s} : s \in S\}$ for $a_{p,s}$ of the "FAIL" in Tables II, then the performance profile for each solver can be defined by

$$\rho_s(\tau) = \frac{1}{n_p} size\{p \in P : \log_2 r_{p,s} \leq \tau\}.$$

Thus $\rho_s(\tau)$ is the probability for the solver $s \in S$ that the $r_{p,s}$ output ratio is inside the $\tau \in \mathbb{R}$ factor. Further, the function $\rho_s(\tau)$ is the distribution function for the performance ratio. In general, the solver whose curve is at the top will win over the rest of the solvers.

From Table II, the MTTBZAU method only failed in the Extended Powel test function, but has solved all other test functions efficiently. However, the TTRMIL was unable to solve 2, 4, 8, 35, 36, 42, 45, 54, 82, and 84 problems , and the MTTPRP has failed to solve 18, 54 and 65 problems. Based on this, we can deduce that the MTTBZAU has the best performance compared to others.

Figs. 1, 2, and 3 illustrates the comparison based on performance profile for all methods in NOI, NOF, and CPU time, respectively. Based on the aspect of computing speed, the all figure indicate that the MTTBZAU method has the best performance compared to the TTRMIL and MTTPRP methods.

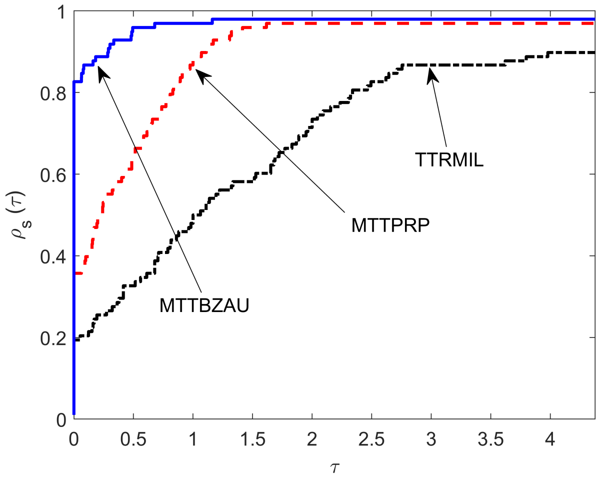

Fig. 1: Comparison performance of TTRMIL, MTTPRP, and MTTBZAU methods based on NOI.

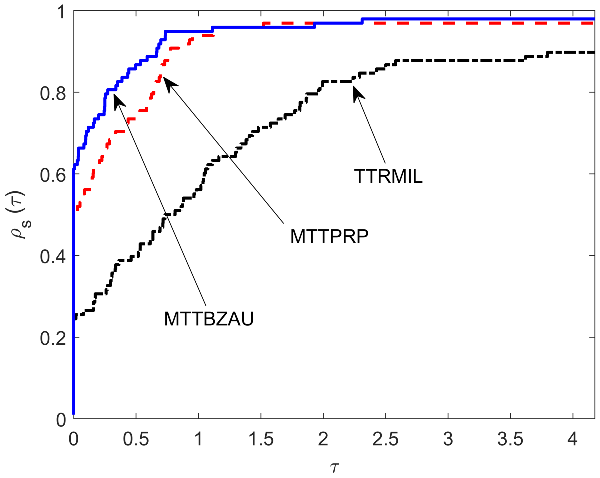

Fig. 2: Comparison performance of TTRMIL, MTTPRP, and MTTBZAU methods based on NOF.

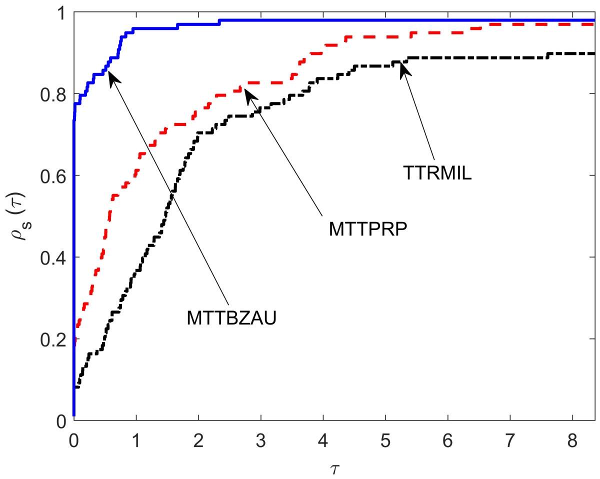

Fig. 3: Comparison performance of TTRMIL, MTTPRP, and MTTBZAU methods based on CPU Time.

TABLE I: List of the test functions, dimensions, and initial points.

Problems	Functions	Dimensions	Initial points	Problems	Functions	Dimensions	Initial points
1	Ext White & Holst	1,000	(-1.2, 1,...,-1.2,1)	50	Ext Maratos	10	(-1,..., -1)
2	Ext White & Holst	1,000	(10,...,10)	51	Six Hump Camel	2	(-1,2)
3	Ext White & Holst	10,000	(-1.2, 1,...,-1.2,1)	52	Six Hump Camel	2	(-5, 10)
4	Ext White & Holst	10,000	(5,...,5)	53	Three Hump Camel	2	(-1,2)
5	Ext Rosenbrock	1,000	(-1.2, 1,...,-1.2,1)	54	Three Hump Camel	2	(2,-1)
6	Ext Rosenbrock	1,000	(10,...,10)	55	Booth	2	(5, 5)
7	Ext Rosenbrock	10,000	(-1.2, 1,...,-1.2,1)	56	Booth	2	(10, 10)
8	Ext Rosenbrock	10,000	(5,...,5)	57	Trecanni	2	(-1,0.5)
9	Ext Freudenstein & Roth	4	(0.5,-2,0.5,-2)	58	Trecanni	2	(-5, 10)
10	Ext Freudenstein & Roth	4	(5,5,5,5)	59	Zettl	2	(-1,2)
11	Ext Beale	1,000	(1,0.8,...,1,0.8)	60	Zettl	2	(10,10)
12	Ext Beale	1,000	(0.5,...,0.5)	61	Shallow	1,000	(0,..., 0)
13	Ext Beale	10,000	(-1,..., -1)	62	Shallow	1,000	(10,..., 10)
14	Ext Beale	10,000	(0.5,...,0.5)	63	Shallow	10,000	(-1,..., -1)
15	Ext Wood	4	(-3,-1,-3,-1)	64	Shallow	10,000	(-10,..., -10)
16	Ext Wood	4	(5,5,5,5)	65	Gen Quartic	1,000	(1,...,1)
17	Raydan 1	10	(1,..., 1)	66	Gen Quartic	1,000	(20,...,20)
18	Raydan 1	10	(10,..., 10)	67	Quadratic QF2	50	(0.5,...,0.5)
19	Raydan 1	100	(-1,...,-1)	68	Quadratic QF2	50	(30,...,30)
20	Raydan 1	100	(-10,...,-10)	69	Leon	2	(2,2)
21	Ext Tridiagonal 1	500	(2,...,2)	70	Leon	2	(8,8)
22	Ext Tridiagonal 1	500	(10,...,10)	71	Gen Tridiagonal 1	10	(2,...,2)
23	Ext Tridiagonal 1	1,000	(1,...,1)	72	Gen Tridiagonal 1	10	(10,...,10)
24	Ext Tridiagonal 1	1,000	(-10,...,-10)	73	Gen Tridiagonal 2	4	(1,1,1,1)
25	Diagonal 4	500	(1,...,1)	74	Gen Tridiagonal 2	4	(10,10,10,10)
26	Diagonal 4	500	(-20,...,-20)	75	POWER	10	(1,...,1)
27	Diagonal 4	1,000	(1,...,1)	76	POWER	10	(10,...,10)
28	Diagonal 4	1,000	(-30,...,-30)	77	Quadratic QF1	50	(1,...,1)
29	Ext Himmelblau	1,000	(1,...,1)	78	Quadratic QF1	50	(10,...,10)
30	Ext Himmelblau	1,000	(20,...,20)	79	Quadratic QF1	500	(1,...,1)
31	Ext Himmelblau	10,000	(-1,...,-1)	80	Quadratic QF1	500	(-5,...,-5)
32	Ext Himmelblau	10,000	(50,...,50)	81	Ext Quad Pen QP2	100	(1, . . . ,1)
33	FLETCHCR	10	(0,...,0)	82	Ext Quad Pen QP2	100	(10,...,10)
34	FLETCHCR	10	(10,...,10)	83	Ext Quad Pen QP2	500	(10,...,10)
35	Ext Powel	100	(3,-1,0,1,...)	84	Ext Quad Pen QP2	500	(50,...,50)
36	Ext Powel	100	(5,...,5)	85	Ext Quad Pen QP1	4	(1,1,1,1)
37	NONSCOMP	2	(3,3)	86	Ext Quad Pen QP1	4	(10,10,10,10)
38	NONSCOMP	2	(10, 10)	87	Quartic	4	(10,10,10,10)
39	Ext DENSCHNB	10	(1, . . . , 1)	88	Quartic	4	(15,15,15,15)
40	Ext DENSCHNB	10	(10,...,10)	89	Matyas	2	(1,1)
41	Ext DENSCHNB	100	(10,...,10)	90	Matyas	2	(20, 20)
42	Ext DENSCHNB	100	(-50,...,-50)	91	Colville	4	(2,2,2,2)
43	Ex Penalty	10	(1,2,...,10)	92	Colville	4	(10,10,10,10)
44	Ex Penalty	10	(-10,...,-10)	93	Dixon and Price	3	(1,1,1)
45	Ex Penalty	100	(5,...,5)	94	Dixon and Price	3	(10,10,10)
46	Ex Penalty	100	(-10,..., -10)	95	Sphere	5,000	(1,...,1)
47	Hager	10	(1,...,1)	96	Sphere	5,000	(10,...,10)
48	Hager	10	(-10,...,-10)	97	Sum Squares	50	(0,1,...,0,1)
49	Ext Maratos	10	(1.1,0.1,...)	98	Sum Squares	50	(10,...,10)

TABLE II: Numerical results of the TTRMIL, MTTPRP, and MTTBZAU methods.

Problem	TTRMIL			MTTPRP			MTTBZAU		
	NOI	NOF	CPU	NOI	NOF	CPU	NOI	NOF	CPU
1	75	316	0.1497	25	159	0.0604	15	106	0.0564
2	FAIL	FAIL	FAIL	57	455	0.169	29	266	0.1185
3	60	262	1.0275	25	159	0.5981	15	106	0.397
4	FAIL	FAIL	FAIL	54	365	1.2764	25	231	0.8574
5	41	191	0.0575	30	148	0.0398	19	123	0.0268
6	107	440	0.1059	47	263	0.0611	19	136	0.0434
7	41	191	0.354	22	123	0.2473	19	123	0.2652
8	FAIL	FAIL	FAIL	23	137	0.2294	29	206	0.3816
9	50	187	0.0195	14	86	0.0099	8	48	9.16E-04
10	54	287	0.1799	18	105	9.48E-04	8	48	9.25E-04
11	33	113	0.0884	15	66	0.0426	10	55	0.0423
12	37	119	0.0907	12	48	0.0262	10	47	0.0384
13	32	114	0.5667	15	62	0.2579	9	43	0.1833
14	39	127	0.5599	12	48	0.2063	10	47	0.2203
15	512	1903	0.0387	120	542	0.0172	148	588	0.0083

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TABLE II – *Continued*

Problem	TTRMIL			MTTPRP			MTTBZAU		
	NOI	NOF	CPU	NOI	NOF	CPU	NOI	NOF	CPU
16	389	1258	0.0146	135	597	0.0052	302	1290	0.0262
17	19	90	0.0154	19	85	0.0058	17	80	0.0014
18	158	876	0.0207	FAIL	FAIL	FAIL	32	179	0.0037
19	110	332	0.0258	72	217	0.0169	76	361	0.0254
20	184	565	0.0453	181	558	0.0251	169	886	0.0484
21	13	65	0.0369	10	53	0.022	14	72	0.0304
22	22	112	0.039	9	43	0.0134	7	46	0.0191
23	16	88	0.0541	10	53	0.0305	16	86	0.0543
24	22	112	0.0686	10	50	0.0212	7	51	0.0359
25	2	6	0.0093	2	6	0.0067	2	6	0.0026
26	2	6	0.002	2	6	0.0019	2	6	0.0022
27	2	6	0.0032	2	6	0.0033	2	6	0.003
28	2	6	0.0032	2	6	0.0029	2	6	0.0036
29	9	32	0.0331	7	26	0.0153	7	31	0.0112
30	15	64	0.0272	12	59	0.0201	6	31	0.0098
31	10	41	0.1126	12	50	0.0976	8	44	0.0963
32	10	47	0.099	10	48	0.1181	7	38	0.0899
33	75	295	0.0183	46	206	0.0089	56	263	0.005
34	165	825	0.0132	88	500	0.0071	79	407	0.0067
35	FAIL	FAIL	FAIL	392	1316	0.0976	FAIL	FAIL	FAIL
36	FAIL	FAIL	FAIL	550	1824	0.1097	FAIL	FAIL	FAIL
37	19	68	0.0019	13	47	0.0032	11	54	0.0013
38	28	107	0.0207	25	115	0.0013	14	85	0.0011
39	8	24	6.00E-04	7	21	0.0039	5	19	6.07E-04
40	9	37	0.0013	10	41	7.88E-04	9	44	7.93E-04
41	9	37	0.0048	10	41	0.0043	9	44	0.0035
42	FAIL	FAIL	FAIL	11	52	0.0047	7	36	0.0023
43	27	95	0.0073	31	106	0.0068	38	134	0.003
44	16	64	0.0268	12	52	8.61E-04	7	39	7.73E-04
45	FAIL	FAIL	FAIL	20	102	0.007	8	47	0.0038
46	20	85	0.0134	19	114	0.0074	15	101	0.0071
47	12	36	7.79E-04	12	36	0.0096	12	37	9.73E-04
48	17	53	0.0087	19	62	0.0017	18	67	0.0011
49	34	169	0.0025	52	285	0.0136	38	281	0.0042
50	27	137	0.0136	36	211	0.0036	24	185	0.0042
51	8	29	6.46E-04	7	26	0.0065	7	34	5.47E-04
52	8	41	0.0111	11	55	7.36E-04	6	34	4.91E-04
53	18	62	0.0217	16	55	0.0081	9	273	0.0021
54	FAIL	FAIL	FAIL	FAIL	FAIL	FAIL	11	264	0.0024
55	2	6	1.63E-04	2	6	0.015	2	6	2.68E-04
56	2	6	0.0075	2	6	0.0029	2	6	1.85E-04
57	1	3	1.49E-04	1	3	0.0118	1	3	1.70E-04
58	16	59	0.0055	11	39	0.0052	5	24	4.02E-04
59	16	58	7.58E-04	14	53	0.0135	10	52	6.59E-04
60	29	108	0.0062	17	78	0.0038	9	50	8.51E-04
61	43	129	0.0443	8	24	0.0286	7	29	0.0112
62	18	72	0.0302	15	58	0.0267	11	56	0.0134
63	33	102	0.2472	9	29	0.0733	8	31	0.0739
64	49	165	0.3467	13	53	0.1293	11	53	0.1261
65	7	218	0.0535	FAIL	FAIL	FAIL	7	219	0.0541
66	16	80	0.0375	32	120	0.0421	12	305	0.0612
67	81	267	0.013	67	223	0.0271	70	250	0.0077
68	95	379	0.0149	77	313	0.0209	54	262	0.0106
69	56	242	0.0026	22	118	0.0083	14	121	0.0017
70	177	725	0.0073	54	418	0.0047	27	260	0.0033
71	22	69	0.0017	23	72	0.0147	23	77	0.0012
72	28	114	0.0083	32	127	0.0037	27	117	0.0029

(Continued on next page)

TABLE II – *Continued*

Problem	TTRMIL			MTTPRP			MTTBZAU		
	NOI	NOF	CPU	NOI	NOF	CPU	NOI	NOF	CPU
73	7	21	3.86E-04	5	15	3.92E-04	4	13	2.98E-04
74	10	38	0.0113	9	36	0.0135	11	57	8.21E-04
75	123	369	0.0042	10	30	7.64E-04	10	30	4.69E-04
76	139	417	0.0089	10	30	0.0083	10	30	5.95E-04
77	69	207	0.0086	38	114	0.0054	38	114	0.0041
78	78	234	0.0076	40	120	0.006	40	120	0.0051
79	447	1341	0.1612	131	393	0.0638	131	393	0.0522
80	500	1500	0.1862	137	411	0.0736	137	411	0.049
81	394	1465	0.0558	33	293	0.0228	25	259	0.0104
82	FAIL	FAIL	FAIL	41	413	0.0307	24	241	0.0111
83	111	844	0.1271	48	461	0.0858	44	488	0.0677
84	FAIL	FAIL	FAIL	47	429	0.089	45	508	0.0737
85	14	46	8.96E-04	8	28	4.18E-04	6	28	2.83E-04
86	20	77	0.0078	13	58	0.0126	7	37	6.13E-04
87	764	2502	0.0215	161	667	0.0119	151	637	0.0072
88	734	2386	0.0206	293	1071	0.0273	156	656	0.006
89	1	8	5.58E-04	1	8	1.60E-04	1	8	1.15E-04
90	1	8	6.33E-04	1	8	0.0068	1	8	1.60E-04
91	861	3094	0.0166	223	776	0.0118	253	1124	0.0085
92	502	1779	0.0127	279	1082	0.0248	91	377	0.0039
93	22	72	9.54E-04	21	68	8.74E-04	10	41	4.19E-04
94	56	216	0.0012	38	157	0.0136	53	252	0.0038
95	1	3	0.0093	1	3	0.0067	1	3	0.0054
96	1	3	0.0069	1	3	0.0179	1	3	0.0064
97	46	138	0.0061	25	75	0.0033	25	75	0.0023
98	81	243	0.0057	41	123	0.0154	41	123	0.0041

TABLE III: Historical Real Time Closing Prices of BBCA, UNVR, BBRI, TLKM, and ICBP (currency in IDR) in Three Years .

Date	BBCA.JK	UNVR.JK	BBRI.JK	TLKM.JK	ICBP.JK
Dec 28, 2020	33,850	7,350	4,170	3,310	9,575
Dec 21, 2020	33,625	7,425	4,160	3,320	9,525
Dec 14, 2020	34,000	7,600	4,280	3,510	9,700
Dec 07, 2020	33,675	7,475	4,280	3,280	9,825
Nov 30, 2020	31,950	7,600	4,300	3,250	9,950
Nov 23, 2020	31,925	7,750	4,270	3,460	10,600
Nov 16, 2020	33,000	7,725	4,020	3,220	10,100
Nov 09, 2020	31,950	7,750	4,000	2,990	9,700
Nov 02, 2020	31,500	8,075	3,560	2,830	9,875
Oct 26, 2020	28,950	7,825	3,360	2,620	9,650
Oct 19, 2020	28,850	7,925	3,290	2,630	9,725
Oct 12, 2020	28,800	8,000	3,250	2,750	9,750
Oct 05, 2020	28,875	8,050	3,150	2,730	10,050
Sep 28, 2020	27,525	8,000	3,100	2,680	10,075
Sep 21, 2020	28,050	7,925	3,160	2,690	10,050
Sep 14, 2020	28,150	8,025	3,220	2,890	10,175
Sep 07, 2020	29,525	8,300	3,250	2,810	10,250
Aug 31, 2020	31,900	8,400	3,550	2,860	10,300
Aug 24, 2020	32,475	8,250	3,690	2,960	10,325
Aug 17, 2020	31,650	8,200	3,560	3,000	10,100
Aug 10, 2020	32,025	8,200	3,340	3,030	10,175
Aug 03, 2020	30,900	8,125	3,110	2,980	10,175
Jul 27, 2020	31,200	8,400	3,160	3,050	9,200
Jul 20, 2020	30,500	8,050	3,090	3,020	9,175
Jul 13, 2020	30,600	8,125	3,100	3,060	9,275
Jul 06, 2020	31,000	7,925	3,110	3,110	9,225

(Continued on next page)

TABLE III – *Continued*

Date	BBCA.JK	UNVR.JK	BBRI.JK	TLKM.JK	ICBP.JK
Jun 29, 2020	29,350	7,900	3,050	3,120	9,425
Jun 22, 2020	28,225	7,900	3,030	3,190	9,225
Jun 15, 2020	27,875	8,050	3,100	3,280	8,925
Jun 08, 2020	28,350	8,200	3,030	3,030	8,600
Jun 01, 2020	28,625	8,050	3,110	3,230	8,625
May 25, 2020	25,950	7,750	2,950	3,150	8,150
May 18, 2020	23,825	8,050	2,480	3,180	9,600
May 11, 2020	23,925	8,575	2,240	3,100	9,700
May 04, 2020	26,225	8,050	2,590	3,190	9,625
Apr 27, 2020	25,850	8,275	2,730	3,500	9,875
Apr 20, 2020	24,600	7,500	2,630	3,090	9,900
Apr 13, 2020	27,125	6,875	2,830	3,230	10,200
Apr 06, 2020	27,975	7,250	2,790	3,120	10,100
Mar 30, 2020	27,475	7,100	2,890	3,200	10,000
Mar 23, 2020	27,550	6,800	3,230	3,090	9,975
Mar 16, 2020	23,675	6,225	2,810	2,880	8,975
Mar 09, 2020	27,800	7,225	3,720	3,310	10,975
Mar 02, 2020	31,000	7,450	4,010	3,750	10,950
Feb 24, 2020	31,450	6,825	4,190	3,490	10,275
Feb 17, 2020	33,075	7,500	4,510	3,690	10,950
Feb 10, 2020	33,400	7,475	4,550	3,640	10,775
Feb 03, 2020	33,800	7,900	4,550	3,790	11,500
Jan 27, 2020	32,400	7,950	4,460	3,800	11,375
Jan 20, 2020	34,050	8,175	4,740	3,920	11,700
Jan 13, 2020	34,375	8,400	4,630	3,810	11,575
Jan 06, 2020	33,625	8,250	4,410	3,980	11,525
Dec 30, 2019	34,000	8,575	4,420	3,980	11,250
Dec 23, 2019	33,475	8,560	4,430	3,990	11,175
Dec 16, 2019	33,300	8,325	4,360	4,020	11,525
Dec 09, 2019	31,800	8,245	4,280	3,990	11,450
Dec 02, 2019	31,975	8,450	4,170	4,100	11,425
Nov 25, 2019	31,400	8,360	4,090	3,930	11,325
Nov 18, 2019	31,525	8,430	4,210	4,050	11,425
Nov 11, 2019	31,375	8,445	4,090	4,080	11,400
Nov 04, 2019	31,400	8,630	3,990	4,110	11,275
Oct 28, 2019	31,625	8,740	4,180	4,080	11,625
Oct 21, 2019	31,000	8,630	4,230	4,280	11,425
Oct 14, 2019	30,800	8,625	4,170	4,190	11,275
Oct 07, 2019	30,625	8,835	3,920	4,170	11,625
Sep 30, 2019	30,225	9,085	3,950	4,190	12,225
Sep 23, 2019	30,350	9,400	4,180	4,310	11,950
Sep 16, 2019	29,950	9,230	4,160	4,290	11,950
Sep 09, 2019	30,150	9,320	4,310	4,160	11,600
Sep 02, 2019	30,125	9,410	4,270	4,210	11,900
Aug 26, 2019	30,500	9,770	4,270	4,450	12,050
Aug 19, 2019	29,975	9,165	4,080	4,380	11,775
Aug 12, 2019	29,800	8,995	4,210	4,280	11,625
Aug 05, 2019	30,325	8,965	4,330	4,260	11,300
Jul 29, 2019	30,825	9,000	4,450	4,280	10,850
Jul 22, 2019	30,975	8,960	4,440	4,160	10,575
Jul 15, 2019	31,000	9,160	4,480	4,270	10,850
Jul 08, 2019	30,050	8,955	4,510	4,180	10,275
Jul 01, 2019	29,850	8,990	4,400	4,280	10,100
Jun 24, 2019	29,975	9,000	4,360	4,140	10,150
Jun 17, 2019	29,400	9,060	4,360	4,040	10,025
Jun 10, 2019	29,000	8,960	4,230	3,990	9,975
May 27, 2019	29,100	8,900	4,100	3,900	9,800
May 20, 2019	28,050	8,705	3,850	3,750	9,700

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TABLE III – *Continued*

Date	BBCA.JK	UNVR.JK	BBRI.JK	TLKM.JK	ICBP.JK
May 13, 2019	25,900	8,320	3,790	3,510	9,300
May 06, 2019	28,050	8,840	4,120	3,790	9,825
Apr 29, 2019	28,375	8,930	4,380	3,820	9,625
Apr 22, 2019	28,100	9,100	4,330	3,910	9,175
Apr 15, 2019	28,125	9,880	4,460	3,860	9,100
Apr 08, 2019	27,500	9,820	4,310	3,830	9,050
Apr 01, 2019	27,650	9,755	4,270	4,060	9,075
Mar 25, 2019	27,550	9,840	4,110	3,960	9,325
Mar 18, 2019	27,425	9,830	4,050	3,800	10,325
Mar 11, 2019	27,700	9,960	3,950	3,750	10,425
Mar 04, 2019	27,200	9,620	3,850	3,740	10,250
Feb 25, 2019	27,700	9,810	3,870	3,910	10,300
Feb 18, 2019	27,450	9,900	3,900	3,840	10,400
Feb 11, 2019	26,800	9,600	3,770	3,790	10,425
Feb 04, 2019	27,600	9,965	3,890	3,850	10,600
Jan 28, 2019	28,175	10,000	3,920	3,870	10,750
Jan 21, 2019	27,500	9,810	3,780	3,880	10,750
Jan 14, 2019	27,125	9,785	3,820	4,020	10,150
Jan 07, 2019	26,250	9,690	3,730	3,860	10,150
Dec 31, 2018	26,025	9,560	3,660	3,710	10,600
Dec 24, 2018	26,000	9,080	3,660	3,750	10,450
Dec 17, 2018	25,850	9,320	3,620	3,760	10,525
Dec 10, 2018	25,825	8,900	3,680	3,730	10,100
Dec 03, 2018	25,950	8,800	3,620	3,670	9,700
Nov 26, 2018	26,050	8,450	3,620	3,680	9,850
Nov 19, 2018	25,100	8,430	3,480	3,990	8,925
Nov 12, 2018	24,825	8,295	3,490	4,050	8,775
Nov 05, 2018	24,000	8,065	3,340	3,920	8,575
Oct 29, 2018	24,000	8,545	3,280	3,940	8,925
Oct 22, 2018	23,600	8,835	2,990	3,630	8,775
Oct 15, 2018	23,375	8,970	3,020	3,730	8,775
Oct 08, 2018	23,250	8,860	2,950	3,680	8,725
Oct 01, 2018	23,050	8,535	2,980	3,530	8,800
Sep 24, 2018	24,150	9,405	3,150	3,640	8,825
Sep 17, 2018	23,700	9,415	3,120	3,600	8,950
Sep 10, 2018	23,975	9,420	3,070	3,590	8,925
Sep 03, 2018	24,850	8,895	3,030	3,390	8,850
Aug 27, 2018	24,800	8,770	3,180	3,490	8,675
Aug 20, 2018	25,075	8,600	3,270	3,290	8,875
Aug 13, 2018	23,375	8,505	3,050	3,350	8,650
Aug 06, 2018	23,875	8,710	3,390	3,500	8,975
Jul 30, 2018	23,450	8,990	3,330	3,460	8,775
Jul 23, 2018	23,225	8,865	3,090	3,940	8,600
Jul 16, 2018	23,100	8,840	2,980	3,990	8,600
Jul 09, 2018	23,025	9,380	2,970	4,020	8,875
Jul 02, 2018	20,925	9,150	2,840	3,860	8,700
Jun 25, 2018	21,475	9,220	2,840	3,750	8,850
Jun 18, 2018	21,925	8,840	2,980	3,580	8,400
Jun 04, 2018	22,250	9,080	3,140	3,610	8,850
May 28, 2018	22,700	9,120	3,080	3,520	8,700
May 21, 2018	22,550	9,360	3,120	3,560	8,300
May 14, 2018	21,700	9,235	2,940	3,490	8,000
May 07, 2018	22,750	9,800	3,160	3,630	8,375
Apr 30, 2018	22,025	9,070	3,030	3,730	8,200
Apr 23, 2018	22,925	9,600	3,490	3,830	8,350
Apr 16, 2018	22,975	10,315	3,660	3,770	8,325
Apr 09, 2018	22,900	10,040	3,550	3,660	8,350
Apr 02, 2018	22,725	10,155	3,480	3,650	8,175

(Continued on next page)

TABLE III – *Continued*

Date	BBCA.JK	UNVR.JK	BBRI.JK	TLKM.JK	ICBP.JK
Mar 26, 2018	23,300	9,905	3,600	3,600	8,275
Mar 19, 2018	23,800	10,195	3,600	3,660	8,700
Mar 12, 2018	23,350	9,940	3,680	3,820	8,750
Mar 05, 2018	23,300	10,110	3,690	4,150	9,000
Feb 26, 2018	22,875	10,780	3,790	4,070	9,100
Feb 19, 2018	24,250	10,890	3,790	4,030	8,925
Feb 12, 2018	23,450	10,910	3,840	4,010	8,975
Feb 05, 2018	23,575	11,080	3,710	3,950	8,925
Jan 29, 2018	23,975	11,005	3,740	4,000	8,800
Jan 22, 2018	22,700	10,910	3,850	4,150	8,825
Jan 15, 2018	22,450	10,890	3,620	4,160	8,700
Jan 08, 2018	22,425	10,850	3,540	4,130	8,850
Jan 01, 2018	22,250	10,800	3,590	4,280	9,275

V. Applications in Portfolio Selection and Robotic Motion Control

In this section, we will discuss the application of the MTTBZAU method in portfolio selection as in [35], [36], [37] and motion control as in [38], [39]. Another application of the conjugate gradient method can be seen in [40], [41], [42], [43], and [44].

A. Minimizing Risk in Portfolio Selection

A portfolio is defined as a collection of investments composed of various types of assets, such as bonds and stocks. One of the goals of investors in investing is to maximize returns, without forgetting the risk factors for investment that may occur. Return is one of the factors that motivates investors to invest and is also a reward for the courage of the investor to take the risk of his investment [45].

1) Return and Risk: Return is the level of profit that investors get in investing. The main source of return in investment consists of two components, namely yield and capital gain (loss). Yield is the return on investment for an investor expressed as a percentage. Yield measures the rate of return on a financial instrument, for example, stocks or bonds, which is based on dividends and interest rates. Capital gain is defined as the profit an investor receives when the selling price is reduced by the purchase price. The difference between the selling price and the buying price is then calculated as capital gain. This profit can occur in many assets such as property, goods, mutual funds, bonds, collectibles and businesses, and options. The opposite of capital gains is capital loss, which is a condition when the difference in selling price is lower than the purchase price. Based on the two sources of return above, we can calculate the total return on an investment with the formula:

$$\text{Total Return} = \text{yield} + \text{capital gain (loss)}$$

Besides calculating returns, investors also need to consider the level of risk of an investment as a basis for making investment decisions. Risk is the possible difference between the actual return received and the expected return. Please note that greater the possible of difference, then greater the investment risk. In its application, there are several sources of risk that can affect the amount of risk in an investment, including market risk, interest rate risk, inflation risk, business risk, liquidity risk, financial risk, country risk, and currency exchange rate risk. To reduce investment risk, investors need to diversify. Diversification is the spread or separation of investments into several assets classes, for example, stocks, currencies, property, options, land, gold, and bonds.

On the other hand, some investors may diversify their portfolios focusing on only one asset class, stocks for example. The problem that arises is which company shares should be included in the portfolio and what percentage of funds will be allocated in each of the selected shares.

Therefore, in this paper we focus on portfolio problems in only one asset class, namely stocks. We have collected five real stocks data for PT Bank Central Asia Tbk (BBCA.JK), PT Unilever Indonesia Tbk (UNVR.JK), PT Bank Rakyat Indonesia (Persero) Tbk (BBRI.JK), PT Telekomunikasi Indonesia Tbk (TLKM.JK), and PT Indofood CBP Sukses Makmur Tbk (ICBP.JK) from the database http://finance.yahoo.com. The website provides, the opening price, the highest price, the lowest price, the closing price, the adjusted price, and the volume of stocks. In this case we use the closing price, which is the price at the end of the trade on that day. The stocks we choose are listed as the 20 best blue chip stocks 2020 on the IDX. We can use daily, weekly, or monthly prices, but in this paper, we consider to use weekly prices [45].

2) Problem Formulations: Portfolio optimization is a process of selecting the proportions of various assets in a portfolio that make the portfolio better than others based on certain criteria. Some of the criteria that can be done to optimize a portfolio include: minimize risk, maximize return, and minimize risk with a certain target return [46], [47]. To adjust problem 1, in this paper we choose the optimal portfolio determination by minimizing risk. Portfolio optimization model formulation here, we will use the closing price only in Table III. First, we define the return of a stock at time t as follows:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \tag{31}$$

where P_t is the closing prices at time t and P_{t-1} is the closing prices at time $t-1$. We will also have formula of the mean of return of stock as follows:

$$\bar{r}_i = \frac{1}{n}\sum_{i=1}^{n} r_{it}, \tag{32}$$

where n is number of stocks and r_{it} is individual return on stock.

Apart from that, we also need the value of the variance and the covariance between two assets. The variance measures how far each number in the set is from the mean. The variance of the return of stock can be calculated by

$$\sigma_v^2 = \frac{1}{n-1}\sum_{t=1}^{n}(r_{it} - \bar{r}_i)^2, \tag{33}$$

where n, r_{it}, and $\bar{r}_i$ are total number of returns on stocks, individual return on stocks, and mean of returns of stocks, respectively [48].

Meanwhile, covariance measures the directional relationship between the returns on two assets. If positive covariance then that asset returns move together while a negative covariance means they move inversely. Covariance can be calculated by formula as follows:

$$cov(r_i, r_j) = \frac{1}{n-1}\sum_{t=1}^{n}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j), \tag{34}$$

where $\bar{r}_i, r_{it}$ are mean of return on stock and returns on stock of asset i, and $\bar{r}_j, r_{ij}$ are mean of return on stock and returns on stock of asset j, where $i \neq j$.

For our cases, portfolio risk is symbolized as σ_p^2 and is defined as the variance of the portfolio (see [46], [47]), i.e.:

$$\sigma_p^2 = \mathbf{X}^T\mathbf{V}\mathbf{X} \tag{35}$$

where $\mathbf{X}^T = \begin{bmatrix} x_1 \ x_2 \ x_3 \ x_4 \ x_5 \end{bmatrix}$, x_1, x_2, x_3, x_4 and x_5 are the invested fractions in BBCA.JK, UNVR.JK, BBRI.JK, TLKM.JK, and ICBP.JK assets, respectively, and $\mathbf{V}$ is variance-covariance matrix

$$\mathbf{V} = \begin{bmatrix} \sigma_{11}^2 & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & \sigma_{22}^2 & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & \sigma_{33}^2 & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & \sigma_{44}^2 & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & \sigma_{55}^2 \end{bmatrix} \tag{36}$$

where $\sigma_{11}^2, \sigma_{22}^2, \sigma_{33}^2, \sigma_{44}^2, \sigma_{5,5}^2$ are the variance of BBCA.JK, UNVR.JK, BBRI.JK, TLKM.JK, and ICBP.JK assets, respectively, which can be calculated using formula (33). Meanwhile, $C_{i,j}$ are the covariance between i asset and j asset, where $i \neq j, i, j = 1, 2, 3, 4, 5$, which can be obtained from (34).

Based on the above discussion, we can formulate a portfolio optimization problem by minimizing risk as:

$$\begin{cases} \text{minimize} : \sigma_p^2 = \mathbf{X}^T\mathbf{V}\mathbf{X} \\ \text{subject to} : \sum_{l=1}^{5} x_l = 1 \end{cases}. \tag{37}$$

The next step is to turn the minimization problem defined (37) into an unconstrained minimization problem. We suppose that $x_5 = 1 - x_1 - x_2 - x_3 - x_4$, then we can write

$$\mathbf{X}^T = \begin{bmatrix} x_1 \ x_2 \ x_3 \ x_4 \ 1 - x_1 - x_2 - x_3 - x_4 \end{bmatrix}. \tag{38}$$

Therefore, the problem (37) changes into a unconstrained optimization problem as follows:

$$\min_{\mathbf{X} \in \mathbb{R}^4} \mathbf{X}^T\mathbf{V}\mathbf{X}. \tag{39}$$

According to Table III, we have mean of return, and variance for BBCA.JK, UNVR.JK, BBRI.JK, TLKM.JK, and ICBP.JK assets as follows:

TABLE IV: Mean of return and variance for Five Stocks

Asset	Mean of Return	Variance
BBCA.JK	-0.00204	0.00134
UNVR.JK	0.00311	0.00127
BBRI.JK	0.00033	0.00273
TLKM.JK	0.00247	0.00166
ICBP.JK	0.00047	0.00142

Based on the value of return and mean return of each asset, we can obtain the value of covariance as in the following table.

TABLE V: Covariance of Five Stocks

	BBCA	UNVR	BBRI	TLKM	ICBP
BBCA	0.00134	0.00052	0.00133	0.00059	0.00049
UNVR	0.00052	0.00127	0.00058	0.00053	0.00062
BBRI	0.00133	0.00058	0.00273	0.00091	0.00059
TLKM	0.00059	0.00053	0.00091	0.00166	0.00048
ICBP	0.00049	0.00062	0.00059	0.00048	0.00142

Then, from (36) and Table V, we have

$$\mathbf{V} = \begin{bmatrix} 0.00134 & 0.00052 & 0.00133 & 0.00059 & 0.00049 \\ 0.00052 & 0.00127 & 0.00058 & 0.00053 & 0.00062 \\ 0.00133 & 0.00058 & 0.00273 & 0.00091 & 0.00059 \\ 0.00059 & 0.00053 & 0.00091 & 0.00166 & 0.00048 \\ 0.00049 & 0.00062 & 0.00059 & 0.00048 & 0.00142 \end{bmatrix}.$$

In this context, we employ our proposed method to solve the problem (39) and compare with TTRMIL and MTTPRP methods. We choose some initial points and we obtain the result as in the following table:

TABLE VI: Test Result of TTRMIL Method for Solving Portfolio Risk Optimization

Initial Point	NOI	NOF	CPU Time
$(0.25, 0.25, 0.25, 0.25)^T$	6	70	0.0013
$(0.35, 0.15, 0.35, 0.15)^T$	7	80	9.9340e-04
$(0.3, 0.3, 0.2, 0.2)^T$	6	70	7.8190e-04

TABLE VII: Test Result of MTTPRP Method for solving portfolio risk optimization

Initial Point	NOI	NOF	CPU Time
$(0.25, 0.25, 0.25, 0.25)^T$	4	48	9.6400e-04
$(0.35, 0.15, 0.35, 0.15)^T$	4	48	8.2000e-04
$(0.3, 0.3, 0.2, 0.2)^T$	4	48	5.6400e-04

TABLE VIII: Test Result of MTTBZAU Method for Solving Portfolio Risk Optimization

Initial Point	NOI	NOF	CPU Time
$(0.25, 0.25, 0.25, 0.25)^T$	4	48	5.7750e-04
$(0.35, 0.15, 0.35, 0.15)^T$	4	48	5.9160e-04
$(0.3, 0.3, 0.2, 0.2)^T$	4	48	5.2750e-04

Table VI, VII, and VIII display the numerical results of MTTPRP, TTRMIL, and MTTBZAU methods in NOI, NOF, and CPU time for, repectively. According to Table VI and Table VIII, it is clear that the MTTBZAU performs the best in NOI, NOF, and CPU time, which implies that the MTTBZAU method is efficient than TTRMIL method.

From Table VII and Table VIII, it is indicates that the MTTBZAU is efficient than MTTPRP under CPU time only. Both gave the same results for NOI and NOF. So, the MTTBZAU method requires less time to obtain optimal values, and applicable for portfolio selection problem.

Based on the test results, each method also gave the following results:

- Minimum point: $(0.3125, 0.2300, -0.1004, 0.3371)^T$.
- Minimum objective function value: $\sigma_p^2 = 0.00071$.
- Proportion of stock invested for BBCA.JK asset: $x_1 = 0.3125$.
- Proportion of stock invested for UNVR.JK asset: $x_2 = 0.2300$.
- Proportion of stock invested for BBRI.JK asset: $x_3 = -0.1004$.
- Proportion of stock invested for TLKM.JK asset: $x_4 = 0.3371$.
- Proportion of stock invested for ICBP.JK asset: $x_5 = 0.2208$.

Hence, selection of a portfolio with a minimum risk, which can be done with the proportion of stock investment in BBCA.JK asset is 31.25%, 23% in UNVR.JK, 33.71% in TLKM.JK and 22.08% in ICBP.JK asset. While for BBRI.JK asset, the value of the proportion of asset is -10.04%. A negative sign indicates that the investor is short shelling. Finally, the portfolio risk value is 0.00071, with expected portfolio return is 0.00098.

B. Robotic Motion Control

In this section, MTTBZAU method is used to solve the robotic motion control problem. The discrete-time kinematics equation of a two-joint planar robot manipulator at the position level is given in [49] as follows:

$$f(\theta_k) = r_k,$$

where $\theta_k = \theta(t_k)$ is the joint angle vector-effector position vector in $\mathbb{R}^2$, r_k is the end-effector position vector in $\mathbb{R}^2$ and $f(.)$ is the kinematics mapping function with known structure and formulated as

$$f(\theta) = \begin{bmatrix} h_1 \cos(\theta_1) + h_2 \cos(\theta_1 + \theta_2) \\ h_1 \sin(\theta_1) + h_2 \sin(\theta_1 + \theta_2) \end{bmatrix},$$

where h_1 and h_2 are the length of the rod links. For the robotic motion control problem, we must solve the minimization problem below at each computational time interval $t_k \in [0, t_f]$:

$$\min_{r_k \in \mathbb{R}^2} \frac{1}{2} \|r_k - r_{dk}\|^2,$$

where r_{dk} is the desired path vector at time instant t_k. In this problem, the end-effector is controlled to track a Lissajous curve, declared as:

$$r_{dk} = \begin{bmatrix} 0.2 \sin\left(\dfrac{\pi t_k}{5}\right) + 1.5 \\ 0.2 \sin\left(\dfrac{2\pi t_k}{5} + \dfrac{\pi}{3}\right) + \dfrac{\sqrt{3}}{2} \end{bmatrix}.$$

The initial point chosen for the joint angle vector is set as $\theta_0 = \left[0, \frac{\pi}{3}\right]^T$. For the length of the rod links, we set $h_1 = h_2 = 1$ and the time duration in the close interval $[0, 10]$ is divided into 200 equal parts.

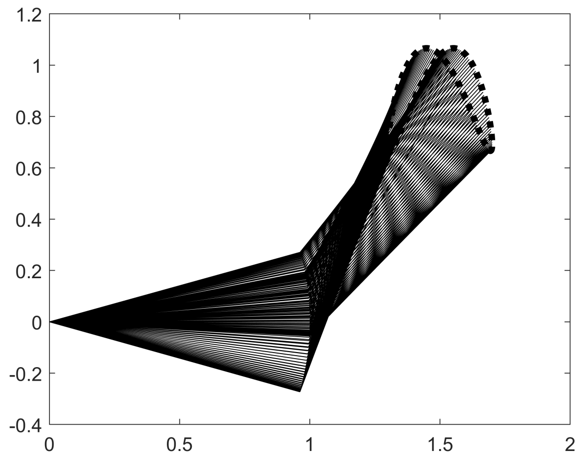

Fig. 4: Robot trajectories synthesized by MTTBZAU.

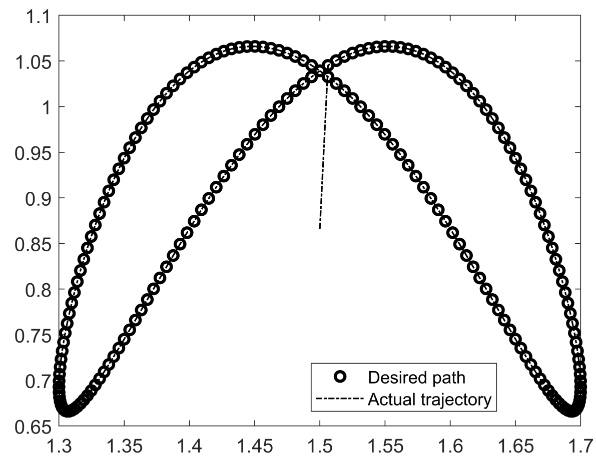

Fig. 5: End effector trajectory and desired path.

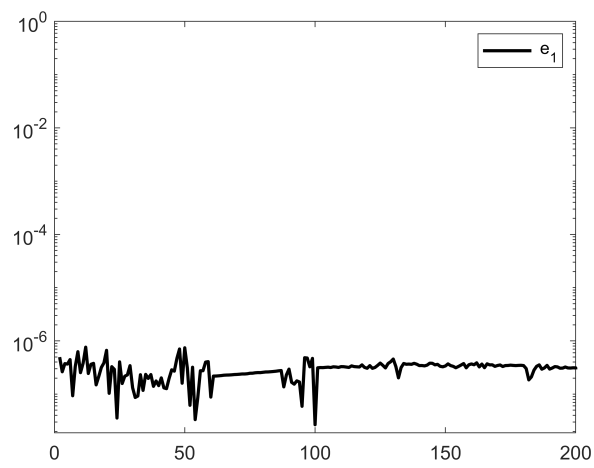

Fig. 6: Tracking residual error by MTTBZAU on x-axis.

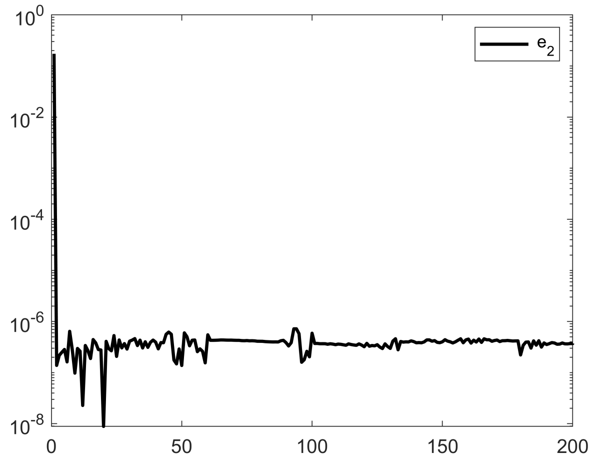

Fig. 7: Tracking residual error by MTTBZAU on y-axis.

We show the numerical results of the motion control problem by MTTBZAU method in Figs. 4, 5, 6 and 7. Fig. 4 represents robot trajectories synthesized. Fig. 5 represents the end effector trajectory and the desired path. Finally, tracking eror on x-axis and y-axis represents in Figs. 6 and 7, respectively. Based on Figs. 4 and 5, we can see that MTTBZAU successfully solves the robotic motion control problem. As shown in Figs. 6 and 7, the residual error produced by the MTTBZAU method is below 10^{-6}. So, it shows the effectiveness of the proposed method.

VI. Conclusion

In this paper, we proposed a new direction of three-term conjugate gradient method and established the descent condition based on some assumptions. The global convergence properties is presented under the strong Wolfe line search. Based on the numerical experiments, we conclude that the our proposed method is the best and efficient for NOI, NOF, and CPU time. The proposed method was extended to solve applications problem of portfolio selection and robotic motion control.

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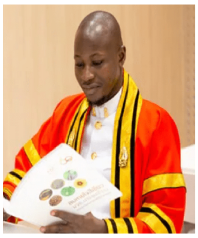
Maulana Malik received the B.Sc. and M.Sc. degrees in mathematics from the Universitas Indonesia (UI), Indonesia. He is currently pursuing the Ph.D. degree with the Universiti Sultan Zainal Abidin (UniSZA), Kuala Terengganu, Malaysia. Since 2016, he has been a Lecturer with the Department of Mathematics, Faculty of Mathematics and Natural Sciences, UI. His current research focuses on nonlinear optimization includes the conjugate gradient method and its application in financial mathematics and motion control.



Auwal Bala Abubakar received the master’s degree in mathematics and the Ph.D. degree in applied mathematics from the King Mongkut’s University of Technology Thonburi, Thailand, in 2015. He is currently a Lecturer with the Department of Mathematical Sciences, Faculty of Physical Sciences, Bayero University Kano, Nigeria. He is the author of more than 40 research articles. His main research interest includes methods for solving nonlinear monotone equations with application in signal recovery and image restoration.

Ibrahim Mohammed Sulaiman is currently a post-doctoral researcher at Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin (UniSZA) Malaysia, from 2019 till date. He obtained his Ph.D. from UniSZA in 2018 specializing in the field of fuzzy systems. He has published research papers in various international journals and attended international conferences. His research interest includes numerical research, Fuzzy nonlinear systems and unconstrained optimization.

Mustafa Mamat is currently a Professor in Faculty of Informatics and Computing at the Universiti Sultan Zainal Abidin since 2013. He obtained his Ph.D. from the UMT in 2007 with specialization in optimization. He was appointed as a Senior Lecturer in 2008 and the as an Associate Professor in 2010 also the UMT. To date, he has published more than 436 research paper in various international journals and conferences. His research interest in applied mathematics, with a field of concentration of optimization include conjugate gradient, steepest descent methods, Broyden’s family and quasi-Newton methods.

Siti Sabariah Abas is a Lecturer at Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin (UniSZA) Malaysia. She obtained her Ph.D. from the Universiti Sains Malaysia (USM) in 2016 with field in numerical analysis include the fluid dynamics.

Sukono (Member) is a Lecturer in the Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran. Master’s in Actuarial Sciences at Institut Teknologi Bandung, Indonesia in 2000, and Ph.D. in Financial Mathematics at the Universitas Gadjah Mada, Yogyakarta Indonesia in 2011. Currently serves as Head of Master Program in Mathematics, the field of applied mathematics, with a field of concentration of financial mathematics and actuarial sciences.