The Structure and Characterizaions of Normal Clifford Semirings

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Abstract—**In this paper, we define normal Clifford semirings, which are generalizations of rectangular Clifford semirings. We also give the necessary and sufficient conditions for a semiring to be normal Clifford semiring and the spined product decomposition of normal Clifford semirings. We also discuss a special case of this kind of semirings, that is strong distributive lattices of rectangular rings.**

Index Terms—**rectangular rings, normal Clifford semirings, Distributive lattice congruence, normal band.**

I. Introduction

A semiring $S = (S, +, \bullet)$ is an algebra with two binary operation "$+$", "$\bullet$" such that the additive reduct $(S.+)$ and the multiplicative reduct $(S, \bullet)$ are semigroups connected by ring-like distributive laws. A semiring $S = (S, +, \bullet)$ is called idempotent semiring, if $(\forall s \in S) s + s = s = s \bullet s$. [2] and [5] discussed left Clifford semirings and rectangular Clifford semirings respectively. In this paper we will generalize the rectangular Clifford semirings to the normal Clifford semirings. A semiring S is called a normal Clifford semiring if S is the distributive lattices of rectangular band semirings and the set of all additive idempotents of S is a normal band. Some structure and characterizations of normal Clifford semirings and a special case of this kind of semirings will be introduced by us.

II. Characterizations and Structure

Definition II.1. *A semiring S is called a normal Clifford semiring if S is a distributive lattice of a rectangular ring and $E^+(S)$ is a normal band.*

Similarly, we can get the definition of left normal Clifford semirings.

Theorem II.1. *A semiring S is a normal Clifford semiring if and only if the additive reduct $(S, +)$ of S is a normal orthogroup in which each maximal subgroup is abelian, $E^+(S) \subseteq E^{\cdot}(S)$ and S satisfies the following conditions.*
(1) $\forall s \in S, V^+(s) + s \supseteq s(s + V^+(s))$;
(2) $\forall s, t \in S, V^+(st) + st \supseteq (t + V^+(t))s$;
(3) $\forall s, t \in S, V^+(s) + s \supseteq s + st + V^+(st)) + V^+(s)$.

Proof: Necessity, if S is a normal Clifford semiring, then S is a distributive lattice D of rectangular rings $S_\alpha, \alpha \in D$, so $E^+(S) \subseteq E^\bullet(S)$ and the additive reduct $(S, +)$ of S is an upper semilattice D of rectangular commutative groups $(R_\alpha, +), \alpha \in D$, since $E^\bullet(S)$ is a normal band, the $(S, +)$ is a normal orthogroup in which each maximal subgroup is abelian. So $S/\overset{+}{\mathcal{D}}$ is the distributive lattice D. It is clear that

$$s \overset{+}{\mathcal{D}} t \Longleftrightarrow V^+(s) + s = t + V^+(t)$$
$$\Longleftrightarrow (V^+(s) + s) \cap (t + V^+(t)) \neq \emptyset.$$

Due to $\overset{+}{\mathcal{D}}$ is the distributive lattice congruence on semiring S, we get $s \overset{+}{\mathcal{D}} s^2$, $st \overset{+}{\mathcal{D}} ts$, $s(s+t) \overset{+}{\mathcal{D}} s$. Foy any $c \in sV^+(s)$, there exists $x \in V^+(s)$ such that $c = sx$, from the law of distribution, we have

$$s^2 + sx + s^2 = s(s + x + s) = ss = s^2;$$
$$sx + s^2 + sx = s(x + s + x) = sx,$$

then $sV^+(s) \subseteq V^+(s^2)$. Hence from

$$V^+(s) + s = s^2 + V^+(s^2) \supseteq s^2 + sV^+(s) = s(s + V^+(s)),$$

we can see

$$V^+(s) + s \supseteq s(s + V^+(s)) \quad (\forall s \in S). \tag{II.1}$$

Also, by

$$V^+(st) + st = ts + V^+(ts) \supseteq ts + V^+(t)s = (t + V^+(t))s,$$

we have

$$V^+(st) + st \supseteq (t + V^+(t))s \quad (\forall s, t \in S). \tag{II.2}$$

From

$$V^+(s) + s = s + st + V^+(s + st) \supseteq (s + st) + V^+(st) + V^+(s),$$

we have

$$V^+(s) + s \supseteq s + st + V^+(st) + V^+(s) \quad (\forall s, t \in S). \tag{II.3}$$

On the other hand, if the additive reduct $(S, +)$ of semiring S is a normal orthogroup in which each maximal subgroup is abelian, then $(S, +)$ is a semilattices $S/\overset{+}{\mathcal{D}}$ of rectangular commutative-groups $(S_\alpha, +)$. From the left and right distributive laws of multiplication over addition, we obtain that $\overset{+}{\mathcal{D}}$ is a congruence on $(S, \bullet)$. If (1) holds, through

$$s^2 + V^+(s^2) \supseteq s^2 + sV^+(s) = s(s + V^+(s))$$

we have

$$(s^2 + V^+(s^2)) \cap (V^+(s) + s) \neq \emptyset.$$

in other words for all $s \in S, s \overset{+}{\mathcal{D}} s^2$. If (2) holds, then, by

$$ts + V^+(ts) \supseteq ts + V^+(t)s = (t + V^+(t))s,$$

we have

$$(ts + V^+(ts)) \cap (V^+(st) + st) \neq \emptyset,$$

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in other words, for $s,t \in S, st \overset{+}{\mathcal{D}} ts$. If (3) holds, then, by

$$s + st + V^+(s+st) \supseteq s + st + V^+(ts) + V^+(s),$$

we have

$$(s + st + V^+(s+st) \cap (V^+(s) + s \neq \emptyset,$$

in other words for all $s,t \in S$, $(s+st)\overset{+}{\mathcal{D}} s$. So $\overset{+}{\mathcal{D}}$ is a distributive lattice congruence on semiring S. Because $E^+(S) \subseteq E^\bullet(S)$, each $\overset{+}{\mathcal{D}}$-class is a rectangular semiring. This indicates that S is a distributive lattice of rectangular semiring S_α. Since $E^+(S)$ is a normal band, the semiring S is a normal Clifford semiring. ■

corollary II.1. *A semiring S is a normal Clifford semiring if and only if $\overset{+}{\mathcal{D}}$ is a distributive lattice congruence on S, every $\overset{+}{\mathcal{D}}$-class is a rectangular semiring and $E^+(S)$ is a normal band.*

With the help of the research method of theoremII.1, we can get the following proposition:

Proposition II.1. *A semiring S is a left normal Clifford semiring if and only if the additive reduct $(S,+)$ of S is a left normal orthogroup in which each maximal subgroup is abelian, $E^+(S) \subseteq E^\cdot(S)$ and S satisfies the following conditions.*
(1) $\forall s \in S, V^+(s) + s \supseteq s(s + V^+(s))$;
(2) $\forall s,t \in S, V^+(st) + st \supseteq (t + V^+(t))s$;
(3) $\forall s,t \in S, V^+(s) + s \supseteq s + st + V^+(st)) + V^+(s)$.

Next, for some distributive lattice skeleton D, let $[D, L_\alpha, \varphi_{\alpha,\beta}]$ be the strong distributive lattice D decomposition of left normal band semiring L into left zero band semirings L_α, $\bigcup_{\alpha\in D} T_\alpha$ be the distributive lattice D decomposition of Clifforg semiring T into rings T_α, $[D, R_\alpha, \psi_{\alpha,\beta}]$ be the strong distributive lattice D decomposition of right normal band semiring R into right zero band semirings R_α, we have:

Theorem II.2. *The spined product $L\underline{\times}_D T\underline{\times}_D R = \bigcup_{\alpha\in D}(L_\alpha \times T_\alpha \times R_\alpha)$ of left normal band semiring $L = [D, L_\alpha, \varphi_{\alpha,\beta}]$, Clifford semiring $T = \bigcup_{\alpha\in D} T_\alpha$ and right normal band semiring $R = [D, R_\alpha, \psi_{\alpha,\beta}]$ with respect to the same distributive skeleton D is a normal Clifford semiring. On the other hand, every normal Clifford semiring can be expressed by such a spined product.*

Proof: The spined product $L\underline{\times}_D T\underline{\times}_D R$ is clear a distributive lattice D of rectangular semiring $L_\alpha \times T_\alpha \times R_\alpha$ and $E^+(S) = E^+(L)\underline{\times}_D E^+(T)\underline{\times}_D E^+(R)$, where $E^+(L)$ is a left normal band, $E^+(T) = \{0_\alpha | \alpha \in D\}$ and $E^+(R)$ is a right normal band, in fact $E^+(S)$ is isomorphic to the spined product of $E^+(L)\underline{\times}_D E^+(R)$, so $E^+(S)$ is a normal band, we can see $L\underline{\times}_D T\underline{\times}_D R$ is a normal Clifford semiring.
Conversely, let S is a normal Clifford semiring, so $(S,+)$ is a normal orthogroup and $(S,+)$ is the spined product of left normal band $(L,+) = [D,(L_\alpha,+),\varphi_{\alpha,\beta}]$, Clifford semigroup $(T,+) = [D,(T_\alpha,+),\phi_{\alpha,\beta}]$ and right normal band $(R,+) = [D,(R_\alpha,+),\psi_{\alpha,\beta}]$, where $(L_\alpha,+)$ is a left zero band, $(T_\alpha,+)$ is a commutative group and $(R_\alpha,+)$ is a right zero band, and hence, in S, if $(i,x,\lambda) \in L_\alpha \times T_\alpha \times R_\alpha$, $(j,y,\mu) \in L_\beta \times T_\beta \times R_\beta$, then we have

$$(i,x,\lambda) + (j,y,\mu) = (i+j, x+y, \lambda+\mu)$$

where $i+j(x+y, \lambda+\mu)$ is the sum of i and j (x and y, λ and μ) in $(L,+)((T,+),(R,+))$. Next, we will study the product of $(i,x,\lambda) \in L_\alpha \times T_\alpha \times R_\alpha$ and $(j,y,\mu) \in L_\beta \times T_\beta \times R_\beta$. Let

$$(i,x,\lambda)(j,y,\mu) = (k,z,c).$$

we will get that $k(z,c)$ only depends i and j (x and y, λ and μ). Let $(i,x,\lambda) \in L_\alpha \times T_\alpha \times R_\alpha$ and $(i,x',\lambda) \in L_\alpha \times T_\alpha \times R_\alpha$, then $(i,x,\lambda)\overset{+}{\mathcal{H}}(i,x',\lambda)$. Actually, $\overset{+}{\mathcal{H}}$ is a congruence on $(S,\bullet)$, so $(i,x,\lambda)(j,y,\mu)\overset{+}{\mathcal{H}}(i,x',\lambda)(j,y,\mu)$. Let

$$(i,x,\lambda)(j,y,\mu) = (k,z,c),$$
$$(i,x',\lambda)(j,y,\mu) = (k',z',c').$$

we have $(k,z,c)\overset{+}{\mathcal{H}}(k',z',c')$. It is clear that $k=k', c=c'$, so we can see k and c are not related to x. Similarly, we can prove k and c is not related to y. Also, in fact, $E^+(S)$ is an ideal of $(S,\bullet)$. So, for $(i,x,\lambda),(i',x,\lambda) \in L_\alpha \times T_\alpha \times R_\alpha$, $(j,y,\mu) \in L_\beta \times T_\beta \times R_\beta$, if

$$(i,x,\lambda)(j,y,\mu) = (k,z,c),$$
$$(i',x,\lambda)(j,y,\mu) = (k',z',c')$$

through the distributive laws of S, we have

$$\begin{aligned}(k,z,c) &= (i,x,\lambda)(j,y,\mu)\\ &= ((i,0,\lambda) + (i',x,\lambda))(j,y,\mu)\\ &= (i,0,\lambda)(j,y,\mu) + (i',x,\lambda)(j,y,\mu)\\ &= (k,0,c) + (k',z',c')\\ &= (k,z',c')\end{aligned}$$

and hence $z=z'$,$c=c'$, that is z and c are not related to i. In the same way, through the distributive laws of S, we will get that z and c are not related to j. For $(i,x,\lambda),(i,x,\lambda') \in L_\alpha \times T_\alpha \times R_\alpha$, $(j,y,\mu) \in L_\beta \times T_\beta \times R_\beta$, if

$$(i,x,\lambda)(j,y,\mu) = (k,z,c),$$
$$(i,x,\lambda')(j,y,\mu) = (k',z',c')$$

in view of the distributive laws of S, we have

$$\begin{aligned}(k,z,c) &= (i,x,\lambda)(j,y,\mu)\\ &= ((i,x,\lambda') + (i,0,\lambda))(j,y,\mu)\\ &= (i,x,\lambda')(j,y,\mu) + (i,0,\lambda)(j,y,\mu)\\ &= (k',z',c') + (k,0,c)\\ &= (k',z',c)\end{aligned}$$

and hence $k=k'$, $z=z'$, that is k and z is not related to λ. In the same way, through the distributive laws of S, we can show that k and z are not related to μ. In summary, we can see k(z,c) only depends on i and j (x and y,λ and μ). Next, we can define a multiplication on L (T, R) as follows: for any $i \in L_\alpha, j \in L_\beta$[$x \in T_\alpha, y \in T_\beta$ and $\lambda \in R_\alpha, \mu \in R_\beta$),

$$ij = k \Leftrightarrow (i,0,\lambda)(j,0,\mu) = (k,0,\lambda\mu);$$

$$[\lambda\mu = c \Leftrightarrow (i,0,\lambda)(j,0,\mu) = (ij,0,c)];$$

$$[xy = z \Leftrightarrow (i, x, \lambda)(j, y, \mu) = (ij, z, \lambda\mu)].$$

It is obvious that $(L, +, \bullet)[(T, +, \bullet), (R, +, \bullet)$ is a semiring and by the result in [1], we know $(L, +, \bullet)$ $[(T, +, \bullet)$, $(R, +, \bullet)]$ is a left normal band semiring [clifford semiring, right normal band semiring].
This illustrates that semiring S is a spined product of left normal band semiring $L = [D, L_\alpha, \varphi_{\alpha,\beta}]$, Clifford semiring $T = \bigcup_{\alpha\in D} T_\alpha$ and right normal band semiring $R = [D, R_\alpha, \psi_{\alpha,\beta}]$. ■

Example 1. *Let $S = \{e, a_1, a_2\}$. Define $+$ and $\bullet$ on S as below:*

$+$	e	a_1	a_2
e	e	a_1	a_2
a_1	a_1	a_1	a_1
a_2	a_2	a_2	a_2

$\bullet$	e	a_1	a_2
e	e	e	e
a_1	e	a_1	a_1
a_2	e	a_2	a_2

It is clearly that $(S, +)$ is the semigroup $(\{a_1, a_2\}, +)^1$ with identity e; $(S, \bullet)$ is the semigroup $(\{a_1, a_2\}, \bullet)^0$ with zero e, where $(\{a_1, a_2\}, +) = (\{a_1, a_2\}, \bullet)$ is a left zero band . We can see that the two side distributive laws of "$+$" over "$\bullet$" hold. So, $(S, +, \bullet)$ is a semiring and apparently it is a left regular band semiring. Obviously, $S_1 = (S, +, \bullet)$ is a left normal Clifford semiring.

If we define $+$ and $\bullet$ on S as below:

$+$	e	a_1	a_2
e	e	a_1	a_2
a_1	a_1	a_1	a_2
a_2	a_2	a_1	a_2

$\bullet$	e	a_1	a_2
e	e	e	e
a_1	e	a_1	a_2
a_2	e	a_1	a_2

It is easy to prove $S_2 = (S, +, \bullet)$ is a right normal Clifford semiring. So the spined product $S_1 \times S_2$ is a normal Clifford semiring.

In the same way, with the help of the research method of theoremII.2, we can get conclusions as follows:

Proposition II.2. *The spined product $L \underline{\times}_D T = \bigcup_{\alpha\in D}(L_\alpha \times T_\alpha)$ of left normal band semiring $L = [D, L_\alpha, \varphi_{\alpha,\beta}]$ and Clifford semiring $T = \bigcup_{\alpha\in D} T_\alpha$ with respect to the same distributive skeleton D is a left normal Clliford semiring. Conversely, every left normal Clifford semiring can be expressed by such a spined product.*

corollary II.2. *[2] For some distributive lattice skeleton D, let $\bigcup_{\alpha\in D} L_\alpha$ be the distributive lattice $D-$decomposition of left regular band semiring L into left zero band semirings L_α, $\bigcup_{\alpha\in D} R_\alpha$ be the distributive lattice $D-$decomposition of Cliforg semiring S into rings R_α, we have:*
The spined product $L \underline{\times}_D S = \bigcup_{\alpha\in D}(L_\alpha \times R\alpha)$ of left normal band semiring L and Clifford semiring S with respect to the same distributive skeleton D is a left Clliford semiring. On the other hand, every left Clifford semiring can be expressed by such a spined product.

III. A Special Case

Definition III.1. *Let D be a distributive lattice. For each $\alpha \in D$, we associate α with a semiring S_α and assume that $S_\alpha \cap S_\beta = \emptyset$, if $\alpha \neq \beta$. Now for any $\alpha, \beta \in D$ with $\alpha \leqslant \beta$, let*

$$\varphi_{\alpha,\beta} : S_\alpha \to S_\beta,$$

be a semiring homomorphism satisfying the following conditions: For any $\alpha, \beta, \gamma \in D$,

$$\varphi_{\alpha,\alpha} = 1_{S_\alpha},\ the\ identity\ mapping\ on\ S_\alpha, \quad \text{(III.1)}$$

$$\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma},\ if\ \alpha \leqslant \beta \leqslant \gamma, \quad \text{(III.2)}$$

$$\varphi_{\alpha,\beta}\ \ is\ injective,\ if\ \alpha \leqslant \beta, \quad \text{(III.3)}$$

$$S_\alpha\varphi_{\alpha,\gamma}S_\beta\varphi_{\beta,\gamma} \subseteq S_{\alpha\beta,\gamma}\varphi_{\alpha\beta,\gamma},\ if\ \alpha + \beta = \gamma. \quad \text{(III.4)}$$

On $S = \bigcup_{\alpha\in D} S_\alpha$, "$+$" and "$\bullet$" are defined as follows: For any $s \in S_\alpha$ and $t \in S_\beta$

$$s + t = s\varphi_{\alpha,\alpha+\beta} + t\varphi_{\beta,\alpha+\beta} \quad \text{(III.5)}$$

and

$$st = (s\varphi_{\alpha,\alpha+\beta}t\varphi_{\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta}. \quad \text{(III.6)}$$

With the above operations S is a semiring and each S_α is a subsemiring of S. Denote the system using $S = [D, S_\alpha, \varphi_{\alpha,\beta}]$ and call it the strong distributive lattice D of semirings S_α

Theorem III.1. *Every strong distributive lattice $S = [D, S_\alpha, \varphi_{\alpha,\beta}]$ of rectangular semirings S_α is a normal Clifford semiring if and only if $E^+(S)$ is left unitary in $(S, +)$.*

Proof: Necessity, obviously, a strong distributive lattice $S = [D, S_\alpha, \varphi_{\alpha,\beta}]$ of rectangular semirings S_α is a normal Clifford semiring. Let $S_\alpha = I_\alpha \times T_\alpha$, where $I_\alpha(T_\alpha)$ is a rectangular band semiring (ring), $\alpha \in D$. $\forall \alpha, \beta \in D$, if $(\lambda, s) \in S_\alpha, (\mu, 0) \in E^+(S_\alpha), (k, 0) \in E^+(S_{\alpha+\beta})$ such that $(\lambda, s) + (\mu, 0) = (k, 0)$, so

$$(\lambda, s)\varphi_{\alpha,\alpha+\beta} + (\mu, 0)\varphi_{\beta,\alpha+\beta} = (k, 0)$$

Denote

$$(\lambda, s)\varphi_{\alpha,\alpha+\beta} = (k', s') \in S_{\alpha+\beta},$$

$$(\mu, 0)\varphi_{\beta,\alpha+\beta} = (k'', 0) \in S_{\alpha+\beta}.$$

We have

$$(k', s') + (k'', 0) = (k' + k'', s') = (k, 0),$$

and so $s' = 0$, that is, $(\lambda, s)\varphi_{\alpha,\alpha+\beta} = (k', 0) \in E^+(S_{\alpha+\beta})$. Because $\varphi_{\alpha,\alpha+\beta}$ is injective, we get $s = 0$, that is, $(\lambda, s) \in E^+(S_\alpha)$. We can see $E^+(S)$ is left unitary in $(S, +)$.

On the other hand, if S is a normal Clifford semiring, then S is a distributive lattice D of the rectangular semirings $S_\alpha = I_\alpha \times T_\alpha$, where $I_\alpha(T_\alpha)$ is a rectangular band semiring (ring), $\alpha \in D$, and $E^+(S)$ is a normal band. For any $\alpha, \beta \in D$ with $\alpha \leqslant \beta$ and a fixed $(\mu, 0) \in E^+(S_{\beta)}$, $\forall(\lambda, s) \in S_\alpha$,

$$\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$$

by

$$(\lambda, s)\varphi_{\alpha,\beta} = (\lambda, s) + (\mu, 0) + (\lambda, 0) \quad (\forall(\lambda, s) \in S_\alpha)$$

If $(\mu^{'},0) \in S_\alpha$, then because $E^+(S_\beta)$ is a rectangular band and $E^+(S)$ is a normal band, we have

$$\begin{aligned}&(\lambda,s)+(\mu,0)+(\lambda,0)\\=&(\lambda,s)+(\lambda,0)+(\mu,0)+(\mu^{'},0)+(\mu,0)+(\lambda,0)\\=&(\lambda,s)+(\lambda,0)+(\mu^{'},0)+(\mu,0)+(\mu,0)+(\lambda,0)\\=&(\lambda,s)+(\lambda,0)+(\mu^{'},0)+(\mu,0)+(\lambda,0)\\=&(\lambda,s)+(\mu^{'},0)+(\mu^{'},0)+(\mu,0)+(\lambda,0)\\=&(\lambda,s)+(\mu^{'},0)+(\mu,0)+(\mu^{'},0)+(\lambda,0)\\=&(\lambda,s)+(\mu^{'},0)+(\lambda,0).\end{aligned}$$

So we can see that the definition of $\varphi_{\alpha,\beta}$ is not dependent on choice of the element in $E^+(S_\beta)$. Next,, for any $(\lambda,s),(\lambda^{'},s^{'}) \in S_\alpha$, we have

$$((\lambda,s)+(\lambda^{'},s^{'}))\varphi_{\alpha,\beta} = (\lambda,s)+(\lambda^{'},s^{'})+(\mu,0)+(\lambda,0)+(\lambda^{'},0)$$

If $(\lambda^{'},s^{'})+(\mu,0) = (\mu^{'},t^{'})$, then

$$\begin{aligned}&((\lambda,s)+(\lambda^{'},s^{'}))\varphi_{\alpha,\beta}\\=&(\lambda,s)+(\mu^{'},0)+(\lambda^{'},s^{'})+(\mu,0)+(\lambda,0)+(\lambda^{'},0)\\=&(\lambda,s)+(\lambda,0)+(\lambda,0)+(\mu^{'},0)+(\lambda^{'},0)+(\lambda^{'},s^{'})\\&+(\lambda^{'},0)+(\lambda^{'},0)+(\mu,0)+(\lambda,0)+(\lambda^{'},0)\\=&(\lambda,s)+(\lambda,0)+(\mu^{'},0)+(\lambda,0)+(\lambda^{'},0)+(\lambda^{'},s^{'})\\&+(\lambda^{'},0)+(\lambda^{'},0)+(\lambda,0)+(\mu,0)+(\lambda^{'},0)\\=&(\lambda,s)+(\mu^{'},0)+(\lambda,0)+(\lambda^{'},s^{'})+(\lambda^{'},0)+(\lambda,0)\\&+(\lambda^{'},0)+(\mu,0)+(\lambda^{'},0)\\=&(\lambda,s)+(\mu^{'},0)+(\lambda,0)+(\lambda^{'},s^{'})+(\lambda^{'},0)+(\mu,0)\\&+(\lambda^{'},0)\\=&(\lambda,s)+(\mu^{'},0)+(\lambda,0)+(\lambda^{'},s^{'})+(\mu,0)+(\lambda^{'},0)\\=&(\lambda,s)\varphi_{\alpha,\beta}+(\lambda^{'},s^{'})\varphi_{\alpha,\beta}.\end{aligned}$$

Alao, because $E^+(S)$ is an ideal of $(S,\bullet)$, we have

$$\begin{aligned}&((\lambda,s)(\lambda^{'},s^{'}))\varphi_{\alpha,\beta}\\=&(\lambda,s)(\lambda^{'},s^{'})+(\mu,0)+(\lambda\lambda^{'},0)\\=&(\lambda,s)(\lambda^{'},s^{'})+(\lambda,s)(\mu,0)+(\lambda,s)(\lambda^{'},0)+(\mu,0)(\lambda^{'},s^{'})\\&+(\mu,0)+(\mu,0)(\lambda^{'},0)+(\lambda,0)(\lambda^{'},s^{'})+(\lambda,0)(\mu,0)\\&+(\lambda,0)(\lambda^{'},0)\\=&[(\lambda,s)+(\mu,0)+(\lambda,0)][(\lambda^{'},s^{'})+(\mu,0)+(\lambda^{'},0)]\\=&(\lambda,s)\varphi_{\alpha,\beta}(\lambda^{'},s^{'})\varphi_{\alpha,\beta}.\end{aligned}$$

This indicates that $\varphi_{\alpha,\beta}$ is a semiring homomorphism. Obviously $\varphi_{\alpha,\beta}$ satisfies(III.1)(III.2). If $(\lambda,s),(\lambda^{'},s^{'}) \in S_\alpha$, we have $(\lambda,s)\varphi_{\alpha,\beta} = (\lambda^{'},s^{'})\varphi_{\alpha,\beta}$, that is

$$(\lambda,s)+(\mu,0)+(\lambda,0) = (\lambda^{'},s^{'})+(\mu,0)+(\lambda^{'},0).$$

Through left-adding $(\lambda,-s^{'})$ on both sides of the above formula, we get

$$(\lambda,s-s^{'})+(\mu,0)+(\lambda,0) = (\lambda+\lambda^{'},0)+(\mu,0)+(\lambda^{'},0).$$

Because $E^+(S)$ is left unitary in $(S,+)$ and $(\mu,0)+(\lambda,0),(\lambda+\lambda^{'},0)+(\mu,0)+(\lambda^{'},0) \in E^+(S)$, $(\lambda,s-s^{'}) \in E^+(S)$, that is $s=s^{'}$, and so

$$(\lambda,s)+(\mu,0)+(\lambda,0) = (\lambda^{'},s)+(\mu,0)+(\lambda^{'},0)$$

Now, by right-adding $(\lambda,-s)+(\mu,0)+(\lambda,0)$ on both sides of the above formula, we get

$$\begin{aligned}&(\lambda,s)+(\mu,0)+(\lambda,0)+(\lambda,-s)+(\mu,0)+(\lambda,0)\\=&(\lambda^{'},s)+(\mu,0)+(\lambda^{'},0)+(\lambda,-s)+(\mu,0)+(\lambda,0)\end{aligned}$$

And hence

$$(\lambda,s)\varphi_{\alpha,\beta}+(\lambda,-s)\varphi_{\alpha,\beta} = (\lambda^{'},s)\varphi_{\alpha,\beta}+(\lambda,-s)\varphi_{\alpha,\beta}.$$

Since $\varphi_{\alpha,\beta}$ is a homomorphism, we can see

$$[(\lambda,s)+(\lambda,-s)]\varphi_{\alpha,\beta} = [(\lambda^{'},s)+(\lambda,-s)]\varphi_{\alpha,\beta},$$

that is

$$(\lambda,0)+(\mu,0)+(\lambda,0) = (\lambda^{'}+\lambda,0)+(\mu,0)+(\lambda^{'}+\lambda,0)$$

Then because $E^+(S)$ is a normal band, we get

$$(\lambda,0)+(\mu,0)+(\lambda,0) = (\lambda^{'},0)+(\mu,0)+(\lambda,0).$$

Through left-multiplying $(\lambda,0)$ on both sides of the above formula, we get

$$(\lambda,0)+(k,0)+(\lambda,0) = (\lambda\lambda^{'},0)+(k,0)+(\lambda,0),$$

where $(k,0) = (\lambda,0)(\mu,0) \in E^+(S_{\alpha\beta}) = E^+(S_\alpha)$, and hence $(\lambda,0) = (\lambda\lambda^{'},0)$. So $\lambda = \lambda\lambda^{'}$. Similary, we can show $\lambda^{'} = \lambda\lambda^{'}$, so we obtain $\lambda = \lambda^{'}$. This indicates that $\varphi_{\alpha,\beta}$ satisfies (III.3). Let $(\lambda,s) \in S_\alpha,(\mu,t) \in S_\beta,(k,0) \in S_\gamma$, where $\alpha+\beta \leqslant \gamma$. Then

$$\begin{aligned}&(\lambda,s)\varphi_{\alpha,\beta}(\mu,t)\varphi_{\beta,\gamma}\\=&[(\lambda,s)+(k,0)+(\lambda,0)][(\mu,t)+(k,0)+(\mu,0)]\\=&(\lambda,s)(\mu,t)+(\lambda,s)(k,0)+(\lambda,s)(\mu,0)+(k,0)(\mu,t)+(k,0)\\&+(k,0)(\mu,0)+(\lambda,0)(\mu,t)+(\lambda,0)(k,0)+(\lambda,0)(\mu,0)\\=&[(\lambda,s)(\mu,t)]\varphi_{\alpha\beta,\gamma}\end{aligned}$$

This is because $(\lambda,s)(k,0)+(\lambda,s)(\mu,0)+(k,0)(\mu,t)+(k,0)+(k,0)(\mu,0)+(\lambda,0)(\mu,t)+(\lambda,0)(k,0) \in E^+(S_\gamma)$, and hence(III.4) holds. For $(\lambda,s) \in S_\alpha,(\mu,t) \in S_\beta$, if

$$(\lambda,s)+(\mu,t) = (k,c) \in S_{\alpha+\beta},$$

then

$$\begin{aligned}(\lambda,s)+(\mu,t) &= (\lambda,s)+(\mu,t)+(\mu,0)\\&= (\lambda,s)+(\mu,t)+(k,0)+(\mu,0)\\&= (\lambda,s)+(\lambda,0)+(\mu,t)+(k,0)+(\mu,0).\end{aligned}$$

We now let

$$(\lambda,0)+(\mu,t) = (l,d) \in S_{\alpha+\beta},$$

then we have

$$\begin{aligned}(\lambda,s)+(\mu,t) &= (\lambda,s)+(l,0)+(\lambda,0)+(\mu,t)+(k,0)+(\mu,0)\\&= (\lambda,s)\varphi_{\alpha,\alpha+\beta}+(\mu,t)\varphi_{\beta,\alpha+\beta}.\end{aligned}$$

This indicates that (III.5) holds. Also, for $(\lambda,s) \in S_\alpha,(\mu,t) \in S_\beta,(k,0) \in E^+(S_{\alpha+\beta})$, we have

$$\begin{aligned}&[(\lambda,s)(\mu,t)]\varphi_{\alpha\beta,\alpha+\beta}\\=&(\lambda,s)(\mu,t)+(k,0)+(\lambda,0)(\mu,0)\\=&(\lambda,s)(\mu,t)+(\lambda,s)(k,0)+(\lambda,s)(\mu,0)+(k,0)(\mu,t)\\&+(k,0)+(k,0)(\mu,0)+(\lambda,0)(\mu,t)+(\lambda,0)(k,0)\\&+(\lambda,0)(\mu,0)\\=&[(\lambda,s)+(k,0)+(\lambda,0)][(\mu,t)+(k,0)+(\mu,0)]\\=&(\lambda,s)\varphi_{\alpha,\alpha+\beta}(\mu,t)\varphi_{\beta,\alpha+\beta}.\end{aligned}$$

Since $\varphi_{\alpha\beta,\alpha+\beta}$ is injective, we obtain

$$(\lambda, s)(\mu, t) = [(\lambda, s)\varphi_{\alpha,\alpha+\beta}(\mu, t)\varphi_{\beta,\alpha+\beta}]\varphi^{-1}_{\alpha\beta,\alpha+\beta}$$

Hence, (III.6) holds and the prove is completed. ∎

Similarly, we can verify the following conclusion by the research method of theoremIII.1:

Proposition III.1. *Every strong distributive lattice* $S = [D, S_\alpha, \varphi_{\alpha,\beta}]$ *of left semiring* S_α *is a left normal Clifford semiring if and only if* $E^+(S)$ *is left unitary in* $(S, +)$.

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