Study on Fractional p-Laplacian Differential Equation with Sturm-Liouville Boundary Value Conditions

Tingting Xue*, Fanliang Kong, and Long Zhang

Abstract—**In this paper, we concern with the fractional p-Laplacian differential equation with Sturm-Liouville boundary value conditions**

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^CD_t^{\alpha}u(t)\right)\right) \\ \quad +a(t)\,\phi_p(u(t)) = f(t,u(t)), a.e. t\in[0,T], \\ \alpha_1\phi_p(u(0)) - \alpha_2\,{}_tD_T^{\alpha-1}\left(\phi_p\left({}_0^CD_t^{\alpha}u(0)\right)\right) = 0, \\ \beta_1\phi_p(u(T)) + \beta_2\,{}_tD_T^{\alpha-1}\left(\phi_p\left({}_0^CD_t^{\alpha}u(T)\right)\right) = 0, \end{cases}$$

where ${}_0^CD_t^{\alpha}$, ${}_tD_T^{\alpha}$ are the left Caputo and right Riemann-Liouville fractional derivatives of order $\alpha\in(\frac{1}{2},1]$, respectively. The Nehari manifold method, Mountain Pass Theorem and the properties of genus are used to study the existence results of solutions of the above-mentioned Sturm-Liouville problem. More general superlinear conditions are used in the proof of the theorems.

Index Terms—**fractional differential equation, Sturm-Liouville boundary value conditions, ground state solution, Nehari manifold.**

I. Introduction

FRACTIONAL differential equations have been extensively applied in mathematical modeling. The theory of fractional differential equations is a hot topic in recent years. Many scholars have developed a strong interest in this kind of problem and achieved some excellent results [1-8]. Recently, scholars have also discussed equations involving left and right fractional differential operators. It has become a new research field of fractional calculus theory. Among them, the variational method is a good way to study such equations. Left and right fractional differential operators are widely used in the physical phenomena of anomalous diffusion, such as fractional convection diffusion equation [9-10]. In [11], Ervin and Roop first proposed a class of steady-state fractional convection-diffusion equations with variational structure

$$\begin{cases} -aD\left(p\,{}_0D_t^{-\beta} + q\,{}_tD_T^{-\beta}\right)Du + b(t)\,Du + c(t)\,u = f, \\ u(0) = u(T) = 0, \end{cases}$$

where $0\le\beta<1$, D is the classical first derivative, ${}_0D_t^{-\beta}$, ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional derivatives. The authors constructed a suitable fractional derivative space. The Lax-Milgram theorem is used to study the above problems. In [12], the authors discussed the following Dirichlet problems

$$\begin{cases} \frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\right) \\ \quad +\nabla F(t,u(t)) = 0, \quad \text{a.e. } t\in[0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $0\le\beta<1$. A suitable variational framework of the above problem was given, and some existence results of the solution were obtained by Mountain Pass Theorem and the minimization principle under the Ambrosetti-Rabinowitz condition. The following year, the authors [13] further investigated the following problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\frac{1}{2}<\alpha\le1$. Under the Ambrosetti-Rabinowitz condition, the existence of the weak solution was obtained by using Mountain Pass Theorem. In addition, the authors also discussed the regularity of the weak solution.

Tian and Nieto [14] studied the following Sturm-Liouville boundary value problems

$$\begin{cases} -\frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\right) \\ \quad = \lambda f(u(t)), \text{ a.e. } t\in[0,T], \\ au(0) - b\left(\frac{1}{2}{}_0D_t^{-\beta}u'(0) + \frac{1}{2}{}_tD_T^{-\beta}u'(0)\right) = 0, \\ cu(T) + d\left(\frac{1}{2}{}_0D_t^{-\beta}u'(T) + \frac{1}{2}{}_tD_T^{-\beta}u'(T)\right) = 0, \end{cases}$$

where $0\le\beta<1$, $a,c>0$, $b,d\ge0$, $\lambda>0$. The variational structure of the problem was established and the existence result of the unbounded sequence of the solution was obtained by the critical point theory. The results of this document are applicable to problems with continuous non-linearity and Dirichlet boundary conditions. Subsequently, Nyamoradi Nemat and Tersian Stepan [15] further studied Sturm-Liouville problems with a p-Laplacian operator

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^CD_t^{\alpha}u(t)\right)\right) \\ \quad +a(t)\,\phi_p(u(t)) = f(t,u(t)), a.e. t\in[0,T], \\ \alpha_1\phi_p(u(0)) - \alpha_2\,{}_tD_T^{\alpha-1}\left(\phi_p\left({}_0^CD_t^{\alpha}u(0)\right)\right) = 0, \\ \beta_1\phi_p(u(T)) + \beta_2\,{}_tD_T^{\alpha-1}\left(\phi_p\left({}_0^CD_t^{\alpha}u(T)\right)\right) = 0, \end{cases} \tag{1}$$

where $\alpha\in(\frac{1}{2},1]$, ${}_0^CD_t^{\alpha}$ is the left Caputo fractional derivative, ${}_tD_T^{\alpha}$ is the right Riemann-Liouville fractional

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derivative. $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $h(t) \in L^{\infty}([0,T], \mathbb{R})$ with $h_0 = \mathrm{essinf}_{[0,T]} h(t) > 0$, $a \in C([0,T], \mathbb{R})$ with $a_0 = \mathrm{essinf}_{[0,T]} a(t) > 0$, there exist a_1, a_2 such that $0 < a_1 \le a(t) \le a_2$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $\phi_p(x) = |x|^{p-2}x \ (x \neq 0)$, $\phi_p(0) = 0$, $p > 1$. By means of variational method, existence result of the solution was obtained.

From the current research status, there is almost no result on the ground state solution of problem (1), so the main purpose of this article is to study existence of the ground state solution to problem (1). Under the variational framework, most studies of this kind of problem need to use Ambrosetti-Rabinowitz condition to estimate the boundedness of the sequence $\{u_n\}$. However, this paper is more interested in using the conditions weaker than Ambrosetti-Rabinowitz type condition to study problem (1). Therefore, in this paper, we first use the Nehari manifold method to obtain the existence of the ground state solutions of problem (1) when the nonlinear term satisfies the condition weaker than Ambrosetti-Rabinowitz type condition. Secondly, by using the Mountain Pass Theorem, we obtain that there is at least one nontrivial weak solution to problem (1) when the nonlinear term satisfies the condition weaker than Ambrosetti-Rabinowitz type condition. Finally, the existence of infinitely many nontrivial weak solutions of problem (1) is obtained by using the properties of genus. Hence, the results of this paper enrich and extend the work of [15] to a certain extent.

II. Preliminaries

For the convenience of readers, this section introduces some definitions and lemmas of fractional calculus theory.

Definition 2.1 ([16]). (Left and Right Riemann-Liouville Fractional Derivatives) Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $0 \le \gamma < 1$ for function u denoted by ${}_aD_t^{\gamma}u(t)$ and ${}_tD_b^{\gamma}u(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^{\gamma}u(t) &= \frac{d}{dt}{}_aD_t^{\gamma-1}u(t) \\ &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\gamma}u(s)ds\right), \\ {}_tD_b^{\gamma}u(t) &= -\frac{d}{dt}{}_tD_b^{\gamma-1}u(t) \\ &= -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\gamma}u(s)ds\right), \end{aligned}$$

where $t \in [a,b]$.

Let $AC([a,b])$ be the space of absolutely continuous functions within $[a,b]$ (see [16]).

Definition 2.2 ([16]). (Left and Right Caputo Fractional Derivatives) Let $0 < \gamma < 1$ and $u \in AC([a,b])$, then the left and right Caputo fractional derivatives of order γ for function u denoted by ${}_a^CD_t^{\gamma}u(t)$ and ${}_t^CD_b^{\gamma}u(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^CD_t^{\gamma}u(t)$ and ${}_t^CD_b^{\gamma}u(t)$ are represented by

$${}_a^CD_t^{\gamma}u(t) = {}_aD_t^{\gamma-1}u'(t) = \frac{1}{\Gamma(1-\gamma)}\int_a^t (t-s)^{-\gamma}u'(s)ds,$$

$${}_t^CD_b^{\gamma}u(t) = -{}_tD_b^{\gamma-1}u'(t) = -\frac{1}{\Gamma(1-\gamma)}\int_t^b (s-t)^{-\gamma}u'(s)ds,$$

where $t \in [a,b]$.

Let us recall that for any fixed $t \in [0,T]$ and $1 \le r < \infty$,

$$\|u\|_{L^r([0,t])} = \left(\int_0^t |u(\xi)|^r d\xi\right)^{\frac{1}{r}},$$

$$\|u\|_{L^r} = \left(\int_0^T |u(\xi)|^r d\xi\right)^{\frac{1}{r}}, \|u\|_{\infty} = \max_{t\in[0,T]} |u(t)|.$$

Definition 2.3 ([14]). Let $\alpha \in (\frac{1}{2}, 1]$, $p \in [1, \infty)$. The fractional derivative space

$$E^{\alpha,p} = \left\{u | u \in AC([0,T], \mathbb{R}), {}_0^CD_t^{\alpha}u \in L^p([0,T], \mathbb{R})\right\}$$

is defined by closure of $C^{\infty}([0,T], \mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T \left[|u(t)|^p + \left|{}_0^CD_t^{\alpha}u(t)\right|^p\right] dt\right)^{\frac{1}{p}}. \quad (2)$$

Lemma 2.1 ([12]). Let $0 < \alpha \le 1$, $1 \le p < \infty$. For $\forall f \in L^p([0,T], \mathbb{R})$, $\forall \xi \in [0,t]$, $t \in [0,T]$, one has

$$\left\|{}_0D_{\xi}^{-\alpha}f\right\|_{L^p([0,t])} \le \frac{t^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])}.$$

Lemma 2.2 ([14]). Let $0 < \alpha \le 1$, $1 \le p < \infty$. For $\forall f \in L^p([0,T], \mathbb{R})$, $\forall \xi \in [t,T]$, $t \in [0,T]$, one has

$$\left\|{}_{\xi}D_T^{-\alpha}f\right\|_{L^p([t,T])} \le \frac{(T-t)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p([t,T])}.$$

Lemma 2.3 ([16]). Let $n \in \mathbb{N}$, $n-1 < \alpha \le n$. If $f \in AC^n([a,b], \mathbb{R})$ or $f \in C^n([a,b], \mathbb{R})$, then

$${}_aD_t^{-\alpha}\left({}_a^CD_t^{\alpha}f(t)\right) = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{j!}(t-a)^j, \forall t \in [a,b],$$

$${}_tD_b^{-\alpha}\left({}_t^CD_b^{\alpha}f(t)\right) = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(b)}{j!}(b-t)^j, \forall t \in [a,b].$$

In particular, if $0 < \alpha < 1$, $f \in AC([a,b], \mathbb{R})$ or $f \in C^1([a,b], \mathbb{R})$, then

$${}_aD_t^{-\alpha}\left({}_a^CD_t^{\alpha}f(t)\right) = f(t) - f(a),$$

$${}_tD_b^{-\alpha}\left({}_t^CD_b^{\alpha}f(t)\right) = f(t) - f(b).$$

Lemma 2.4 ([15]). Let $\frac{1}{2} < \alpha \le 1$, $1 \le p < \infty$. If $u \in E^{\alpha,p}$, then

$$\|u\|_{\infty} \le M\|u\|_{\alpha,p},$$

where

$$M := \left(\max\left\{\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}, 1\right\} + \left[\frac{2^{p-1}}{T}\max\left\{1, \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)^p\right\}\right]^{\frac{1}{p}}\right), \ \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.5 ([15]). Let $1/p < \alpha \leq 1$, $1 < p < \infty$, by Lemma 2.4, one has

$$\|u\|_\infty \leqslant \frac{M}{(\min\{a_0,h_0\})^{\frac{1}{p}}} \times \left(\int_0^T a(t)|u(t)|^p dt + \int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}},$$

where $h_0 = \mathrm{essinf}_{[0,T]} h(t) > 0$, $a_0 = \mathrm{essinf}_{[0,T]} a(t) > 0$.

Remark 2.1 It's also easy to check that, if $a \in C([0,T],\mathbb{R})$ is such that $0 < a_1 \leq a(t) \leq a_2$, an equivalent norm in $E^{\alpha,p}$ is the following:

$$\|u\|_a = \left(\int_0^T a(t)|u(t)|^p dt + \int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}}. \tag{3}$$

By combining Lemma 2.5, we can see that, for $\forall u \in E^{\alpha,p}$, if $1/p < \alpha \leq 1$, then

$$\|u\|_\infty \leq \frac{M}{\Lambda^{1/p}}\|u\|_a,\ \Lambda = \min\{a_0,h_0\}. \tag{4}$$

Lemma 2.6 ([14]). Let $0 < \alpha \leq 1$, $1 < p < \infty$. The fractional derivative space $E^{\alpha,p}$ is a reflexive and separable Banach space.

Lemma 2.7 ([14]). Let $1/p < \alpha \leq 1$, $1 < p < \infty$. Assume that the sequence $\{u_k\}$ converges weakly to u in $E^{\alpha,p}$, i.e., $u_k \rightharpoonup u$, then $u_k \to u$ in $C([0,T],\mathbb{R})$, i.e., $\|u_k - u\|_\infty \to 0$, $\quad k \to \infty$.

Lemma 2.8 ([15]). Assume that $1/p < \alpha \leq 1$, $1 < p < \infty$, then $E^{\alpha,p}$ compactly embedded in $C([0,T],\mathbb{R})$.

Lemma 2.9 ([16]). Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, $1/p+1/q < 1+\alpha$ or $p \neq 1$, $q \neq 1$, $1/p+1/q = 1+\alpha$. If $u \in L^p([a,b],\mathbb{R})$, $v \in L^q([a,b],\mathbb{R})$, then

$$\int_a^b \left[{}_aD_t^{-\alpha}u(t)\right]v(t)\,dt = \int_a^b u(t)\left[{}_tD_b^{-\alpha}v(t)\right]dt. \tag{5}$$

By multiplying the equation in (1) by any $v \in E^{\alpha,p}$, and then integrating on $[0,T]$, one has

$$\begin{aligned}&\int_0^T {}_tD_T^\alpha\left(\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right)\right)\cdot v(t)\,dt\\&+\int_0^T a(t)\,\phi_p(u(t))\,v(t)\,dt = \int_0^T f(t,u(t))v(t)\,dt.\end{aligned} \tag{6}$$

From Definition 2.1, 2.2 and Lemma 2.9, we can get

$$\begin{aligned}&\int_0^T {}_tD_T^\alpha\left(\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right)\right)\cdot v(t)\,dt\\
&=-\int_0^T \frac{d}{dt}\left[{}_tD_T^{\alpha-1}\left(\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right)\right)\right]v(t)\,dt\\
&=\frac{\beta_1 h(T)}{\beta_2}\phi_p(u(T))\,v(T)+\frac{\alpha_1 h(0)}{\alpha_2}\phi_p(u(0))\,v(0)+\\
&\int_0^T\left[{}_tD_T^{\alpha-1}\left(\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right)\right)\right]v'(t)\,dt\\
&=\frac{\beta_1 h(T)}{\beta_2}\phi_p(u(T))\,v(T)+\frac{\alpha_1 h(0)}{\alpha_2}\phi_p(u(0))\,v(0)\\
&+\int_0^T\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right){}_0D_t^{\alpha-1}v'(t)\,dt\\
&=\frac{\beta_1 h(T)}{\beta_2}\phi_p(u(T))\,v(T)+\frac{\alpha_1 h(0)}{\alpha_2}\phi_p(u(0))\,v(0)\\
&+\int_0^T\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right){}_0^C D_t^\alpha v(t)\,dt.\end{aligned} \tag{7}$$

Getting the similar result for the second part of equation (1), then we can give the weak solution definition of (1), which is as follows:

Definition 2.4. Let $u \in E^{\alpha,p}$ be a weak solution of problem (1), if

$$\begin{aligned}&\frac{\beta_1 h(T)}{\beta_2}\phi_p(u(T))\,v(T)+\frac{\alpha_1 h(0)}{\alpha_2}\phi_p(u(0))\,v(0)\\
&+\int_0^T\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right){}_0^C D_t^\alpha v(t)\,dt\\
&+\int_0^T a(t)\,\phi_p(u(t))\,v(t)\,dt=\int_0^T f(t,u(t))v(t)\,dt,\end{aligned}$$

holds for $\forall v \in E^{\alpha,p}$.

Define the corresponding functional $I : E^{\alpha,p} \to \mathbb{R}$ as below

$$\begin{aligned}I(u)&=\frac{1}{p}\int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt+\frac{1}{p}\int_0^T a(t)|u(t)|^p dt\\
&+\frac{\beta_1 h(T)}{p\beta_2}|u(T)|^p+\frac{\alpha_1 h(0)}{p\alpha_2}|u(0)|^p-\int_0^T F(t,u(t))dt\\
&=\frac{1}{p}\|u\|_a^p+\frac{\beta_1 h(T)}{p\beta_2}|u(T)|^p+\frac{\alpha_1 h(0)}{p\alpha_2}|u(0)|^p\\
&-\int_0^T F(t,u(t))dt.\end{aligned} \tag{8}$$

According to the definition of f, it is easy to prove that the above functional $I \in C^2(E^{\alpha,p},\mathbb{R})$. For $\forall v \in E^{\alpha,p}$, we have

$$\begin{aligned}\langle I'(u),v\rangle&=\int_0^T\frac{1}{(h(t))^{p-2}}\phi_p\left(h(t)\,{}_0^C D_t^\alpha u(t)\right){}_0^C D_t^\alpha v(t)\,dt\\
&+\int_0^T a(t)|u(t)|^{p-2}u(t)\,v(t)\,dt+\frac{\beta_1 h(T)}{\beta_2}\phi_p(u(T))\,v(T)\\
&+\frac{\alpha_1 h(0)}{\alpha_2}\phi_p(u(0))\,v(0)-\int_0^T f(t,u(t))v(t)\,dt.\end{aligned} \tag{9}$$

Then

$$\begin{aligned}\langle I'(u), u\rangle = \|u\|_a^p - \int_0^T f(t,u(t))u(t)\,dt \\ + \frac{\beta_1 h(T)}{\beta_2}|u(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|u(0)|^p.\end{aligned} \tag{10}$$

Therefore, the critical point of functional I corresponds to the weak solution of (1). The ground state solution here refers to the minimum energy solution of the functional I.

III. Main result

The theorem and proof process of the existence of ground state solutions of (1) are as follows.

Theorem 3.1. Let $f \in C^1([0,T]\times\mathbb{R},\mathbb{R})$. Assume that the following conditions (H_1)-(H_4) hold

(H_1) the map $x \to f(t,x)/|x|^{p-1}$ is increasing on $\mathbb{R}\setminus\{0\}$, for $\forall t\in[0,T]$;

(H_2) $f(t,x) = o(|x|^{p-1})$ as $|x|\to 0$ uniformly with respect to $t\in[0,T]$;

(H_3) there exist $\Lambda_0>0,\ R>0,\ \theta>p$ such that

$$xf(t,x) - \theta F(t,x) \ge -\Lambda_0|x|^p,\ \forall t\in[0,T],\ |x|\ge R,$$

where $F(t,x) = \int_0^x f(t,s)ds$;

(H_4) $\lim\limits_{|x|\to\infty} \frac{F(t,x)}{|x|^\theta} = \infty$, $\forall t\in[0,T]$.

Then problem (1) has at least a nontrivial ground state solution.

Remark 3.1. Reference [15] used the following classic Ambrosetti-Rabinowitz type condition (F) to estimate the sequence $\{u_n\}$ bounded, while this article uses the conditions (H_3) and (H_4).

(F) There exist constants $R>0,\ \theta>p$ such that

$$0<\theta F(t,x)\le xf(t,x),\ \forall t\in[0,T],\ |x|\ge R.$$

Because the Ambrosetti-Rabinowitz type condition (F) contains the conditions (H_3) and (H_4) in Theorem 3.1, the conditions (H_3) and (H_4) are weaker than the Ambrosetti-Rabinowitz type condition.

The set is defined as follows

$$\mathcal{N} = \{u\in E^{\alpha,p}\setminus\{0\}\,|G(u)=0\},\ G(u) = \langle I'(u),u\rangle.$$

Then any non-zero critical point of I must be in $\mathcal{N}$. By (H_1), for $\forall\,(t,u)\in[0,T]\times\mathbb{R}\setminus\{0\}$, we have

$$(p-1)f(t,u(t))u(t) < \frac{\partial f(t,u(t))}{\partial u}u^2(t). \tag{11}$$

So, for $u\in\mathcal{N}$, by (10), (11), one has

$$\begin{aligned}&\langle G'(u),u\rangle\\ &= p\,\|u\|_a^p - \int_0^T \frac{\partial f(t,u(t))}{\partial u}\cdot u^2(t)dt - \int_0^T f(t,u(t))u(t)dt\\ &< \int_0^T \left[(p-1)\,f(t,u(t))u(t) - \frac{\partial f(t,u(t))}{\partial u}\cdot u^2(t)\right]dt\\ &<0.\end{aligned} \tag{12}$$

The formula indicates that $\mathcal{N}$ has a C^1 structure, which is a Nehari manifold.

Here are some lemmas to prove Theorem 3.1.

Lemma 3.1. Suppose that the condition (H_1) holds. If $u\in\mathcal{N}$ is one critical point of $I|_{\mathcal{N}}$, then $I'(u)=0$. In other words, $\mathcal{N}$ is a natural constraint on I.

Proof. If $u\in\mathcal{N}$ is one critical point of $I|_{\mathcal{N}}$, then there is a Lagrange multiplier $\lambda\in\mathbb{R}$, which makes the following equation true

$$I'(u) = \lambda G'(u).$$

Therefore,

$$\langle I'(u),u\rangle = \lambda\,\langle G'(u),u\rangle = 0.$$

Combining with (12), we know that $\lambda=0$, so $I'(u)=0$. □

Next, let's examine the structure of $\mathcal{N}$.

Lemma 3.2. If the conditions (H_1)-(H_4) are met. For $\forall u\in E^{\alpha,p}\setminus\{0\}$, there is one unique $s=s(u)>0$ such that $su\in\mathcal{N}$ and $I(su) = \max_{s\ge 0} I(su) > 0$.

Proof. The first step, we will prove that there are $\rho,\sigma>0$ such that the following inequality holds

$$I(u)>0,\ \forall u\in B_\rho\setminus\{0\};\ I(u)\ge\sigma,\ \forall u\in\partial B_\rho.$$

It is easy to know that 0 is a strict local minimizer of I. From (H_2), one has

$$\forall\varepsilon>0,\ \exists\delta>0,\ F(t,u)\le\varepsilon|u|^p,\ |u|\le\delta.$$

Thus, for $\forall u\in E^{\alpha,p}\setminus\{0\}$, by (4), we get

$$\|u\|_\infty \le \frac{M}{\Lambda^{1/p}}\|u\|_a = \delta,\ \|u\|_a = \rho,$$

Then by (3), (8), one has

$$\begin{aligned}I(u)&\ge\frac{1}{p}\|u\|_a^p - \int_0^T F(t,u(t))dt \ge \frac{1}{p}\|u\|_a^p - \varepsilon\int_0^T |u|^p dt\\ &\ge\frac{1}{p}\|u\|_a^p - \varepsilon\cdot\frac{1}{a_0}\int_0^T a(t)\,|u|^p dt \ge \frac{1}{p}\|u\|_a^p - \frac{\varepsilon}{a_0}\|u\|_a^p.\end{aligned}$$

Select $\varepsilon = \frac{a_0}{2p}$, we can get

$$I(u)\ge\frac{1}{2p}\|u\|_a^p.$$

Let $\rho = \frac{\delta\Lambda^{1/p}}{M}$, $\sigma = \frac{\rho^p}{2p}$. So, for $u\in\partial B_\rho$, one has $I(u)\ge\sigma$.

Secondly, we prove that $I(\xi u)\to-\infty$, as $\xi\to\infty$. In fact, by (H_4), there are $c_1,\ c_2>0$ such that the following inequality holds

$$F(t,u)\ge c_1|u|^\theta - c_2,\ (t,u)\in[0,T]\times\mathbb{R}.$$

Therefore, combining (4), (8) and Hölder inequality, we have

$$\begin{aligned}
I(\xi u) &\le \frac{\xi^p}{p}\|u\|_a^p + \frac{\xi^p}{p}\|u\|_\infty^p\left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right) \\
&\quad - c_1\xi^\theta\int_0^T |u|^\theta dt + c_2 T \\
&\le \frac{\xi^p}{p}\|u\|_a^p + \frac{\xi^p}{p}\frac{M^p}{\Lambda}\|u\|_a^p\left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right) \\
&\quad - c_1\xi^\theta\left(T^{\frac{p-\theta}{\theta}}\int_0^T |u(t)|^p dt\right)^{\frac{\theta}{p}} + c_2 T \\
&\le \frac{1}{p}\xi^p\left[1 + \frac{M^p}{\Lambda}\left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right)\right]\|u\|_a^p \\
&\quad - c_1\xi^\theta T^{\frac{p-\theta}{p}}\|u\|_{L^p}^\theta + c_2 T.
\end{aligned}$$

Because $\theta > p$, $I(\xi u) \to -\infty$ $(\xi \to \infty)$.

Let $g_u(s) := I(su)$, $\forall s > 0$. It can be seen from the above proof that g_u has at least a maximum point $s(u)$, and the corresponding maximum value is greater than σ. The following proof shows that when $s > 0$, g_u has a unique critical point, which must be the global maximum point. In fact, if s is the critical point of g, we can obtain

$$\begin{aligned}
&g'_u(s) = \langle I'(su), u\rangle \\
&= s^{p-1}\|u\|_a^p + s^{p-1}\left[\frac{\beta_1 h(T)}{\beta_2}|u(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|u(0)|^p\right] \\
&\quad - \int_0^T f(t, su(t))u(t)dt \\
&= 0.
\end{aligned}$$

Combined with (3.1), we get

$$\begin{aligned}
&g''_u(s) = (p-1)s^{p-2}\|u\|_a^p + (p-1)s^{p-2} \\
&\quad \times\left[\frac{\beta_1 h(T)}{\beta_2}|u(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|u(0)|^p\right] \\
&\quad - \int_0^T \frac{\partial f(t,su)}{\partial(su)}\cdot u^2(t)dt \\
&= \frac{p-1}{s}\left\{\int_0^T f(t, su(t))u(t)dt\right. \\
&\quad \left. - s^{p-1}\left[\frac{\beta_1 h(T)}{\beta_2}|u(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|u(0)|^p\right]\right\} \\
&\quad + (p-1)s^{p-2}\left[\frac{\beta_1 h(T)}{\beta_2}|u(T)|^p + \frac{\alpha_1 h(0)}{\alpha_2}|u(0)|^p\right] \\
&\quad - \int_0^T \frac{\partial f(t,su)}{\partial(su)}\cdot u^2(t)dt \\
&= \frac{p-1}{s}\int_0^T f(t, su(t))u(t)dt - \int_0^T \frac{\partial f(t,su)}{\partial(su)}\cdot u^2(t)dt \\
&= \frac{1}{s^2}\int_0^T\left[(p-1)f(t,su(t))su(t) - \frac{\partial f(t,su(t))}{\partial(su)}(su(t))^2\right]dt \\
&< 0,
\end{aligned} \tag{13}$$

This means that if s is a critical point of g, then s must be a strict local maximum point of g and the critical point is unique. □

Remark 3.2. According to

$$g'_u(s) = \frac{1}{s}\langle I'(su), su\rangle, \ \forall s > 0, \tag{14}$$

if s is the critical point of g_u, then $su \in \mathcal{N}$.

Define $m = \inf_{\mathcal{N}} I$, so we can get

$$m \ge \inf_{\partial B_\rho} I \ge \sigma > 0.$$

Lemma 3.3. If the conditions (H_1)-(H_4) are satisfied, then there is $u \in \mathcal{N}$ such that $I(u) = m$ holds.

Proof. First, we prove that I and G are weakly lower semi-continuous. From Lemma 2.7, if $u_k \rightharpoonup u$ in $E^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R})$. So, $F(t, u_k(t)) \to F(t, u(t))$ $a.e. t \in [0, T]$. According to Lebesgue control convergence theorem, $\int_0^T F(t, u_k(t))dt \to \int_0^T F(t, u(t))dt$. That is, $u \to \int_0^T F(t, u(t))dt$ is weakly continuous on $E^{\alpha,p}$. Similarly, $u \to \int_0^T f(t, u(t))u(t)dt$ is also weakly continuous on $E^{\alpha,p}$. So I and G are weakly lower semi-continuous. Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{N}$ be the minimization sequence of I, then

$$I(u_k) = m + o(1),\ G(u_k) = 0.$$

Next, we prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $E^{\alpha,p}$. Otherwise, $\|u_k\|_a \to \infty$ as $k \to \infty$. For $\forall u \in E^{\alpha,p}\setminus\{0\}$, choose $v_k = \frac{u_k}{\|u_k\|_a}$, then $\|v_k\|_a = 1$. Since $E^{\alpha,p}$ is a reflexive Banach space, there exists a subsequence of $\{v_k\}$ (still denoted as $\{v_k\}$) such that $v_k \rightharpoonup v$ in $E^{\alpha,p}$, then $v_k \to v$ in $C([0,T],\mathbb{R})$. On the one hand, by (4) and (8), we have

$$\begin{aligned}
&\int_0^T F(t, u_k)dt \\
&= \frac{1}{p}\|u_k\|_a^p + \left[\frac{\beta_1 h(T)}{p\beta_2}|u_k(T)|^p + \frac{\alpha_1 h(0)}{p\alpha_2}|u_k(0)|^p\right] - I(u_k) \\
&\le \frac{1}{p}\|u_k\|_a^p + \frac{1}{p}\left[\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right]\|u_k\|_\infty^p + C_0 \\
&\le \frac{1}{p}\|u_k\|_a^p\left[1 + \left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2}\right)\frac{M^p}{\Lambda}\right] + C_0,
\end{aligned}$$

where $C_0 > 0$. That means that when $k \to \infty$, we get

$$\int_0^T \frac{F(t, u_k)}{\|u_k\|_a^\theta}dt \le o(1). \tag{15}$$

On the other side, according to the continuity of f, there is $\Lambda_1 > 0$ such that

$$|uf(t,u) - \theta F(t,u)| \le \Lambda_1,\ \forall |u| \le R,\ t \in [0,T].$$

Combining the condition (H_3), we have

$$uf(t,u) - \theta F(t,u) \ge -\Lambda_0|u|^p - \Lambda_1,\ \forall |u| \in \mathbb{R},\ t \in [0,T]. \tag{16}$$

So, by (8), (10), (16), we have

$$
\begin{aligned}
& m+o(1)=I(u_k) \\
& =\frac{1}{p}\|u_k\|_a^p+\frac{\beta_1 h(T)}{p\beta_2}|u_k(T)|^p+\frac{\alpha_1 h(0)}{p\alpha_2}|u_k(0)|^p \\
& -\int_0^T F(t,u_k(t))dt \\
& \geq \frac{1}{p}\|u_k\|_a^p+\frac{\beta_1 h(T)}{p\beta_2}|u_k(T)|^p+\frac{\alpha_1 h(0)}{p\alpha_2}|u_k(0)|^p \\
& -\frac{1}{\theta}\int_0^T u_k f(t,u_k(t))dt-\frac{\Lambda_0}{\theta}\int_0^T|u_k|^p dt-\frac{\Lambda_1 T}{\theta} \\
& \geq\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_k\|_a^p+\frac{1}{\theta}\langle I'(u_k),u_k\rangle \\
& +\left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\frac{\beta_1 h(T)}{\beta_2}|u_k(T)|^p+\frac{\alpha_1 h(0)}{\alpha_2}|u_k(0)|^p\right] \\
& -\frac{\Lambda_0}{\theta}\int_0^T|u_k|^p dt-\frac{\Lambda_1 T}{\theta} \\
& \geq\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_k\|_a^p-\frac{T\Lambda_0}{\theta}\|u_k\|_\infty^p-\frac{\Lambda_1 T}{\theta}.
\end{aligned}
$$

This means that there exists $\Lambda_2>0$ such that

$$\lim_{k\to\infty}\|v_k\|_\infty=\lim_{k\to\infty}\frac{\|u_k\|_\infty}{\|u_k\|_a}\geq\Lambda_2>0.$$

Therefore, $v\neq 0$. Let

$$\Delta_1=\{t\in[0,T]:v\neq 0\},\ \Delta_2=[0,T]\backslash\Delta_1.$$

According to the condition (H_4), there is $\Lambda_3>0$ such that

$$F(t,u)\geq 0,\ \forall t\in[0,T],\ |u|\geq\Lambda_3.$$

Combining with the condition (H_2), there exist $\Lambda_4,\ \Lambda_5>0$ such that

$$F(t,u)\geq-\Lambda_4 u^p-\Lambda_5,\ \forall t\in[0,T],\ u\in\mathbb{R}.$$

According to the Fatou lemma, one has

$$\liminf_{k\to\infty}\int_{\Delta_2}\frac{F(t,u_k)}{\|u_k\|_a^\theta}dt>-\infty.$$

Combining with the condition (H_4), we can obtain

$$
\begin{aligned}
& \liminf_{k\to\infty}\int_0^T\frac{F(t,u_k)}{\|u_k\|_a^\theta}dt \\
& =\liminf_{k\to\infty}\left(\int_{\Delta_1}+\int_{\Delta_2}\right)\frac{F(t,u_k)}{|u_k|^\theta}|v_k|^\theta dt\to\infty.
\end{aligned}
\tag{17}
$$

This contradicts (15). So, $\{u_k\}_{k\in\mathbb{N}}$ is bounded. Assuming that $\{u_k\}_{k\in\mathbb{N}}$ has a subsequence, still denoted as $\{u_k\}_{k\in\mathbb{N}}$, there is $u\in E^{\alpha,p}$ such that $u_k\rightharpoonup u$ in $E^{\alpha,p}$, so $u_k\to u$ in $C([0,T],\mathbb{R})$. Thus, when $k\to\infty$, we can get

$$
\begin{cases}
\langle I'(u_k)-I'(u),u_k-u\rangle\to 0, \\
\int_0^T[f(t,u_k(t))-f(t,u(t))][u_k(t)-u(t)]dt\to 0, \\
|u_k(T)-u(T)|^p\to 0,\ |u_k(0)-u(0)|^p\to 0.
\end{cases}
$$

Because

$$
\begin{aligned}
& \|u_k-u\|_a^p=\langle I'(u_k)-I'(u),u_k-u\rangle \\
& +\int_0^T[f(t,u_k(t))-f(t,u(t))][u_k(t)-u(t)]dt \\
& -\frac{\beta_1 h(T)}{\beta_2}|u_k(T)-u(T)|^p-\frac{\alpha_1 h(0)}{\alpha_2}|u_k(0)-u(0)|^p,
\end{aligned}
$$

so $\|u_k-u\|_a\to 0$ $(k\to\infty)$. This means that $u_k\to u$ in $E^{\alpha,p}$. Since G is weakly semi-continuous and $\{u_k\}\subset\mathcal{N}$, so

$$G(u)\leq\underline{\lim}_{k\to\infty}G(u_k)=0.$$

Therefore, $u\neq 0$. Otherwise, if $u=0$, then $u_k\to 0$ in $C([0,T],\mathbb{R})$. By $G(u_k)=0$, one has $\|u_k\|_a\to 0$. This contradicts $\{u_k\}\subset\mathcal{N}$.

According to Lemma 3.2, there is a unique $s>0$, such that $su\in\mathcal{N}$. Combined with I is weakly lower semi-continuous, one has

$$m\leq I(su)\leq\underline{\lim}_{k\to\infty}I(su_k)\leq\lim_{k\to\infty}I(su_k). \tag{18}$$

Last, for $\forall u_k\in\mathcal{N}$, by (13) and (14), we obtain that $s=1$ is a global maximum point of g_{u_k}, so $I(su_k)\leq I(u_k)$. Combined with (18), one has

$$m\leq I(su)\leq\lim_{k\to\infty}I(u_k)=m.$$

Therefore, m is obtained at $su\in\mathcal{N}$. □

The proof process of Theorem 3.1 is given below.

Proof. By Lemma 3.3, there is $u\in\mathcal{N}$ such that $I(u)=m=\inf_{\mathcal{N}}I>0$, i.e., u is the non-zero critical point of $I|_{\mathcal{N}}$. By Lemma 3.1, one has $I'(u)=0$, thus u is the non-trivial ground state solution of problem (1). □

In order to prove the existence of another solution of problem (1), the key lemma is given.

Lemma 3.4. ([17]). (Mountain Pass Theorem) Let X be a real Banach space and $I\in C^1(X,\mathbb{R})$. $I(u)$ satisfies the (PS) condition, if a sequence $\{u_n\}_{n\in\mathbb{N}}\subset X$ which satisfies the conditions $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'(u_n)\to 0$ as $n\to\infty$, has a convergent subsequence. Suppose that $I(0)=0$ and

(i) there exist constants ρ, $\sigma>0$ such that $I|_{\partial B_\rho}\geq\sigma$;

(ii) there exists an $e\in X\big/\overline{B_\rho}$ such that $I(e)\leq 0$.

Then I possesses a critical value $c\geq\sigma$. Moreover c can be characterized as

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}I(g(s)),$$

where

$$\Gamma=\{g\in C([0,1],X):g(0)=0,\ g(1)=e\}.$$

Theorem 3.2. Let $f\in C([0,T]\times\mathbb{R},\mathbb{R})$. Assume that (H_2)-(H_4) hold. Then problem (1) has at least one nontrivial weak solution.

Proof. Step 1. Obviously, $I(0)=0$. According to Lemma 3.3, $I\in C^1(E^{\alpha,p},\mathbb{R})$ satisfies the (PS) condition.

Step 2. We will prove that the condition (i) in Lemma 3.4 holds. From (H_2), one has

$$\forall\varepsilon>0,\ \exists\delta>0,\ F(t,u)\leq\varepsilon|u|^p,\ |u|\leq\delta.$$

For $\forall u\in E^{\alpha,p}\backslash\{0\}$, by (3), (4), (8), one has

$$
\begin{aligned}
I(u) & \geq\frac{1}{p}\|u\|_a^p-\int_0^T F(t,u(t))dt\geq\frac{1}{p}\|u\|_a^p-\varepsilon\int_0^T|u|^p dt \\
& \geq\frac{1}{p}\|u\|_a^p-\varepsilon\cdot\frac{1}{a_0}\int_0^T a(t)|u|^p dt\geq\frac{1}{p}\|u\|_a^p-\frac{\varepsilon}{a_0}\|u\|_a^p.
\end{aligned}
$$

Choose $\varepsilon = \frac{a_0}{2p}$, we get

$$I(u) \geq \frac{1}{2p} \|u\|_a^p .$$

Let $\rho = \frac{\delta \Lambda^{1/p}}{M}$, $\sigma = \frac{\rho^p}{2p}$. So, for $u \in \partial B_\rho$, one has $I(u) \geq \sigma > 0$.

Step 3. We will prove that there exist $e \in E^{\alpha,p}$ and $\|e\|_a > \rho$ such that $I(e) < 0$, where ρ is defined in Step 2. By (H_4), there are $c_1,\ c_2 > 0$ such that the following inequality holds

$$F(t,u) \geq c_1|u|^\theta - c_2,\ (t,u) \in [0,T] \times \mathbb{R}.$$

Therefore, combining (4), (8) and Hölder inequality, we have

$$\begin{aligned}
I(\xi u) &\leq \frac{\xi^p}{p} \|u\|_a^p + \frac{\xi^p}{p} \|u\|_\infty^p \left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2} \right) \\
&\quad - c_1 \xi^\theta \int_0^T |u|^\theta dt + c_2 T \\
&\leq \frac{\xi^p}{p} \|u\|_a^p + \frac{\xi^p}{p} \frac{M^p}{\Lambda} \|u\|_a^p \left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2} \right) \\
&\quad - c_1 \xi^\theta \left(T^{\frac{p-\theta}{\theta}} \int_0^T |u(t)|^p dt \right)^{\frac{\theta}{p}} + c_2 T \\
&\leq \frac{1}{p} \xi^p \left[1 + \frac{M^p}{\Lambda} \left(\frac{\beta_1 h(T)}{\beta_2} + \frac{\alpha_1 h(0)}{\alpha_2} \right) \right] \|u\|_a^p \\
&\quad - c_1 \xi^\theta T^{\frac{p-\theta}{p}} \|u\|_{L^p}^\theta + c_2 T.
\end{aligned}$$

Since $\theta > p$, the above formula implies that when ξ_0 is sufficiently large, $I(\xi_0 u) \to -\infty$. Let $e = \xi_0 u$, one has $I(e) < 0$, so condition (ii) in Lemma 3.4 holds. From Lemma 3.4, we know that I has one critical value $c \geq \sigma > 0$, as follows:

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \left\{ g \in C([0,1], E^{\alpha,p}) : g(0) = 0, g(1) = e \right\}.$$

Therefore, there exists $0 \neq u \in E^{\alpha,p}$ such that

$$I(u) = c \geq \sigma > 0,\ I'(u) = 0.$$

That is, problem (1) has at least one nontrivial weak solution. □

Remark 3.3. Clearly, the conditions (H_3) and (H_4) are weaker than the Ambrosetti-Rabinowitz type condition. Consequently, our conclusion generalizes Theorem 1.0 in [15].

The following result shows that there are infinitely many nontrivial weak solutions to problem (1) by using the properties of genus.

Theorem 3.3. Suppose the following conditions hold.

(H_5) There is a constant $1 < r_1 < p$ and function $b \in L^1([0,T], \mathbb{R}^+)$ such that

$$|f(t,x)| \leq r_1 b(t) |x|^{r_1 - 1},\ \forall (t,x) \in [0,T] \times \mathbb{R};$$

(H_6) There is an open interval $\Pi \subset [0,T]$ and $\eta,\ \delta > 0$, $1 < r_2 < p$ such that

$$F(t,x) \geq \eta |x|^{r_2},\ \forall (t,x) \in \Pi \times [-\delta, \delta];$$

(H_7) $f(t,x) = -f(t,-x),\ \forall (t,x) \in [0,T] \times \mathbb{R};$

Then problem (1) has infinitely many nontrivial weak solutions.

IV. Conclusion

In this paper, firstly, the Nehari manifold method is used to study the existence of ground state solutions of the fractional p-Laplacian equation with Sturm-Liouville boundary conditions. When the nonlinear term satisfies the condition weaker than the Ambrosetti-Rabinowitz type condition, the existence theorem (see Theorem 3.1) that problem (1) has at least one nontrivial ground state solution is obtained. Secondly, this paper also uses the Mountain Pass Theorem to study the existence of at least one nontrivial weak solution of the above problem. Under the condition that the nonlinear term satisfies the condition weaker than the Ambrosetti-Rabinowitz type condition, the existence result of weak solution for problem (1) is obtained (see Theorem 3.2). Finally, the existence of infinitely many nontrivial weak solutions of problem (1) is obtained by using the properties of genus (see Theorem 3.3). Therefore, the work of this paper enriches and promotes the results of [15] to a certain extent.

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