Stability and Bifurcation Analysis in a Fractional-order Epidemic Model with Sub-optimal Immunity, Nonlinear Incidence and Saturated Recovery Rate

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***Abstract*—Recently, many deterministic mathematical model have been extended to fractional model, using some fractional differential equation. Numerous studies had shown that these fractional models are more realistic to represent the daily life phenomena. This paper focused on extending the model of a SIR epidemic to fractional model. More specifically, the study discussed the fractional SIR epidemic model with sub-optimal immunity, nonlinear incidence and saturated recovery rate. The fractional ordinary differential equations were defined in the sense of the Caputo derivative. Stability analysis of the equilibrium points of the models for the fractional models were presented. Furthermore, we investigated the Hopf bifurcation analysis. The result obtained showed that the model undergo Hopf bifurcation for some values, and further confirmed that choosing an appropriate figure of the fractional $\alpha \in (0, 1]$ increase the stability region of the equilibrium points.**

***Index Terms*—Saturated recovery rate, fractional-order, stability, Hopf bifurcation.**

I. Introduction

THE study of epidemiology, which involves the transmission of diseases within a population, has recently gained more attention from researchers in various fields. The severe active respiratory syndrome (SARS) outbreak in 2003 and Ebola outbreak in 2014 had further lead to numerous research and publications most of which have advanced the topic in many areas. Furthermore, numerous infectious diseases models have been developed in order to study the dynamical process of epidemics. These models are able to integrate the realistic aspects of how the diseases is been spread.

One of the classical deterministic models was introduced in 1927 by Kermack and McKendrick. This is a very simple but powerful model. It simplicity has led other researchers to advanced the model in many areas. The researchers established a simple susceptible-infected-recovered model known as SIR Model. Susceptible individuals are those that are liable to be infected with transmission rate that is proportional to the infected persons within the population. The infected individuals recover or get delivered at a constant rate.

In this research directions, mathematicians are among the prominent scholars who have been contributing to the skills of epidemiology by modeling various outbreak of diseases. One of the major areas of epidemiological research is focus on the rate based differential equations models. Numerous researches have been carried out in mathematics and epidemiology with the aim of introducing more realistic epidemic models, including works by [1], [2], [3]. In recent years, the research on ordinary differential equations (ODEs) epidemic modelling had been shifted to fractional differential equations (FDEs) model. Several fractional epidemic models have been studied [4], [5], [6], [7], [8].

Fractional calculus, which is an important branch of mathematics has been in existence since 1695. Numerous investigation has shown that the fractional calculus has the superiority accuracy when describing several non-classical phenomena in engineering applications and basic science, such as biological system [9], finance system [10], than the integer-order. On the other hand, the well-known Caputo fractional derivative defined by Michele Caputo in 1967 and famous Riemann-Liouville fractional integral are the main subjects of many studies in fractional calculus [11], [12], [13]. The research work in this area is has been gaining a lot of attention. These include the study of theory of fractional calculus [14], [15], efficient numerical schemes [16], [6], [17] and application on physical problem [18]. Furthermore, the stability region of the system is increased using the fractional derivative, which is more suitable than integer order [19].

Recently, [19] studied the fractional-order of epidemic models with constant recruitment rate, mass action incidence and variable population size, and discussed the reasons for considering fractional order to include that fractional differential equations used to obtain errors arising from the neglected parameters in modelling real phenomena to a bearing minimum. Their research concluded that fractional epidemic models are good in modelling biology, economic and social system than integer order.

A fractional derivative SIR model was recently studied

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by [20]. The research focused on a stochastic process of an infected individual with a time-since-infection. The researchers built more on what was established by Kermack and McKendrick. The model was extended to fractional order and solved. The obtain result indicate that the system parameters allow the disease-free and endemic. However, the authors failed to identify any kind of disease that would give rise to the epidemic model in fractional order.

A SIS model with vaccination was developed and the report obtained show that the model exhibits a backward bifurcation [21]. Also, a stochastic SIR model with vaccination [22] and stoichiometric [38] was introduced and the result confirmed that the model has multiple endemic equilibria. Many of these SIR and SIS models are yet to be studied under the fractional order equation (FODs), rather, researchers only focused on models based on ordinary differential equations (ODEs). Thus, this paper extend a SIR to a fractional-order and obtained it Hopf bifurcation analysis .

A. Model Description

Ruan and Wang [53] in [24] proposed a model on SIR epidemic, that describe a specific nonlinear incidence rate $(\beta SI^2)/(1+aI^2)$, with

$$\frac{dS}{dt} = A - \beta SI^2 1 + aI^2 - \mu S + wR, \tag{1}$$
$$\frac{dI}{dt} = \beta SI^2 1 + aI^2 - (\mu+\gamma)I,$$
$$\frac{dR}{dt} = \gamma I - (\mu + w)R.$$

The susceptible recruitment and the disease transmission rate are A and β, respectively. The nature death is represented as μ, while the individual rate of recovery of infective and the individual rate of removed are represented as γ and w, respectively.

An attempt was done for the dynamic simulation of the disease transmission in [25]. The authors introduced r as the constant remove as a result of the infectious treatment as given bellow;

$$\frac{dS}{dt} = -IS\beta - \mu\, S + A, \frac{dI}{dt} = IS\beta - I\,(\gamma+\mu) - hI, \tag{2}$$
$$\frac{dR}{dt} = \gamma\, I + hI - R\mu$$

where $I > 0$ for $r = h(I)$ and with $I = 0$, $I(h) = 0$.

However, a wider epidemic model that is more specific with sub-optimal immunity, incidence nonlinear rate with rate of saturated recovery/treatment was further introduced by [27] as follow;

$$\frac{dS}{dt} = A - \beta SI^2 + \sigma T(I) - \mu S, \frac{dI}{dt} = \beta SI^2 - T(I) - \mu I, \tag{3}$$

$$\frac{dR}{dt} = (1-\sigma)T(I) - \mu R$$

This paper aim to extend the above model (3) to fractional model

We present the definitions of fractional integral-order and the Caputo fraction derivative-order as follow:

B. Definition 1

The fractional integral with fractional order $\beta \in \Re^t$ of function $x\,(t)\,, t > 0$ is defined as:

$$I^{\beta} x\,(t) = \int_0^t \frac{(t-s)^{\beta-1}\, x\,(s)}{\Gamma\,(\beta)}\,\mathrm{d}s \tag{4}$$

where $t = t_0$ refers to initial time and $\Gamma\,(\beta)$ is the function of Euler's gamma.

C. Definition 2

The Caputo fractional derivative with order $\alpha \in n-1, n$ of function $x\,(t)\,, t > 0$ is described as:

$$cD_t{}^{\alpha} x\,(t) = I^{n-\alpha} D^n x\,(t)\,, D_t = \frac{d}{dt}. \tag{5}$$

II. Stability Analysis of Fractional Order System

In this section, we examine the local stability analysis, established on stability theory of fractional-order system. It is worthy to understand that the point of equilibrium of fractional order is the same with the corresponding integer, but their conditions are quite different. For integer order, the equilibrium point is not stable when the eigenvalue is non-negative, while that of the fractional order can still be stable even when the eigenvalue is non-negative.

Theorem .1: The necessary and sufficient condition for Caputo fractional derivative to be asymptotically stable locally, with system (7) where $\alpha \in (0, 1]$ is if and only if λ_i of the Jacobian, $\frac{\partial}{\partial y} f\,(t, y)\,,$ computed at the points of equilibrium is contented by $\left|\arg\, \lambda_i\right| > \frac{\alpha\pi}{2}$, $i = 1, 2, 3$.

Proof: Consider the commensurate fraction-order system given below:

$$cD_t{}^{\alpha} y_i\,(t) = f\,(t, y_i\,(t))\,, y_i\,(t_o) = y_0 \tag{6}$$

where $cD_t{}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1]$.

In order to evaluate the equilibrium points, let put

$$cD_t{}^{\alpha} y_i\,(t) = 0 \Rightarrow f_i\,(f_1{}^{eqn}, f_2{}^{eqn}, f_3{}^{eqn}) = 0. \tag{7}$$

for which we can get the equilibrium points $f_1{}^{eqn}, f_2{}^{eqn}, f_3{}^{eqn}$.

Now, to obtain the asymptotic stability, we consider the system $cD_t{}^{\alpha} f\,(x) = f\,(x, y)$ in the sense of Caputo and to find the asymptotic stability, let $y_i\,(t) = y_i{}^{eqn} \epsilon_i\,(t)$. The equilibrium point $(f_1{}^{eqn}, f_2{}^{eqn}, f_3{}^{eqn})$ is locally asymptotically stable when the Jacobian eigenvalues

$$\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix}$$

evaluated at the equilibrium point is satisfied by $|\arg\, \lambda_{1,2,3}| > \frac{\alpha\pi}{2}$ [19], [39], [40], [41]. ■

III. Existence of equilibria

System (3) was reduced to two equations as given in system (2) after summation. i.e. from $N(t) = S(t) + I(t) + R(t)$ where $\frac{dN}{dt} = -\mu t + A$ was got and resulted into $N(t) = \frac{A}{\mu(1-e^{-\mu t}) + N_0 e^{-\mu t}}$. As $t \to \infty$. We arrive $N(t)$ as a constant value, $\frac{A}{\mu} = S + I + R$.

$$\frac{dI}{dt} = \beta(A\mu - I - R)I^2 - vI - \frac{cI}{1+aI} - \mu I, \quad (8)$$

$$\frac{dR}{dt} = k(vI + \frac{cI}{1+aI}) - \mu R.$$

The conditions below are for the non-negative initial:

$$I(0) = I_0, R(0) = R_0 \quad (9)$$

The above integer-order derivatives of system (8) is replaced in the sense of Caputo with the fractional derivatives of order $0 < \alpha \le 1$ as follows:

$$cD_t{}^{\alpha} I(t) = \beta(A\mu - I - R)I^2 - vI - \frac{cI}{1+aI} - \mu I, \quad (10)$$

$$cD_t{}^{\alpha} R(t) = k(vI + \frac{cI}{1+aI}) - \mu R.$$

where $0 < \alpha \le 1$. All parameters used represents positive constants.

We shall use R_0 to establish the stability and existence conditions of both endemic and disease-free for the equilibrium points, which is the number of people that one sick person will infect on average.

There are two equilibria in the system (10) when equating them to zero, namely, the active of endemic equilibrium, E^e points and the disease-free equilibrium, E^0 point. To establish the stability of the equilibrium points for both E^0 and E^e of system (10), we apply the lemma defined below.

Lemma 3.1: Let $K = \frac{\beta(kv+kc+\mu-Aa)}{\beta a(kv+\mu)}$, $M = \frac{-\beta A+v\mu a+\mu^2 a}{\beta a(kv+\mu)}$,$N = \frac{c\mu+\mu^2+v\mu}{\beta a(kv+\mu)}$ and $q = M - \frac{1}{3}K^2$, $p = \frac{2}{27}K^3 - \frac{1}{3}KM + N$,$= \frac{p^2}{4} + \frac{q^3}{27}$. Assuming $\alpha < \frac{ck+vc+\mu}{A}$ or $\alpha < \frac{A\beta}{\mu^2+v\mu}$, then,

(a) If < 0 , model (10) possesses two non-negative equilibria, $E_j\left(I_i, \frac{kI_i(c+avI_i+v)}{(aI_i+1)\mu}\right)$ for $i = 1, 2$.
(b) If $= 0$, model (10) possesses a unique non-negative equilibrium, $E^*\left(I^*, \frac{kI^*(c+avI^*+v)}{(1+aI^*)\mu}\right)$.
(c) If > 0, model (10) possesses no non-negative equilibrium..

Proof: Assume equilibrium occurs at t_e , then $cD_t^{\alpha} I(t)_e = 0$, $cD_t^{\alpha} R(t)_e = 0$ with $N(t_e) = A/\mu(1 - e^{-\mu t_e}) + N_0 e^{-\mu t_e} := A_e$. Thus from the second equation of (10), we obtain $R = \frac{kI(v(1+aI)+c)}{\mu(1+aI)}$. We arrive that

$$[(vk+\mu)\beta a]I^3 + [(-Aa - ck + vk + \mu)\beta]I^2$$
$$+[\mu^2 a + v\mu a + A\beta]I + \mu^2 + c\mu + \nu\mu = 0$$

which is simplify as

$$I^3 + KI^2 + MI + N = 0 \quad (11)$$

where $K = \frac{\beta(kv+kc+\mu-Aa)}{\beta a(kv+\mu)}$, $M = \frac{-\beta A+v\mu a+\mu^2 a}{\beta a(kv+\mu)}$,and $N = \frac{c\mu+\mu^2+v\mu}{\beta a(kv+\mu)}$. where

$$a > 1A(kv+kc+\mu) \qquad or \qquad a < \beta Av\mu + \mu^2 \quad (12)$$

Further from (11), suppose $I = x - \frac{K}{3}$, we have that

$$x^3 + dx + p = 0$$

where $d = M - \frac{1}{3}K^2$,and $p = \frac{2}{27}K^3 - \frac{1}{3}KM + N$. Let

$$= p^2 4 + q^3 27. \quad (13)$$

then we have

(a) suppose > 0, there exist one real root.
(b) suppose $= 0$, there exist two real distinct roots. (or two equal, and all roots are real)
(c) suppose < 0, there exist three real distinct roots.

The outcome below is shown after combining (12) and (13), which is from Descartes rule of sign. Assuming that < 0, then model (10) possesses two non-negative equilibria, $E_j\left(I_i, \frac{kI_i(c+avI_i+v)}{(aI_i+1)\mu}\right)$ for $i = 1, 2$. Suppose $= 0$, system (8) possesses a unique non-negative equilibrium, $E^*\left(I^*, \frac{kI^*(c+avI^*+v)}{(1+aI^*)\mu}\right)$. Assumig > 0, model (8) possesses no positive equilibrium. ∎

Remark 3.1: Graphically, the surface $= p^2/4 + q^3/27 = 0$ or $1/108(4K^3N - K^2M^2 - 18KMN + 27N^2 + 4M^3)$ displays a saddle node bifurcation surface. i.e. it gives a two non-negative equilibria on the one side of the surface.

1) Disease-free equilibrium, E^0: In this subsection, we demonstrate the asymptotic stability of the disease-free equilibrium, E^0. when $R_0 < 1$. The basic number for reproduction, R_0, of the model (10) is obtain as:

$$R_0 = \frac{\beta A}{\mu(c+\mu+v)},$$

The disease-free equilibrium is:

$$E^0 = (I^* = 0, R^* = 0) \quad (14)$$

System (10) at E^0 is asymptotically stable if after obtaining the Jacobian matrix, it's two eigenvalues are satisfied by using

$$|\arg \lambda_1| > \frac{\alpha\pi}{2}, |\arg \lambda_2| > \frac{\alpha\pi}{2}, \quad (15)$$

which was described in Section 3.2. This ensures that the E^0 is locally asymptotically stable if $R_0 < 1$, or otherwise unstable when $R_0 > 1$.

However, condition in system (15) is satisfied for the disease-free equilibrium, E_{E^0}, as given in the Theorem (III-1)

[disease-free equilibrium] A sufficient condition for the system (10) to be locally asymptotically stable at E^0 is if and only if

$$R_0 = \frac{\beta A}{\mu(c+\mu+v)} < 1. \quad (16)$$

Proof: To prove Theorem (III-1), it is sufficient to illustrate that all eigenvalues of Jacobian of (10) at E^0 have a negative real part. Hence, the Jacobian is

$$\left[\begin{array}{cc} \psi - v - \frac{c}{Ia+1} + \frac{cIa}{(Ia+1)^2} - \mu & -I^2\beta \\ k\left(v + \frac{c}{Ia+1} - \frac{cIa}{(Ia+1)^2}\right) & -\mu \end{array}\right]$$

where

$$\psi = -I^2\beta + 2\,\beta\left(\frac{A}{\mu} - I - R\right) I$$

Then for $I_{eqn}, R_{eqn} = (I^* = 0, R^* = 0)$ we find that

$$A = \left[\begin{array}{cc} -v - c - \mu & 0 \\ k\,(v + c) & -\mu \end{array}\right],$$

and its eigenvalues are

$$\lambda_1 = -\mu, \lambda_2 = -v - c - \mu. \tag{17}$$

It follows that Equation λ_1 and λ_2 are less than zero, which implies that $R_0 < 0$ and satisfy the condition in Equation (15). Hence, the eigenvalues of the system (10) is always negative (due to all the parameters are positive). So O is locally asymptotically stable. Then, the disease-free equilibrium, E^0, is locally asymptotically stable. Conversely, it becomes unstable when

$$R_0 = \frac{\beta\,A}{\mu\,(c + \mu + v)} > 1. \tag{18}$$

■

2) Endemic equilibrium, E^e: According to the system (10), the endemic equilibrium points are obtained via solving the following quadratic equation , $\lambda^2 + A\lambda + B = 0$, where A=$-2\beta I(\frac{A}{\mu} - I - R) + v + \beta I^2 + \frac{c}{(1+aI)^2} + 2\mu$, B=$\left(-2\beta I(\frac{A}{\mu} - I - R) + \beta I^2 + v + \frac{c}{(1+aI)^2} + \mu\right)\mu + \beta I^2 k\left(v + \frac{c}{(1+aI)^2}\right)$

Now, if $B < 0$, then the eigenvalue becomes $1/2\left(B \pm \sqrt{B^2 - 4\,A}\right)$.

If I^* is a non-negative real root of the above equation, then the $E^e = (I^*, R^*)$ is the point of endemic equilibrium of our equation (10). However, if $\beta \le \mu + v$, then (10) posses no point of endemic equilibrium.

Also, if $B > 0$, we obtain that the eigenvalues are the negative real part if 1c+$\mu + v\left(R\beta + 3\beta I^2 + 2\mu + v2I + c2I(1 + aI)^2\right) > R_0$ Thus, $E(I, R)$, is locally asymptotically stable, if 1c+$\mu + v\left(R\beta + 3\beta I^2 + v + 2\mu 2I + c2I(1 + aI)^2\right) > R_0$ and $P_2 > 0$.

A. Experimental Simulation Calculation

In this subsection, we employ the Adams-type predictor-corrector (ATPC) implicit numerical method. This method was investigated in [31], and further studied in [32]. The ATPC method gives the error-free means of solving a problem with a sensible and logical choice of the time step [7]. To illustrate the fractional epidemic model stability as in Equation (10), we choose the following parameters $\beta = \frac{1}{2}$, $v = 1.27$, $c = 2$, $a = 4$, $\mu = 1$, $k = \frac{1}{2}$ and $A = 6$ with the following initial values $(I, R) = (2, 1)$. By direct solving and using Maple 18 software, we obtain the model (10) equilibrium points as follow,

$$E_1(I_1, R_1) = (1.615353698, 1.242243888)$$

and

$$E_2(I_2, R_2) = (2.046474276, 1.522295468)$$

.

Hence, the Jacobian matrix for the corresponding equilibrium point (R_1, I_1) is given as

$$J = \left[\begin{array}{cc} \xi^* & -1/2\,J^2 \\ \xi^{**} & -1 \end{array}\right]$$

where,

$$\xi^* = -1/2\,J^2 + (6 - J - R)\,J - 2.27 - 2\,(4\,J + 1)^{-1} + 8\,\frac{J}{(4\,J + 1)^2}$$

$$\xi^{**} = 0.6350000000 + (4\,J + 1)^{-1} - 4\,\frac{J}{(4\,J + 1)^2}$$

and its eigenvalues for disease-free, E^0 are

$$\lambda_1 = -1 + 0.\,I,$$
$$\lambda_2 = -4.27000000000000 + 0.\,I,$$

that of the endemic, E^e are

$$\lambda_1 = -0.206140119950000 + 0.851064558561176\,I,$$
$$\lambda_2 = -0.206140119950000 - 0.851064558561176\,I,$$

and
while the characteristic equation of the fractional epidemic model as in Equation (10) is:

$$P(\lambda) = -0.4106172840 - 0.7053086420\,\lambda + \lambda^2$$

Therefore, the argument $|\arg\,\lambda_{1,2,3}|$ of matrix J at $\alpha = 0.8$ fall with the range of values, 3.141592654. The values of $|\arg\,\lambda_1|$ of the $E_1(I_1, R_1)$ points is said to be stable and the system asymptotically stable because all the eigenvalues satisfy $|\arg\,\lambda_1| > \frac{\alpha\pi}{2}$. That is, $|\arg\,\lambda_1| = 3.141592654 > 1.256800000 = \frac{\alpha\pi}{2}$.

Also, by direct calculation, it is obvious to show that

$$R_0 = \frac{\beta\,A}{\mu\,(c + \mu + v)} = 0.7025761124,$$

whose obtained result are in agreement and compatible with Theorem (III-1), (disease-free equilibrium).

where $R_0 < 0.7025761124$. The active and stability reality of the model equilibrium points were established via R_0 parameter. This indicates that whenever $R_0 > 1$, there would be average increases in confirmed case, leading to an epidemic or pandemic. It would be a reverse case whenever $R_0 < 1$, which indicate that the new confirm case cannot increase. Therefore, R_0 is a parameter that indicate the threshold.

[61] stated that the condition $R_0 < 1$ is a necessary and sufficient condition for the eradication when disease for the forward bifurcation occurs. However, it is no more a sufficient criterion for the occurrence of backward bifurcation. Whenever there is an existence of backward bifurcation, this condition is no longer sufficient because the

backward bifurcation involves the active of a $R_0 = 1$ and $R_0 = R_0^c < 1$ for subcritical transcritical and saddle-node bifurcation, respectively. There would be possibility of appearance of endemic equilibria E_e whenever $R_0 < 1$ and an appearance of backward bifurcation at R 0 = 1. This is in accordance of what was earlier discussed for the existence and asymptotically stable

Fig. 1 displays the stable endemic equilibrium with the following fixed parameters $\beta = 1/2, v = 1.27, c = 2, a = 4, \mu = 1, k = 1/2$ and $A = 6$.

For $E_2(I_2, R_2)$, we have $P_2 = 0.766804631 > 0$ and 1c+μ + $v\left(R\beta + 3\beta I^2 + v + 2\mu 2I + c2I(1+aI)^2\right) = 0.7261661363 > R_0$ where $R_0 = 0.7025761124$. It satisfies the existence and asymptotically stability as discussed.

Fig. 1 displays the phase portrait plot of the infected, recovered and both individual, respectively, in a particular time, t in a stable endemic equilibrium with coefficients fixed as $\beta = \frac{1}{2}, v = 1.27, c = 2, a = 4, \mu = 1, k = \frac{1}{2}, A = 6$. The values of equilibrium are $(2.046474276, 1.522295468)$

IV. Hopf Bifurcation

This section studied how the model in (10) perform Hopf bifurcation for several defined values. For simplicity, suppose $\mu = 1$, where the Jacobian of (10) is M=$\begin{bmatrix} \psi - v - \frac{c}{Ia+1} + \frac{cIa}{(Ia+1)^2} - \mu & -I^2\beta \\ k\left(v + \frac{c}{Ia+1} - \frac{cIa}{(Ia+1)^2}\right) & -\mu \end{bmatrix}$ where

$$\psi = -I^2\beta + 2\,\beta\left(\frac{A}{\mu} - I - R\right)I$$

Theorem .2: Suppose that the criteria of (12) are justified and given $\mu = 1$. If there exist a limit cycle for (10), it must contain a positive equilibrium, $E^+(I^+, R^+)$.

Proof: If there exist a limit cycle, it implies $tr(M) = 0$, then we get $2\,\mathrm{i}\beta\,(-R - i + A) - \frac{c}{ia+1} - v + \frac{ica}{(ia+1)^2} + \beta - 2 = 0$ At equilibrium,$R = \frac{Ik[v(Ia+1)+c]}{[\mu\,(Ia+1)]}$. So we get A=p$_e\frac{}{[2\,I\beta\,(Ia+1)^2]}$ where P$_3 = a^2\beta(2kv+3)I^4 + \beta a(6 + 4kv + 2kc)I^3$
$+ \ (\beta(2kc + 2kv + 3) + a^2(v + 2))I^2 + 2a(v + 2)I + 2 + c + v$ Substitute the above into (10), after some simplification, we get 12(aI+1)$\big(-a^2\beta I^4 - 2\beta a I^3 +$ 12(aI+1)$\big((va^2 - \beta)I^2 + 2a(v+c)I + c + v\big)\big) = 0$ By Descarteso rule of sign, the case of getting any complex root with a non-negative part is ruled out. This is due to the fact that the entire parameters are both real and non-nagative and in addition the quadratic possess real coefficient. Thus, for a limit cycle in (10), it must contain a positive equilibrium,$E^+(I^+, R^+)$, if (12) is satisfied. ∎

By Theorem (Analysis .2) and avoid the condition in (12), consider ($\beta = 1/2, v = 8, c = 8, a = 3, \mu = 1, k = 1/2$), the positive real root is approximated to 4.270040762 or $\frac{12571}{2944}$. Thus, we obtain the positive endemic equilibrium $E^+(I^+, R^+)$ is $(\frac{12571}{2944}, \frac{548108171}{29923552})$. With the values as mentioned, we obtain $A = \frac{3312503940005049139}{122351215219281152}$, or ≈ 27.07373142.

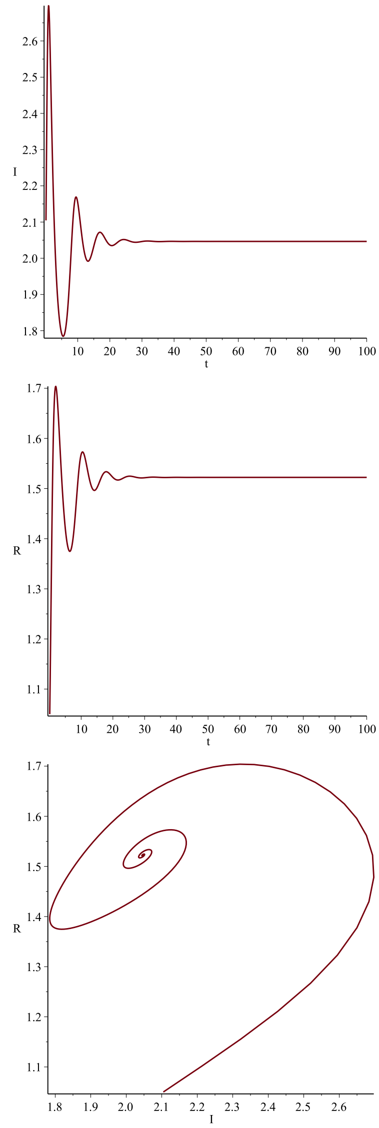

Fig. 1. Phase portrait plot of the model (10) with $\alpha = 1$ (Infected, Recovered and both Stable endemic equilibrium.

Write I and R in term of x and y, hence $(I^+, R^+) = (x^+, y^+) = (\frac{12571}{2944}, \frac{548108171}{29923552})$, we have

$cD_t{}^\alpha x\,(t) = 12\,(3312503940 1223512152 - x - y)\,x^2 - 9x - 8x1 + 3x,$

$cD_t{}^\alpha y\,(t) = 4x + 4x1 + 3x - y.$

Next, we set, $X = x - \frac{12571}{2944}$ and $Y = y - \frac{548108171}{29923552}$ rename X,Y as x,y respectively, to translate (x^+, y^+) to the origin. Then

$cD_t^\alpha x\,(t) = 12\left(\frac{3312503940}{1223512152} - (x + \frac{12571}{2944}\right)$ - (y+548108171$\frac{}{29923552)(x+\frac{12571}{2944})^2 - 9(x+\frac{12571}{2944})}$ - 8(x+12571$\frac{}{2944)1+3(x+\frac{12571}{2944}),}$

$cD_t^\alpha y\,(t) = 4(x + \frac{12571}{2944})$
$+\,4(x + \frac{12571}{2944})1 + 3(x + \frac{12571}{2944}) - (y + \frac{548108171}{29923552}).$

Using the Taylor expansion for the above equation, we have $cD_t^\alpha x\,(t) = -\frac{2331843754}{1687112596} - \frac{158030041}{17334272}y$ +(1-12571$\frac{}{2944y)x - (\frac{2007241309}{9948866714} + \frac{1}{2}y)x^2 - \frac{2743198563}{5464762783}x^3}$

$+4776862929_{1110904302x^4+O(|x,y|^5)}$, $cD_t^{\alpha}y\left(t\right) = -y + \frac{6646635140}{1652991649}x - \frac{306192580608}{6720568147}x^2 + 2704292872_{2732381392x^3 - \frac{2388431464}{1110904302}x^4+O(|x,y|^5)}$.

The Jacobian for the above equation at points $(x,y) = (0,0)$ is $M=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{158030041}{17334272} \\ \frac{6646635140}{1652991649} & -1 \end{pmatrix}$ and hence, $tr(M) = 0$ with $det(M) \approx 35.658 > 0$. In this instance, the eigenvalues are $\pm\frac{\sqrt{710750664623833315864674}}{44645939584}i$ (or $\pm 5.971406960i$), which illustrates the occurrence of Hopf bifurcation.

With the transformation $X = x$, $Y = m_{11}x + m_{12}y$, where $Y = x - \frac{158030041}{17334272}y$, and then renaming X,Y as x,y respectively. Then by approximation it becomes

$cD_t^{\alpha}x\left(t\right) = -2331843816871126 + y - 2473226099488667x^2$
$+ 588812571xy - 4808716286359669x^3 + 8667136158030041x^2y$
$+ 4776862911109043x^4 + O(|x,y|^5)$,
$cD_t{}^{\alpha}y\left(t\right) = -2331843816871126 - 2554286571633517x$
$- 2431902599488667x^2 + 588812571xy$
$- 4886637886359669x^3 + 8667136158030041x^2y$
$+ 2655129511109043x^4 + O(|x,y|^5)$.

Let $k_1 = \frac{255428654207186553}{7163351714373632}$. Changing the variables $u = -x$, $v = \frac{1}{\sqrt{k_1}}y$, we get

$$cD_t{}^{\alpha}u\left(t\right) = -\sqrt{k_1}v + F_1(u,v),\ cD_t{}^{\alpha}v\left(t\right) = \sqrt{k_1}u + F_2(u,v). \quad (19)$$

where $F_1(u,v) = 23318437541687112596 + 24732259349948866714u^2$
$+ 588812571\sqrt{k_1}uv - 48087162378635966867u^3$
$- 8667136158030041\sqrt{k_1}u^2v - 47768629291110904302u^4 + O(|x,y|^5)$,
$F_2(u,v) = -23318437541687112596\sqrt{k_1} - 24319025079948866714\sqrt{k_1}u^2$
$- 588812571uv + 48866377578635966867\sqrt{k_1}u^3$
$- 8667136158030041u^2v + 26551294661110904302u^4 + O(|x,y|^5)$.

The first Liapunov constant, σ can be obtained as $\sigma = -0.01781784460$.

Fig. 2 displays a periodic orbit that is stable with A increasing from $\frac{3312503940}{1223512152}$ ($\approx$ 27.07373142) . It shows both phase patriot are stable orbit for (10) when $\beta = \frac{1}{2}, v = 8, c = 8, a = 3, \mu = 1, k = \frac{1}{2}$ while $(I^+, R^+) = (\frac{12571}{2944}, \frac{548108171}{29923552})$ and $A = 27.1$ [24].

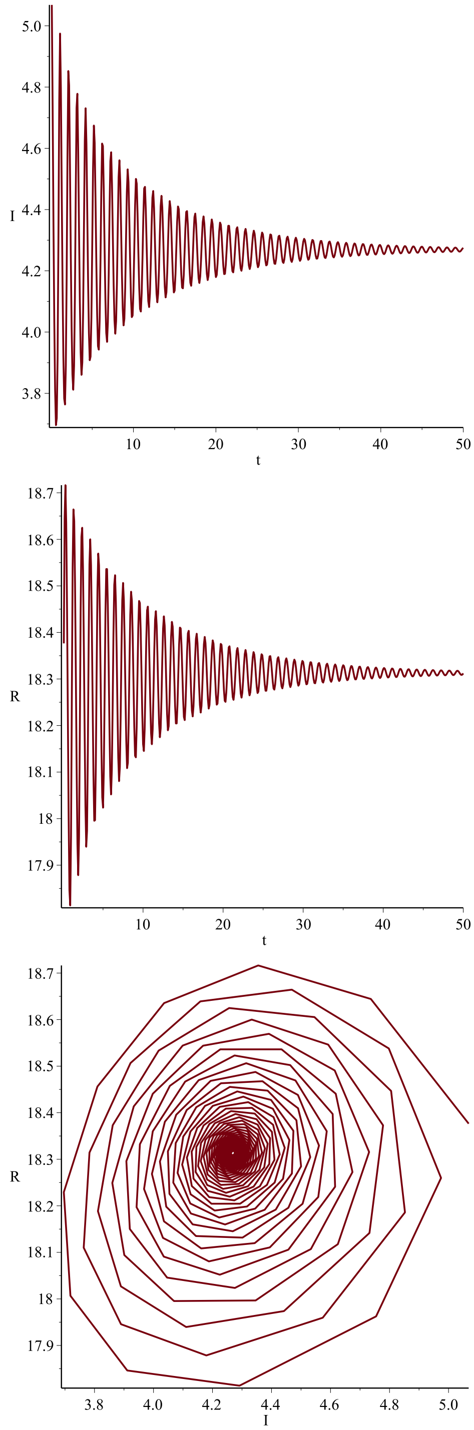

Fig. 2. Infected, Recovered and stable waveform plot of the fractional-order model for the system (10).

Suppose A is chosen as the parameter bifurcation. Let $A = A_0 + \epsilon$, where $A_0 = \frac{33125039400005049139}{122351215219281152}$. From system (10), we get,

$$cD_t{}^{\alpha}I\left(t\right) = \beta(\frac{A_0+\epsilon}{\mu} - I - R)I^2 - vI - \frac{cI}{1+aI} - \mu I, \quad (20)$$

$$cD_t{}^{\alpha}R\left(t\right) = k(vI + \frac{cI}{1+aI}) - \mu R.$$

Let $\mu = 1$, similar to (10), we obtain $[\beta a(kv+1)]I^3 + [\beta(kv+kc+1-(A_0+\epsilon)a)]I^2 + [-\beta(A_0+\epsilon)+va+a]I + [c+1+v] = 0$

$$[-\beta(A_0+\epsilon)+va+a]I + [c+1+v] = 0$$

Rewriting I and R as x and y, and $(\beta, v, c, a, \mu, k) = (1/2, 8, 8, 3, 1, 1/2)$, then by (10), we obtain,

$$cD_t{}^{\alpha}x\left(t\right) = 12(A_0+\epsilon-x-y)x^2 - 9x - 8x1 + 3x, \quad (21)$$

$$cD_t{}^{\alpha}y\left(t\right) = 4x + 4x1 + 3x - y.$$

Suppose the non-negativity equilibrium of the (21) (x^+, y^+),

the Jacobian matrix is M= $\begin{pmatrix} m_{11} & -\frac{(x^+)^2}{2} \\ m_{21} & -1 \end{pmatrix}$ where $m_{11} = x^+(A_0+\epsilon-x^+-y^+)-9-\frac{(x^+)^2}{2}-\frac{8}{1+3x^+}+\frac{24x^+}{(1+3x^+)^2}$, $m_{21} = 4+\frac{4}{1+3x^+}-\frac{12x^+}{(1+3x^+)^2}$ The characteristic equation is $\lambda^2+(1-m_{11})\lambda-m_{11}+(x^+)^2 2m_{21} = 0$ and $\lambda = \frac{1}{2}(m_{11}-1) \pm \sqrt{m_{11}^2+2m_{11}+1-2m_{21}(x^+)^2}$
So, we have

(a) $Re\lambda(\epsilon) = 0$ when $\epsilon = 0$.
(b) $Im\lambda(\epsilon) = 5.971406945 \neq 0$ when $\epsilon = 0$.
(c) $Redde\epsilon\lambda(\epsilon) = -0.3575405923 \neq 0$ when $\epsilon = 0$.

Theorem .3: There exist a $\sigma_1 > 0$ and a function $\epsilon = \epsilon(x_1)$ defined on $0 < x_1 - \frac{12571}{2944} \leq \sigma_1$, satisfying $\epsilon(\frac{12571}{2944}) = 0$ and when $\epsilon = \epsilon(x_1) < 0$, (21) has a unique stable limit cycle that goes through $(x_1, \frac{548108171}{29923552})$.

It is interesting to note that there exist periodic orbit which is unstable when $A = 27.0$ with the initial condition $(I, R) = (5, 18)$. For instance, Fig. 3 show phase patriot and stable waveform as an unstable periodic orbit for system (10) when $\beta = \frac{1}{2}, v = 8, c = 8, a = 3, \mu = 1, k = \frac{1}{2}$. With the eigenvalue $\lambda_{1,2} = \frac{81}{64} \pm \frac{I}{64}\sqrt{205311}$. When this also pass through a critical value, α^* then, (10) gains its stability.

However, at $A = 27.0$, the system undergo Hopf bifurcation with α increasing past $\alpha^* = 0.887385139$. These are displayed in Fig. 4. Similarly, we fixed $A = 27.07$ and the model also undergo Hopf birfurcation, which is show in Fig. 5.

So, the equilibrium of the endemic $E^{^e}$ confirmed loses its stability at the stable limit cycle bifurcation from it at the value of Hopf bifurcation

V. Some effects of the fractional-order α on the behavior of dynamical systems of the epidemic model

In this section, we show clearly the effect that α has on system (10) by taking the value which is appropriate for the fractional order of the α. However, the fractional-order system is achieved in the steady state when parameters affecting the value of α are appropriately controlled [9]

Theorem .4: [Some effects of the fractional-order α on the dynamical behavior] Suppose I^* is the quadratic equation positive real root, then E^* is the point of endemic equilibrium of (10) as introduced in Lemma (3.1):

(a) The endemic equilibrium point E^* is unstable.
(b) If $\alpha \leq \frac{2}{3}$, the point of endemic equilibrium E^* are locally asymptotically α stable.
(c) If $\alpha > \frac{2}{3}$ and $v \leq k$, the endemic equilibrium point E^* are locally asymptotically.

Proof: To prove Theorem (Analysis .4), it is sufficient to illustrate that all eigenvalues of Jacobian of (10) at E^* satisfy the condition (15). Hence, the Jacobian matrix is

$$\begin{bmatrix} -I^2\beta + 2\beta\left(\frac{A}{\mu} - I - R\right)I - v - T - \mu & -I^2\beta \\ k\left(v + \frac{c}{Ia+1} - \frac{cIa}{(Ia+1)^2}\right) & -\mu \end{bmatrix}.$$

where,

$$T = \frac{c}{Ia+1} + \frac{cIa}{(Ia+1)^2}$$

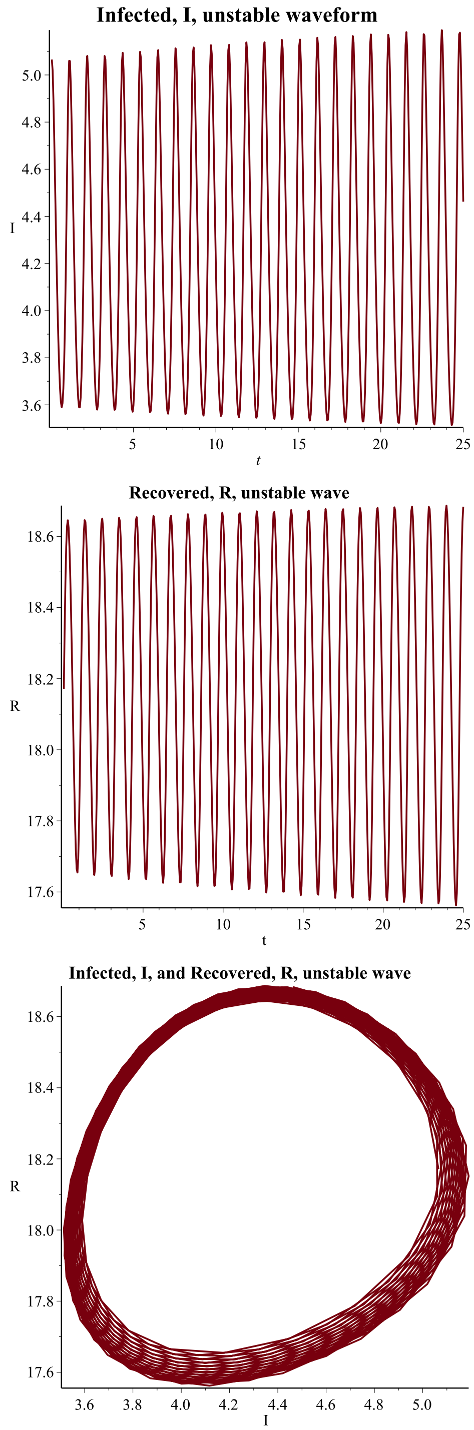

Fig. 3. Infected and recovered unstable waveform plot of the fractional-order model for the system (10) at $A = 27.0$ and $\alpha = 1$.

From the second equation (10), the characteristic of $P(I) = I^3 + KI^2 + MI + N = 0$ where

$$K = \frac{\beta(kv + kc + \mu - Aa)}{\beta a(kv + \mu)},$$

$$M = \frac{-\beta A + v\mu a + \mu^2 a}{\beta a(kv + \mu)},$$

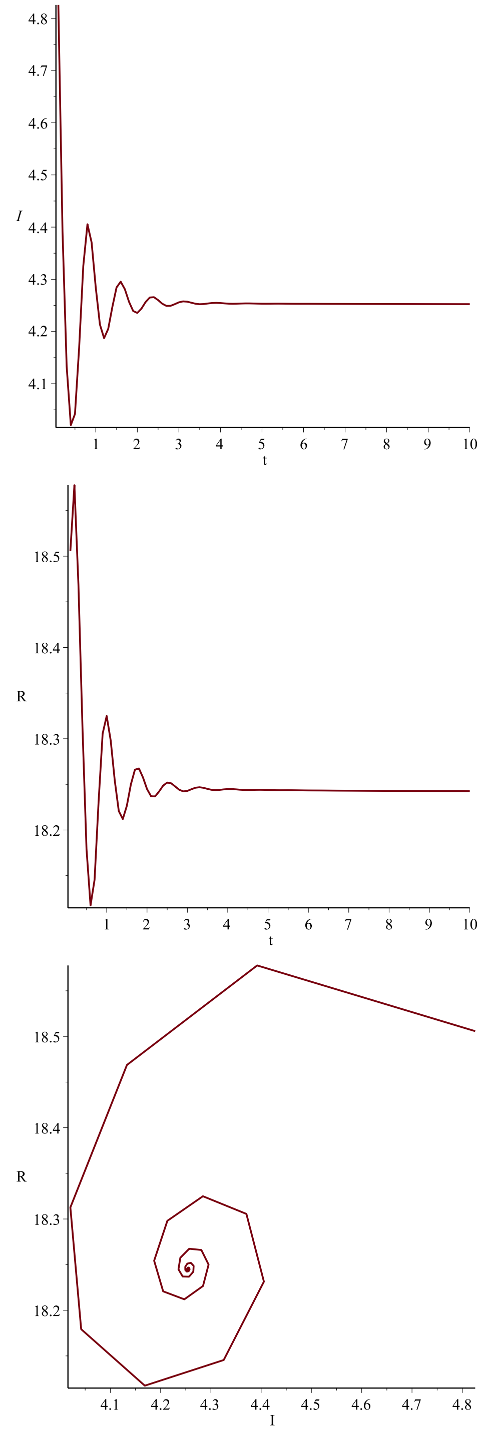

Fig. 4. Infected and recovered stable waveform plot of the fractional-order model for the system (10) at $A = 27.0$ and $\alpha = 0.887385139$.

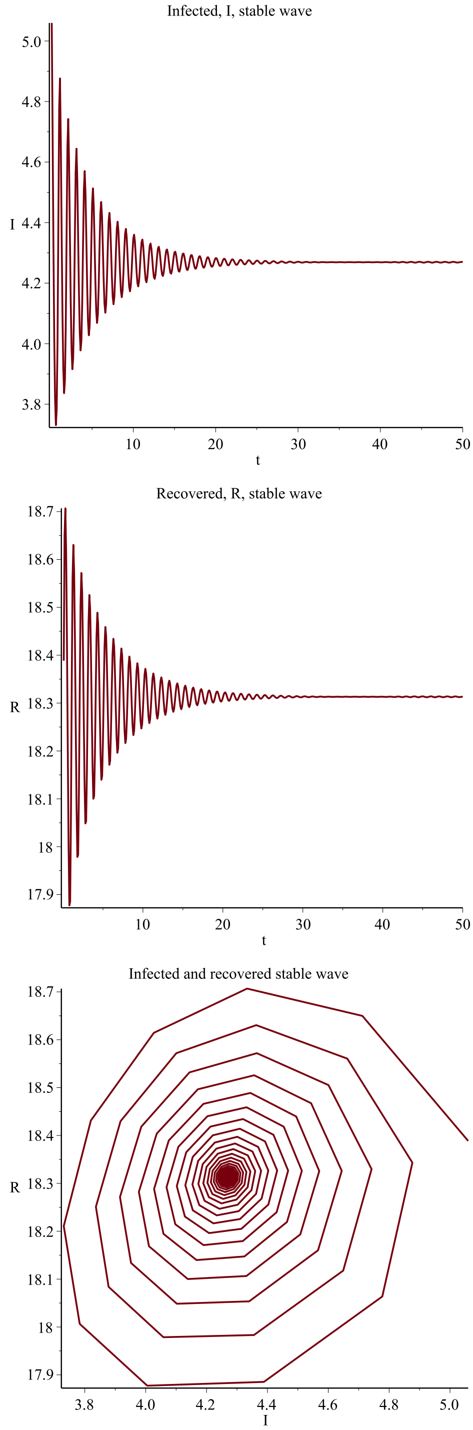

Fig. 5. Infected and recovered stable waveform plot of the fractional-order model for the system (10) at $A = 27.07$.

and

$$N = \frac{c\mu + \mu^2 + v\mu}{\beta a(kv + \mu)}.$$

We basically follow the fundamental fractional order Routh-Hurwitz conditions in [45]. For every point of the endemic equilibrium, E^*, it is clear that the condition for (15) is $0 \leq K, 0 \leq M, 0 \leq N$ and $\alpha < \frac{2}{3}$. As earlier stated, by Descartes' rule of sign, we now assume that $P(I) = 0$ posses one non-positive real root says, $\lambda_1 = -t$ and a pair of complex value roots says $\lambda_{2,3} = 0 \pm bi$ as thus;

$$P(I) = I^3 + I^2(-2a + t) + I(a^2 + b^2 - 2at) + t(a^2 + b^2)$$

It implies that

$$K = -2a + t, M = a^2 + b^2 - 2at, N = t(a^2 + b^2)$$

We know that $0 \leq K, 0 \leq M$. It implies that $2at \leq a^2 + b^2$ and $2a \leq t$, from which we obtain

$$4\,a^2 \leq 2\,at \leq a^2\left(1 + \frac{a^2}{b^2}\right) \tag{22}$$

It then show from equation (22) that $4 \leq sec^2\,(Arg\lambda_{2,3})$ and $\frac{\pi}{3} \leq (Arg\lambda_{2,3}) \leq \frac{2\pi}{3}$. Therefore, if $\alpha \leq \frac{2}{3}$, then condition (15) is satisfied and E^* are considered stable. Likewise, if $KM - N > 0$, then $0 < -2\,a_{(a-t)^2+b^2}$. In the same manner, if $KM - N > 0$, then $\lambda_{2,3}$ must have non-positive real parts. It can be shown that if $v \leq k$, then $KM - N > 0$ and the root of the equation $P(I) = 0$ have a non-positive real parts. ∎

In order to make comparison with the uncontrolled fractional-order for the system (10), we discus the control model with different value. Fig. 6-8 show that at $A = 27.0$ and for lowering the parameter of α namely, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$ it has effect on the stability, and as a result can stabilize the stable fixed point. Fig. $9 - 10$ established our (Analysis .4), where they display the some effects of the fractional-order α on the behavior of dynamical systems of the epidemic model. From the epidemiological point of view, this feature is very important because the meaning gives a longer periodic, which infected persons can effect the health system.

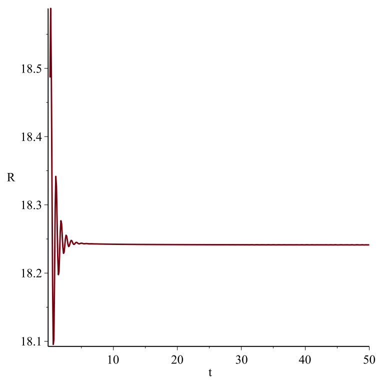

Fig. 6. Size of the stable point of infected and recovered classes over time in system (10) with $\alpha = 0.90$.

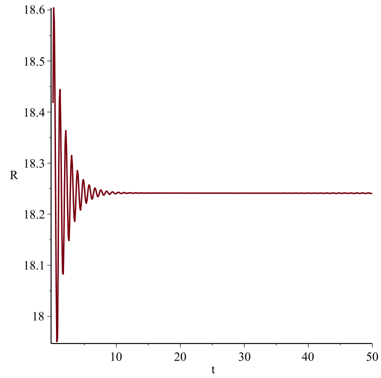

Fig. 7. Size of the stable point of infected and recovered classes over time in system (10) with $\alpha = 0.95$.

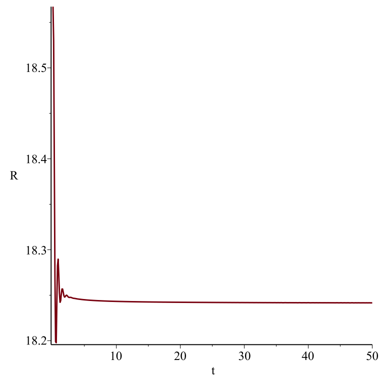

Fig. 8. Size of the stable point of infected and recovered classes over time in system (10) with $\alpha = 0.85$.

VI. Conclusion

In this paper, we extended an epidemic model in the sense of Caputo derivative of order $\alpha \in (0, 1]$. The model indicates that the spread of a disease depends on the contact rates with infected individual within the population. Basic reproduction number, R_0, affects the model behaviour. We used R_0 to establish the stability and existence conditions at the points of equilibrium. For simple epidemic processes, this parameter determines a threshold whenever $R_0 > 1$, a typical infective on average, gives rise to more than one secondary infection, and thus, lead to an epidemic. The system in (10) admits Hopf bifurcations using varying fractional order and parameters. We applied Adams-type predictor-corrector method to the numerical solutions of the models. We also show that the disease will be extinct when the bifurcation parameters are within certain regions.

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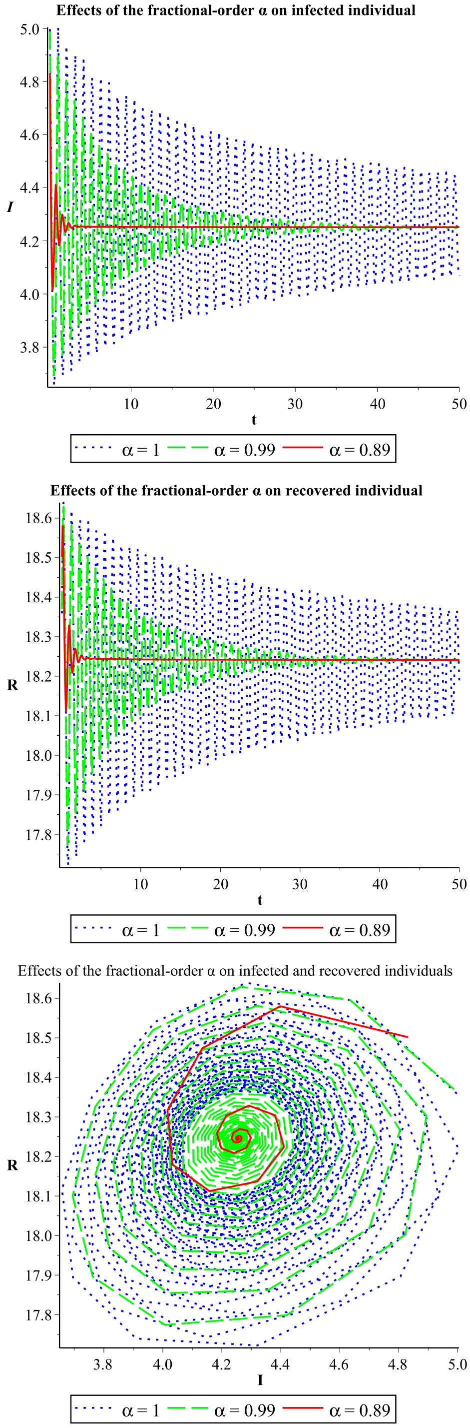

Fig. 9. Effects of the fractional-order α on the behavior of dynamical systems of the epidemic model.

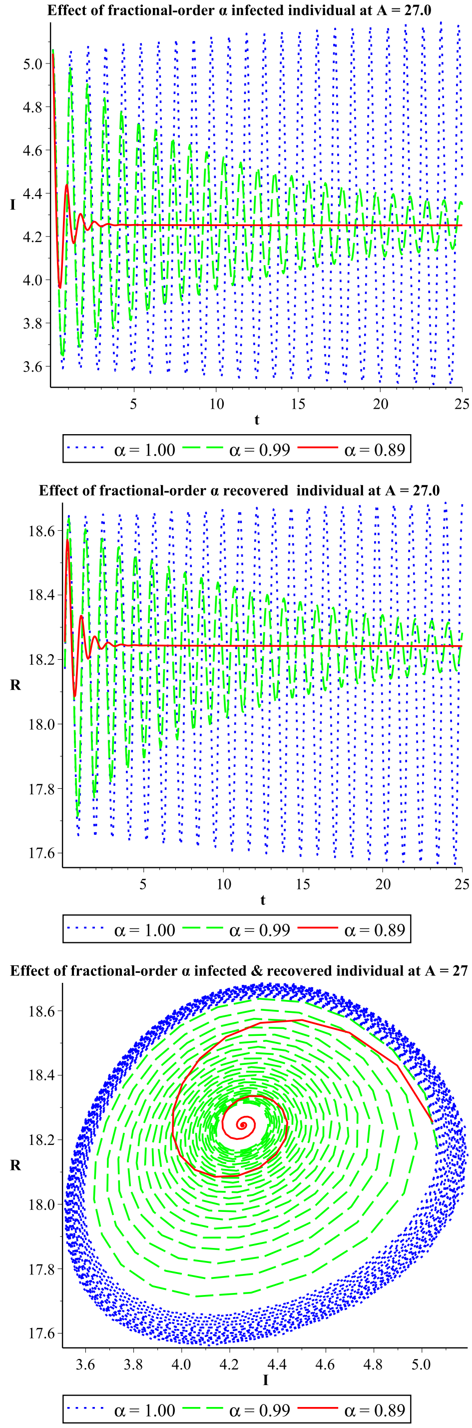

Fig. 10. Effects of the fractional-order α on the behavior of dynamical systems of the epidemic model.

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