Incoming Local Exponent for a Two-cycle Bicolour Hamiltonian Digraph with a Difference of $2n + 1$

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***Abstract*—A bicolour digraph $D^{(2)}$ is a directed graph with every arc coloured in one of two colours, red or black. Suppose r and k are nonnegative integers representing the number of red and black arcs, respectively. The smallest sum of r and k such that every node on $D^{(2)}$ has a walk to node x is called the incoming local exponent of node d_x. For primitive bicolour digraphs with a difference of $2n+1$, there will be three or four red arcs. This article discusses the incoming local exponent for a primitive bicolour Hamiltonian digraph with a difference of $2n + 1$.**

***Index Terms*—primitive-digraph, bicolour-digraph, incoming-local-exponent.**

I. Introduction

A Directed graph (digraph) D consists of a finite set N, which has elements called nodes, and the set A, which contains all the pairs of nodes in N (each pair is called an arc). The bicolour digraph $D^{(2)}$ is a directed graph with every arc coloured in one of two colours, red or black. Let r and k be nonnegative integers representing the number of red and black arcs, respectively. A walk consisting of positive integers $r + k$ in a bicolour digraph is called an (r, k)-walk. For a walk W in bicolour digraph $D^{(2)}$, $p(W)$ and $q(W)$ denote the number of red arcs and the number of black arcs contained in walk W, respectively. The column matrix $\begin{bmatrix} p(W) \\ q(W) \end{bmatrix}$ is the composition of the walk W, and $\ell(W) = p(W) + q(W)$ is the length of the walk W. A primitive bicolour digraph is a bicolour digraph in which each pair of nodes has an (r, k)-walk [1]. The smallest sum of r and k over all pairs of nonnegative integers is called the exponent of $D^{(2)}$ [2]. Whereas, the smallest sum of r and k such that every node on $D^{(2)}$ has a walk to node x is called the incoming local exponent of node d_x and denoted by $\text{inexp}\big(d_x, D^{(2)}\big)$.

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Local exponent research was initiated by Gao [3] using the Wielandt bicolour digraph with cycles of length n and $n-1$. Suwilo [4] found the local exponents for a two-cycle bicolour digraph with cycles $n-1$ and $n-2$. Syahmarani and Suwilo [5] investigated the local exponents of a Hamiltonian digraph with cycles n and $n-2$ for odd n and $n \geq 5$. Suwilo and Syafrianty [6] discussed the local exponents of a two-cycle bicolour digraph with cycles $n-1$ and $n-3$ for even number vertices. Sahara et al. [7] found the local exponents for a two-cycle bicolour digraph with cycles n and one loop. Sumardi and Suwilo [8] determined a local exponent for a bicolour digraph with cycle lengths of $2s+1$ and s for $s \geq 5$ and an allied node. Mardiningsih et al. [9] conducted research on incoming local exponents of bicolour digraphs with cycles of length $s+1$ and s. Mardiningsih et al. [10] investigated the incoming local exponents of primitive two-cycle bicolour digraph with cycles s and $2s-1$. Mardiningsih et al. [11] discussed incoming local exponents for a two-cycle bicolour Hamiltonian digraph with cycles n and $n-3$.

This paper discusses the incoming local exponent of a Hamiltonian digraph with cycle lengths n and $3n+1$. In other words, the difference between cycle lengths is $2n+1$. In Chapter 2, the primitivity of the bicolour digraph is discussed. Chapter 3 discusses how to determine the bounds of incoming local exponents for a bicolour digraph. Chapter 4 presents the results.

II. Primitivity

Fornasini and Valcher [1] provide the characteristics for a primitive bicolour digraph. A bicolour digraph is said to be primitive iff the content of the cycle matrix is equal 1. The cycle matrix's content is the greatest common divisor of the 2×2 submatrix determinant of the cycle matrix. The cycle matrix for a two-cycle bicolour digraph is $M = \begin{bmatrix} p(L_1) & p(L_2) \\ q(L_1) & q(L_2) \end{bmatrix}$, with L_1 and L_2 representing the first and second cycles.

Corollary II.1. *Suppose that $D^{(2)}$ is a strongly connected bicolour digraph with two cycles of length n and $3n+1$. If $D^{(2)}$ is primitive, then the cycle matrix $M = \begin{bmatrix} 1 & 3 \\ n-1 & 3n-2 \end{bmatrix}$ or $M = \begin{bmatrix} n-1 & 3n-2 \\ 1 & 3 \end{bmatrix}$.*

Proof: Note that the shape of the cycle matrix of $D^{(2)}$ is a strongly connected bicolour digraph with two cycles with lengths n and $3n+1$. If $D^{(2)}$ has the cycle matrix $M = \begin{bmatrix} y & z \\ n & 3n+1 \end{bmatrix}$ with $0 \leq y \leq n$ and $0 \leq z \leq 3n+1$, then, because $D^{(2)}$ is primitive, $\det(M) = \pm 1$. If $\det(M) = 1$, then $(3y - z)\,n + y = 1$. Since $0 \le z \le 3n + 1$, we get $3y - z = 0$. Hence, $y = 1$ and $z = 3$. So, $M = \begin{bmatrix} 1 & 3 \\ n-1 & 3n-2 \end{bmatrix}$. If $\det(M) = -1$, then $(z - 3y)\,n - y = 1$. Since $0 \le z \le 3n + 1$, we have $z - 3y = 1$. Consequently, $y = n - 1$ and $z = 3n - 2$. Thus, $M = \begin{bmatrix} n-1 & 3n-2 \\ 1 & 3 \end{bmatrix}$.

■

Because changing all of the arcs from red to black and vice versa does not change the incoming local exponent, without loss of generality, we can assume that the cycle matrix of $D^{(2)}$ is $M = \begin{bmatrix} 1 & 3 \\ n-1 & 3n-2 \end{bmatrix}$. Therefore, $D^{(2)}$ has three or four red arcs.

III. Bounds for the Incoming Local Exponent

This chapter starts with the results obtained in [4] because they will help determine the lower and upper bounds of the incoming local exponent.

Proposition III.1. *[4] Suppose that $D^{(2)}$ is a bicolour digraph with two cycles and d_x is any node on $D^{(2)}$ found in both cycles. If for some nonnegative integers r and k, there is a path P_{d_u,d_x} from d_u to d_x such that system*

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_x}) \\ q(P_{d_u,d_x}) \end{bmatrix} = \begin{bmatrix} r \\ k \end{bmatrix}$$

has a nonnegative integer completion, then $\mathrm{inexp}(d_x, D^{(2)}) \le r + k$.

Lemma III.1. *[4] Suppose that $D^{(2)}$ is a primitive bicolour digraph and d_u is any node on $D^{(2)}$ with the incoming local exponent* $\mathrm{inexp}(d_u, D^{(2)})$. *Then for every $x = 1, 2, \ldots, 3n+1$,* $\mathrm{inexp}(d_x, D^{(2)}) \le \mathrm{inexp}(d_u, D^{(2)}) + \delta(d_u, d_x)$.

Lemma III.2. *[9] Suppose that $D^{(2)}$ is a primitive bicolour digraph with two cycle of length L_1 and L_2 with cycle matrix $M = \begin{bmatrix} p(L_1) & p(L_2) \\ q(L_1) & q(L_2) \end{bmatrix}$ and that* $\det(M) = 1$. *If* $\mathrm{inexp}(d_x, D^{(2)})$ *is obtained via the (r_x, k_x)-walk, then*

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \ge M \begin{bmatrix} q(L_2)p(P_{d_u,d_x}) - p(L_2)q(P_{d_u,d_x}) \\ p(L_1)q(P_{d_w,d_x}) - q(L_1)p(P_{d_w,d_x}) \end{bmatrix}$$

for some path P_{d_u,d_x} and P_{d_w,d_x}.

IV. Results

This article discusses a Hamiltonian two-cycle bicolour digraph with a difference of $2n+1$ (see Fig. 1). The first cycle with length n is $L_1 : d_1 \to d_2 \to \cdots \to d_{n-1} \to d_n \to d_1$, and the second cycle with length $3n + 1$ is $L_2 : d_1 \to d_2 \to \cdots \to d_{n-1} \to d_n \to d_{n+1} \cdots \to d_{3n} \to d_{3n+1} \to d_1$. By Corollary 1, this primitive bicolour digraph has three or four red arcs.

First, we will examine the incoming local exponent for the digraph with four red arcs. The red arcs on $D^{(2)}$ are $d_n \to d_1$, $d_b \to d_{b+1}$, $d_c \to d_{c+1}$ and $d_a \to d_{a+1}$ for $n \le b < c < a \le 3n + 1$. The distance from node $(a + 1)$ to node 1 is denoted by $\delta_1 = \delta(d_{a+1}, d_1)$, whereas the distance from node $(b + 1)$ to node 1 is denoted by $\delta_2 = \delta(d_{b+1}, d_1)$. Finally, the distance from node $(c + 1)$ to node 1 is denoted by $\delta_3 = \delta(d_{c+1}, d_1)$.

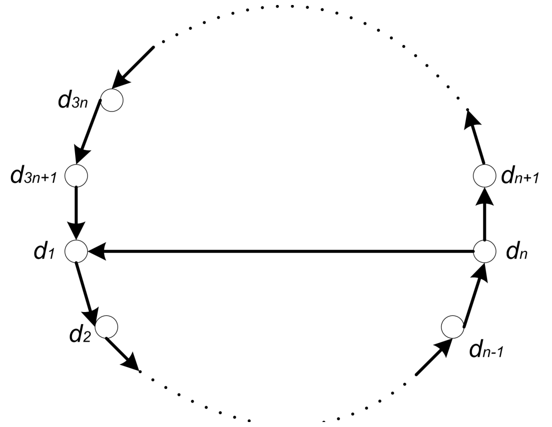

Fig. 1. Hamiltonian two-cycle digraph with a difference of $2n + 1$

Theorem IV.1. *Let a primitive bicolour digraph $D^{(2)}$ have two cycles of length n and $3n+1$. If $D^{(2)}$ has four red arcs, then for every $x = 1, 2, ..., 3n + 1$,*
$\mathrm{inexp}(d_x, D^{(2)}) =$

$$\begin{cases} 9n^2 + 3n\,(\delta_1 - \delta_2) + \delta_1 + \delta\,(d_1, d_x)\,, \\ \quad \text{for } \delta_3 - \delta_1 \le n,\ \ \delta_2 - \delta_1 \le n{+}1 \\ n^2 - 4n + \delta_1 + \delta\,(d_1, d_x)\,, \\ \quad \text{for } \delta_3 - \delta_1 \le n,\ \ n{+}1 < \delta_2 - \delta_1 \le 2n \\ (3n+1)\,\delta_2 - 6n + \delta\,(d_1, d_x)\,, \\ \quad \text{for } \delta_3 - \delta_1 \le n,\ \ \delta_2 - \delta_1 = 2n{+}1 \\ 6n^2 - 4n + \delta_3 + \delta\,(d_1, d_x)\,, \\ \quad \text{for } n < \delta_3 - \delta_1 < 2n \\ (3n+1)\,\delta_3 - 3n + \delta\,(d_1, d_x)\,, \\ \quad \text{for } \delta_3 - \delta_1 = 2n \end{cases}$$

Proof: Suppose that for every $x = 1, 2, ..., 3n + 1$, $\mathrm{inexp}(d_x, D^{(2)})$ is obtained using the (r_x, k_x)-walk. The proof is divided into five cases as follows.

Case 1. (for $\delta_3 - \delta_1 \le n$, $\delta_2 - \delta_1 \le n + 1$)

First, it will be shown that $\mathrm{inexp}(d_x, D^{(2)}) \ge 9n^2 + 3n\,(\delta_1 - \delta_2) + \delta_1 + \delta\,(d_1, d_x)$. We examine the paths P_{d_b,d_x} and P_{d_{a+1},d_x} and define $g_1 = q(L_2)p(P_{d_b,d_x}) - p(L_2)q(P_{d_b,d_x})$ and $g_2 = p(L_1)q(P_{d_{a+1},d_x}) - q(L_1)p(P_{d_{a+1},d_x})$. Four subcases must be examined.

The node d_x is located on the path $d_1 \to d_b$. Utilizing path P_{d_b,d_x}, that is, the $(3, \delta_2 - 2 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 3\delta_2 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(0, \delta_1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \ge M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_1 - 3\delta_2 \\ 9n^2 + 3n(\delta_1 - \delta_2) - 9n - 2\delta_1 + 3\delta_2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\mathrm{inexp}(d_x, D^{(2)}) \ge 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x) \quad (1)$$

for every node d_x located on the path $d_1 \to d_b$.

The node d_x is located on the path $d_{b+1} \to d_c$. Utilizing path P_{d_b,d_x}, that is, the $(1, \delta_2 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 12n - 3\delta_2 + 1 - 3\delta(d_1, d_x)$. Utilizing path

P_{d_{a+1},d_x}, that is, the $(1, \delta_1 - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 - n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_1 - 3\delta_2 + 1 \\ 9n^2 + 3n(\delta_1 - \delta_2) - 9n - 2\delta_1 + 3\delta_2 - 1 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x) \quad (2)$$

for every node d_x located on the path $d_{b+1} \to d_c$.

The node d_x is located on the path $d_{c+1} \to d_a$. Utilizing P_{d_b,d_x}, that is, the $(2, \delta_2 - 3n - 2 + \delta(d_1, d_x))$-path, we obtain $g_1 = 15n - 3\delta_2 + 2 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(2, \delta_1 - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 - 2n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_1 - 3\delta_2 + 2 \\ 9n^2 + 3n(\delta_1 - \delta_2) - 9n - 2\delta_1 + 3\delta_2 - 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x) \quad (3)$$

for every node d_x located on the path $d_{c+1} \to d_a$.

The node d_x is located on the path $d_{a+1} \to d_{3n+1}$. Utilizing path P_{d_b,d_x}, that is, the $(3, \delta_2 - 3n - 3 + \delta(d_1, d_x))$-path, we obtain $g_1 = 18n - 3\delta_2 + 3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(0, \delta_1 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 - 3n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_1 - 3\delta_2 \\ 9n^2 + 3n(\delta_1 - \delta_2) - 12n - 2\delta_1 + 3\delta_2 - 1 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 9n + 3\delta_1 - 3\delta_2$ and $m_2 = 9n^2 + 3n(\delta_1 - \delta_2) - 12n - 2\delta_1 + 3\delta_2 - 1 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{a+1} to d_x. Note that the path is P_{d_{a+1},d_x}, that is, the $(0, \delta_1 - 3n - 1 + \delta(d_1, d_x))$-path. Furthermore, the completion of the system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{a+1},d_x}) \\ q(P_{d_{a+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 9n + 3\delta_1 - 3\delta_2$ and $v_2 = 0$. The path P_{d_{a+1},d_x} located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{a+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{a+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x) \quad (4) \end{aligned}$$

for every node d_x located on the path $d_{a+1} \to d_{3n+1}$.

From (1), (2), (3) and (4), it can be concluded that $\text{inexp}(d_x, D^{(2)}) \geq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\text{inexp}(d_x, D^{(2)}) \leq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\text{inexp}(d_1, D^{(2)}) = 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1$ and then use Lemma III.1 to guarantee that $\text{inexp}(d_x, D^{(2)}) \leq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (1), we obtain $\text{inexp}(d_1, D^{(2)}) \geq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1$. Furthermore, it is enough to show that $\text{inexp}(d_1, D^{(2)}) \leq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_1 - 3\delta_2 \\ 9n^2 + 3n(\delta_1 - \delta_2) - 9n - 2\delta_1 + 3\delta_2 \end{bmatrix} \quad (5)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (5) is $v_1 = 9n - 3\delta_2 - (3n - 2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_1 - (1 - n)p(P_{d_u,d_1}) - q(P_{d_u,d_1})$.

If d_u is located on the $d_1 \to d_b$ path, then there is a $(3, 3n - 2 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 9n - 3(\delta_2 + \delta(d_1, d_u)) \geq 0$ since $\delta_2 + \delta(d_1, d_u) \leq 3n$ and $v_2 = \delta_1 + \delta(d_1, d_u) - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq 1$. If d_u is located on the $d_{b+1} \to d_c$ path, then there is a $(2, 3n - 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 12n + 1 - 3(\delta_2 + \delta(d_1, d_u)) \geq 1$ since $\delta_2 + \delta(d_1, d_u) \leq 4n$ and $v_2 = \delta_1 + \delta(d_1, d_u) - n - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq 2n$ with $n \geq 1$. If d_u is located on the $d_{c+1} \to d_a$ path, then there is a $(1, 3n - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 15n + 2 - 3(\delta_2 + \delta(d_1, d_u)) \geq 2$ since $\delta_2 + \delta(d_1, d_u) \leq 4n + 1$ for $n \geq 1$ and $v_2 = \delta_1 + \delta(d_1, d_u) - 2n - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq 2n + 1$. If d_u is located on the $d_{a+1} \to d_{3n+1}$ path, then there is a $(0, 3n + 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 18n + 3 - 3(\delta_2 + \delta(d_1, d_u)) \geq 3$ since $\delta_2 + \delta(d_1, d_u) \leq 5n + 1$ for $n \geq 1$ and $v_2 = \delta_1 + \delta(d_1, d_u) - 3n - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq 3n + 1$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (5) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n+1$, there is $d_u \overset{(r,k)}{\to} d_1$ walk with $r = 9n + 3\delta_1 - 3\delta_2$ and $k = 9n^2 + 3n(\delta_1 - \delta_2) - 9n - 2\delta_1 + 3\delta_2$. Consequently, $\text{inexp}(d_1, D^{(2)}) \leq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1$. So, $\text{inexp}(d_1, D^{(2)}) = 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1$. By Lemma III.1, we can conclude that $\text{inexp}(d_x, D^{(2)}) \leq 9n^2 + 3n(\delta_1 - \delta_2) + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 2. (for $\delta_3 - \delta_1 \leq n, \quad n + 1 < \delta_2 - \delta_1 \leq 2n$)
First, it will be shown that $\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_1 + \delta(d_1, d_x)$. We examine the paths P_{d_c,d_x} and P_{d_{a+1},d_x} and define $g_1 = b(L_2)r(P_{d_c,d_x}) - r(L_2)b(P_{d_c,d_x})$ and $g_2 = r(L_1)b(P_{d_{a+1},d_x}) - b(L_1)r(P_{d_{a+1},d_x})$. Four subcases must be examined.

The node d_x is located on the path $d_1 \to d_b$. Utilizing path P_{d_c,d_x}, that is, the $(2, \delta_1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 6n - 4 - 3\delta_1 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(0, \delta_1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 4 \\ 6n^2 - 10n + \delta_1 + 4 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\mathrm{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_1 + \delta\left(d_1, d_x\right) \qquad (6)$$

for every node d_x located on the path $d_1 \to d_b$.

The node d_x is located on the path $d_{b+1} \to d_c$. Utilizing path P_{d_c,d_x}, that is, the $(3, \delta_1 - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 3 - 3\delta_1 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(1, \delta_1 - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 - n + \delta(d_1, d_x)$. By Lemma III.2, we have

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 3 \\ 6n^2 - 10n + \delta_1 + 3 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\mathrm{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_1 + \delta(d_1, d_x) \qquad (7)$$

for every node d_x located on the $d_{b+1} \to d_c$ path.

The node d_x is located on $d_{c+1} \to d_a$ path. Utilizing path P_{d_c,d_x}, that is, the $(1, \delta_1 - 3n + \delta(d_1, d_x))$-path, we obtain $g_1 = 12n - 2 - 3\delta_1 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(2, \delta_1 - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 - 2n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 2 \\ 6n^2 - 10n + \delta_1 + 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\mathrm{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_1 + \delta(d_1, d_x) \qquad (8)$$

for every node d_x located on the $d_{c+1} \to d_a$ path.

The node d_x is located on the path $d_{a+1} \to d_{3n+1}$. Utilizing P_{d_c,d_x}, that is, the $(2, \delta_1 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 15n - 1 - 3\delta_1 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, $(0, \delta_1 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_1 - 3n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 4 \\ 6n^2 - 13n + \delta_1 + 3 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 6n - 4$ and $m_2 = 6n^2 - 13n + \delta_1 + 3 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{a+1} to d_x. Note that the path is P_{d_{a+1},d_x}, that is, the $(0, \delta_1 - 3n - 1 + \delta(d_1, d_x))$-path. Furthermore, the completion of system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{a+1},d_x}) \\ q(P_{d_{a+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 6n - 4$ and $v_2 = 0$. The path P_{d_{a+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{a+1} to d_x. Hence, $\mathrm{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{a+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \mathrm{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= 6n^2 - 4n + \delta_1 + \delta(d_1, d_x) \qquad (9) \end{aligned}$$

for every node d_x located on the path $d_{a+1} \to d_{3n+1}$.

From (6), (7), (8) and (9), it can be concluded that $\mathrm{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\mathrm{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\mathrm{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + \delta_1$ and then use Lemma III.1 to guarantee that $\mathrm{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (6), we obtained $\mathrm{inexp}(d_1, D^{(2)}) \geq 6n^2 - 4n + \delta_1$. Furthermore, it is enough to show that $\mathrm{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + \delta_1$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{z} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} = \begin{bmatrix} 6n - 4 \\ 6n^2 - 10n + \delta_1 + 4 \end{bmatrix} \qquad (10)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (10) is $v_1 = 6n - 4 - 3\delta_1 - (3n - 2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_1 - (1 - n)p(P_{d_u,d_1}) - q(P_{d_u,d_1})$.

If d_u is located on the $d_1 \to d_b$ path, then there is a $(3, 3n - 2 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 6n - 4 - 3(\delta_1 + \delta(d_1, d_u) \geq 2$ since $\delta_1 + \delta(d_1, d_u) \leq 2n - 2$ and $v_2 = \delta_1 + \delta(d_1, d_u) - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq 1$. If d_u is located on the $d_{b+1} \to d_c$ path, then there is a $(2, 3n - 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 9n - 3 - 3(\delta_1 + \delta(d_1, d_u)) \geq 0$ since $\delta_1 + \delta(d_1, d_u) \leq 3n - 1$ and $v_2 = \delta_1 + \delta(d_1, d_u) - n - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq n + 1$. If d_u is located on the $d_{c+1} \to d_a$ path, then there is a $(1, 3n - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 12n - 2 - 3(\delta_1 + \delta(d_1, d_u)) \geq 1$ since $\delta_1 + \delta(d_1, d_u) \leq 3n$ for $n \geq 1$ and $v_2 = \delta_1 + \delta(d_1, d_u) - 2n - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq 3n$ for $n \geq 1$. If d_u is located on the $d_{a+1} \to d_{3n+1}$ path, then there is a $(0, 3n + 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain$v_1 = 15n - 1 - 3(\delta_1 + \delta(d_1, d_u)) \geq 5$ since $\delta_1 + \delta(d_1, d_u) \leq 4n - 1$ for $n \geq 1$ and $v_2 = \delta_1 + \delta(d_1, d_u) - 3n - 1 \geq 0$ since $\delta_1 + \delta(d_1, d_u) \geq 3n + 1$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (10) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n + 1$, there is a $d_u \overset{(r,k)}{\to} d_1$ walk with $r = 6n - 4$ and $k = 6n^2 - 10n + \delta_1 + 4$. Consequently, $\mathrm{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + \delta_1$. So, $\mathrm{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + \delta_1$. By Lemma III.1, we can conclude that $\mathrm{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_1 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 3. (for $\delta_3 - \delta_1 \leq n, \quad \delta_2 - \delta_1 = 2n + 1$)
First, it will be shown that $\mathrm{inexp}(d_x, D^{(2)}) \geq (3n + 1)\,\delta_2 - 6n + \delta\left(d_1, d_x\right)$. We examine the paths P_{d_c,d_x} and P_{d_{b+1},d_x} and define $g_1 = q(L_2)p(P_{d_c,d_x}) - p(L_2)q(P_{d_c,d_x})$ and $g_2 = p(L_1)q(P_{d_{b+1},d_x}) - q(L_1)p(P_{d_{b+1},d_x})$. It is necessary to examine three subcases.

The node d_x is located on the path $d_1 \to d_b$. Utilizing pathP_{d_c,d_x}, that is, $(2, \delta(d_1, d_x))$-path, we obtain $g_1 = 6n - 4 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{b+1},d_x}, that is, the $(2, \delta_2 - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_2 - 2n + \delta(d_1, d_x)$. By Lemma III.2, we have

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 3\delta_2 - 4 \\ -6n + 4 + 3n\delta_2 - 2\delta_2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq (3n+1)\delta_2 - 6n + \delta(d_1, d_x) \qquad (11)$$

for every node d_x located on the path $d_1 \to d_b$.

The node d_x is located on the path $d_{b+1} \to d_c$. Utilizing path P_{d_c,d_x}, that is, the $(3, -1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{b+1},d_x}, that is, $(0, \delta_2 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_2 - 3n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 3\delta_2 - 6 \\ -9n + 3n\delta_2 - 2\delta_2 + 5 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 3\delta_2 - 6$ and $m_2 = -9n + 3n\delta_2 - 2\delta_2 + 5 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{b+1} to d_x. Note that the path is P_{d_{b+1},d_x}, that is, the $(0, \delta_2 - 3n - 1 + \delta(d_1, d_x))$-path. Furthermore, the completion of the system $M\mathbf{v} + \begin{bmatrix} p(P_{v_{y+1},d_x}) \\ q(P_{v_{y+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 3\delta_2 - 6$ and $v_2 = 0$. The path P_{d_{b+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{b+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{b+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= (3n+1)\delta_2 - 6n + \delta(d_1, d_x) \quad (12) \end{aligned}$$

for every node d_x located on the path $d_{b+1} \to d_c$.

The node d_x is located on the path $d_{c+1} \to d_{a=3n+1}$. Utilizing path P_{d_c,d_x}, that is, the $(1, -3n + \delta(d_1, d_x))$-path, we obtain $g_1 = 12n - 2 - 3\delta(d_1, d_x)$. Utilizing P_{d_{b+1},d_x}, that is, the $(1, \delta_2 - 3n - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_2 - 4n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 3\delta_2 - 5 \\ -9n + 3n\delta_2 - 2\delta_2 + 4 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 3\delta_2 - 5$ and $m_2 = -9n + 3n\delta_2 - 2\delta_2 + 4 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{b+1} to d_x. Note that the path is P_{d_{b+1},d_x}, that is, the $(1, \delta_2 - 3n - 2 + \delta(d_1, d_x))$-path. Furthermore, the completion of system $M\mathbf{v} + \begin{bmatrix} p(P_{v_{y+1},d_x}) \\ q(P_{v_{y+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 3\delta_2 - 6$ and $v_2 = 0$. The path P_{d_{b+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{b+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{b+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= (3n+1)\delta_2 - 6n + \delta(d_1, d_x) \quad (13) \end{aligned}$$

for every node d_x located on the path $d_{c+1} \to d_{a=3n+1}$.

From (11), (12) and (13), it can be concluded that $\text{inexp}(d_x, D^{(2)}) \geq (3n+1)\delta_2 - 6n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\text{inexp}(d_x, D^{(2)}) \leq (3n+1)\delta_2 - 6n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\text{inexp}(d_1, D^{(2)}) = (3n+1)\delta_2 - 6n$ and then use Lemma III.1 to guarantee that $\text{inexp}(d_x, D^{(2)}) \leq (3n+1)\delta_2 - 6n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (11), we obtain $\text{inexp}(d_1, D^{(2)}) \geq (3n+1)\delta_2 - 6n$. Furthermore, it is enough to show that $\text{inexp}(d_1, D^{(2)}) \leq (3n+1)\delta_2 - 6n$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} = \begin{bmatrix} 3\delta_2 - 4 \\ 3n\delta_2 - 2\delta_2 - 6n + 4 \end{bmatrix} \quad (14)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (14) is $v_1 = 6n - 4 - (3n-2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_2 - 2n - (1 - n)q(P_{d_u,d_1}) - p(P_{d_u,d_1})$.

If d_u is located on the path $d_1 \to d_b$, then there is a $(3, 3n-2-\delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 6n - 4 - 3(\delta(d_1, d_u)) \geq 2$ since $\delta(d_1, d_u) \leq n - 1$ for $n \geq 1$ and $v_2 = \delta_2 + \delta(d_1, d_u) - 2n - 1 \geq 0$ since $\delta_2 + \delta(d_1, d_u) \geq 2n + 1$. If d_u is located on the path $d_{b+1} \to d_c$, then there is a $(2, 3n - 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 9n - 3 - 3\delta(d_1, d_u) \geq 0$ since $\delta(d_1, d_u) \leq 3n - 1$ and $v_2 = \delta_2 + \delta(d_1, d_u) - 3n - 1 \geq 0$ since $\delta_2 + \delta(d_1, d_u) \geq 3n+1$. If d_u is located on the path $d_{c+1} \to d_{a=3n+1}$, then there is a $(1, 3n-\delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 12n - 2 - 3\delta(d_1, d_u) \geq 1$ since $\delta(d_1, d_u) \leq 3n$ for $n \geq 1$ and $v_2 = \delta_2 + \delta(d_1, d_u) - 4n - 1 \geq 1$ since $\delta_2 + \delta(d_1, d_u) \geq 5n + 1$ with $n \geq 1$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (14) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n + 1$, there is a $d_u \overset{(r,k)}{\to} d_1$ walk with $r = 3\delta_2 - 4$ and $k = 3n\delta_2 - 2\delta_2 - 6n + 4$. Consequently, $\text{inexp}(d_1, D^{(2)}) \leq (3n+1)\delta_2 - 6n$. So, $\text{inexp}(d_1, D^{(2)}) = (3n+1)\delta_2 - 6n$. By Lemma III.1, we can conclude that $\text{inexp}(d_x, D^{(2)}) \leq (3n+1)\delta_2 - 6n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 4. (for $n < \delta_3 - \delta_1 < 2n$)
First, we will show that $\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$. We examine the paths P_{d_b,d_x} and P_{d_{c+1},d_x} and define $g_1 = q(L_2)p(P_{d_b,d_x}) - p(L_2)q(P_{d_b,d_x})$ and $g_2 = p(L_1)q(P_{d_{c+1},d_x}) - q(L_1)p(P_{d_{c+1},d_x})$. Four subcases must be considered.

The node d_x is located on path $d_1 \to d_b$. Utilizing path P_{d_b,d_x}, that is, the $(3, \delta_3 - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 3 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(1, \delta_3 - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 3 \\ 6n^2 - 10n + \delta_3 + 3 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \qquad (15)$$

for every node d_x located on the path $d_1 \to d_b$.

The node d_x is located on path $d_{b+1} \to d_c$. Utilizing path P_{d_b,d_x}, that is, the $(1, \delta_3 - 3n + \delta(d_1, d_x))$-path, we obtain $g_1 = 12n - 2 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(2, \delta_3 - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 2n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 2 \\ 6n^2 - 10n + \delta_3 + 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \quad (16)$$

for every node d_x located on the path $d_{b+1} \to d_c$.

The node d_x is located on path $d_{c+1} \to d_a$. Utilizing path P_{d_b,d_x}, that is, $(2, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 15n - 1 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 3n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 4 \\ 6n^2 - 13n + \delta_3 + 3 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 6n - 4$ and $m_2 = 6n^2 - 13n + \delta_3 + 3 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{c+1} to d_x. Note that the path is P_{d_{c+1},d_x}, that is, the $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path. Furthermore, the completion of system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{c+1},d_x}) \\ q(P_{d_{c+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 6n - 4$ and $v_2 = 0$. The path P_{d_{c+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{c+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{c+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \quad (17) \end{aligned}$$

for every node d_x located on the path $d_{c+1} \to d_a$.

The node d_x is located on path $d_{a+1} \to d_{3n+1}$. Utilizing path P_{d_b,d_x}, that is, the $(3, \delta_3 - 3n - 2 + \delta(d_1, d_x))$-path, we obtain $g_1 = 18n - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(1, \delta_3 - 3n - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 4n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 3 \\ 6n^2 - 13n + \delta_3 + 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 6n - 3$ and $m_2 = 6n^2 - 13n + \delta_3 + 2 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{c+1} to d_x. Note that the path is P_{d_{c+1},d_x}, that is, the $(1, \delta_3 - 3n - 2 + \delta(d_1, d_x))$-path. Furthermore, the completion of system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{c+1},d_x}) \\ q(P_{d_{c+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 6n - 4$ and $v_2 = 0$. The path P_{d_{c+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{c+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{c+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \quad (18) \end{aligned}$$

for every node d_x located on the path $d_{a+1} \to d_{3n+1}$.

From (15), (16), (17) and (18), it can be concluded that $\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\text{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + \delta_3$ and then use Lemma III.1 to guarantee that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (15), we obtain $\text{inexp}(d_1, D^{(2)}) \geq 6n^2 - 4n + \delta_3$. Furthermore, it is enough to show that $\text{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + \delta_3$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} = \begin{bmatrix} 6n - 3 \\ 6n^2 - 10n + \delta_3 + 3 \end{bmatrix} \quad (19)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (19) is $v_1 = 9n - 3 - 3\delta_3 - (3n - 2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_3 - n - (1 - n)p(P_{d_u,d_1}) - q(P_{d_u,d_1})$.

If d_u is located on the path $d_1 \to d_b$, then there is a $(3, 3n - 2 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 9n - 3 - 3(\delta_3 + \delta(d_1, d_u)) \geq 0$ since $\delta_3 + \delta(d_1, d_u) \leq 3n - 1$ and $v_2 = \delta_3 + \delta(d_1, d_u) - n - 1 \geq 0$ since $\delta_3 + \delta(d_1, d_u) \geq 2n$ for $n \geq 1$. If d_u is located on the path $d_{b+1} \to d_c$, then there is a $(2, 3n - 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 12n - 2 - 3(\delta_3 + \delta(d_1, d_u)) \geq 1$ since $\delta_3 + \delta(d_1, d_u) \leq 3n$ for $n \geq 1$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 2n - 1 \geq 0$ since $\delta_3 + \delta(d_1, d_u) \geq 3n$ for $n \geq 1$. If d_u is located on the path $d_{c+1} \to d_a$, then there is a $(1, 3n - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 15n - 1 - 3(\delta_3 + \delta(d_1, d_u)) \geq 2$ since $\delta_3 + \delta(d_1, d_u) \leq 5n - 1$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 3n - 1 \geq 0$ since $\delta_3 + \delta(d_1, d_u) \geq 3n + 1$. If d_u is located on the path $d_{a+1} \to d_{3n+1}$, then there is a $(0, 3n + 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 18n - 3(\delta_3 + \delta(d_1, d_u)) \geq 3$ since $\delta_3 + \delta(d_1, d_u) \leq 5n$ for $n \geq 1$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 4n - 1 \geq 1$ since $\delta_3 + \delta(d_1, d_u) \geq 4n + 2$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (19) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n + 1$, there is a $d_u \overset{(r,k)}{\to} d_1$ walk with $r = 6n - 3$ and $k = 6n^2 - 10n + \delta_3 + 3$. Consequently, $\text{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + \delta_3$. So, $\text{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + \delta_3$. By Lemma III.1, we can conclude that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 5. (for $\delta_3 - \delta_1 = 2n$)
First, we will show that $\text{inexp}(d_x, D^{(2)}) \geq (3n + 1)\delta_3 - 3n + \delta(d_1, d_x)$. We examine the paths P_{d_a,d_x} and P_{d_{c+1},d_x} and define $g_1 = q(L_2)p(P_{d_a,d_x}) - p(L_2)q(P_{d_a,d_x})$ and $g_2 =$

$p(L_1)q(P_{d_{c+1},d_x}) - q(L_1)p(P_{d_{c+1},d_x})$. Three subcases must be examined.

The node d_x is located on path $d_1 \rightarrow d_b$. Utilizing path P_{d_a,d_x}, that is, the $(1, \delta(d_1, d_x))$-path, we obtain $g_1 = 3n - 2 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, $(1, \delta_3 - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 3\delta_3 - 2 \\ -3n + 3n\delta_3 - 2\delta_3 + 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq (3n + 1)\delta_3 - 3n + \delta(d_1, d_x) \quad (20)$$

for every node d_x located on the path $d_1 \rightarrow d_b$.

The node d_x is located on path $d_{b+1} \rightarrow d_c$. Utilizing path P_{d_a,d_x}, that is, the $(2, -1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 6n - 1 - 3\delta(d_1, d_x)$. Utilizing P_{d_{c+1},d_x}, that is, the $(2, \delta_3 - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 2n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 3\delta_3 - 1 \\ -3n + 3n\delta_3 - 2\delta_3 + 1 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq (3n + 1)\delta_3 - 3n + \delta(d_1, d_x) \quad (21)$$

for every node d_x located on the path $d_{b+1} \rightarrow d_c$.

The node d_x is located on path $d_{c+1} \rightarrow d_{a=3n+1}$ path. Utilizing P_{d_a,d_x}, that is, the $(3, -2 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 3n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 3\delta_3 - 3 \\ -6n + 3n\delta_3 - 2\delta_3 + 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 3\delta_3 - 3$ and $m_2 = -6n + 3n\delta_3 - 2\delta_3 + 2 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{c+1} to d_x. Note that the path is P_{d_{c+1},d_x}, that is, the $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path. Furthermore, the completion of system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{c+1},d_x}) \\ q(P_{d_{c+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 3\delta_3 - 3$ and $v_2 = 0$. The path P_{d_{c+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{c+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{c+1} \rightarrow d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= (3n + 1)\delta_3 - 3n + \delta(d_1, d_x) \quad (22) \end{aligned}$$

for every node d_x located on the path $d_{c+1} \rightarrow d_{a=3n+1}$.

From (20), (21) and (22), we can conclude that $\text{inexp}(d_x, D^{(2)}) \geq (3n + 1)\delta_3 - 3n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\text{inexp}(d_x, D^{(2)}) \leq (3n + 1)\delta_3 - 3n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\text{inexp}(d_1, D^{(2)}) = (3n + 1)\delta_3 - 3n$ and then use Lemma III.1 to guarantee that $\text{inexp}(d_x, D^{(2)}) \leq (3n + 1)\delta_3 - 3n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (20), we obtain $\text{inexp}(d_1, D^{(2)}) \geq (3n+1)\delta_3 - 3n$. Furthermore, it is enough to show that $\text{inexp}(d_1, D^{(2)}) \leq (3n + 1)\delta_3 - 3n$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} = \begin{bmatrix} 3\delta_3 - 2 \\ -3n + 3n\delta_3 - 2\delta_3 + 2 \end{bmatrix} \quad (23)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (23) is $v_1 = 3n - 2 - (3n-2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_3 - n - (1 - n)p(P_{d_u,d_1}) - q(P_{d_u,d_1})$.

If d_u is located on the path $d_1 \rightarrow d_b$, then there is a $(3, 3n-2-\delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 3n-2-3\delta(d_1, d_u) \geq 1$ since $\delta(d_1, d_u) \leq n-1$ and $v_2 = \delta_3 + \delta(d_1, d_u) - n - 1 \geq 0$ since $\delta_3 + \delta(d_1, d_u) \geq 2n$ with $n \geq 1$. If d_u is located on the path $d_{b+1} \rightarrow d_c$, then there is a $(2, 3n - 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 6n - 1 - 3\delta(d_1, d_u) \geq 2$ since $\delta(d_1, d_u) \leq n$ for $n \geq 1$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 2n - 1 \geq 0$ since $\delta_3 + \delta(d_1, d_u) \geq 3n$ for $n \geq 1$. If d_u is located on the path $d_{c+1} \rightarrow d_{a=3n+1}$, then there is a $(1, 3n-\delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 9n - 3\delta(d_1, d_u) \geq 0$ since $\delta(d_1, d_u) \leq 3n$ and $v_2 = \delta_3+\delta(d_1, d_u)-3n-1 \geq 0$ since $\delta_3+\delta(d_1, d_u) \geq 3n+1$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (23) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n+1$, there is a $d_u \overset{(r,k)}{\rightarrow} d_1$ walk with $r = 3\delta_3 - 2$ and $k = -3n+3n\delta_3 - 2\delta_3+2$. Consequently, $\text{inexp}(d_1, D^{(2)}) \leq (3n+1)\delta_3 - 3n$. So, $\text{inexp}(d_1, D^{(2)}) = (3n+1)\delta_3 - 3n$. By Lemma III.1, we can conclude that $\text{inexp}(d_x, D^{(2)}) \leq (3n+1)\delta_3 - 3n + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

■

Next, we will examine the incoming local exponent for the digraph with three red arcs. The three red arcs in $D^{(2)}$ are the first arc $d_a \rightarrow d_{a+1}$ where $1 \leq a \leq n - 1$. The second and third arcs are $d_b \rightarrow d_{b+1}$ and arcs $d_c \rightarrow d_{c+1}$, respectively, where $n \leq b < c \leq 3n + 1$. δ_{11} represents the distance from node d_{a+1} to node d_1 in L_1. δ_{12} represents the distance from node d_{a+1} to node d_1 in L_2. δ_2 represents the distance from node d_{b+1} to node d_1. δ_3 represents the distance from node d_{c+1} to node d_1.

Theorem IV.2. *Let a primitive bicolour digraph $D^{(2)}$ have two cycles of length n and $3n + 1$. If $D^{(2)}$ has three red arcs with two consecutive red arcs at L_2, then for every $x = 1, 2, ..., 3n + 1$,*
$\text{inexp}(d_x, D^{(2)}) =$

$$\begin{cases} 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x), \\ \quad \text{for } \delta_{12} - \delta_2 \leq n \\ 6n^2 - 4n + \delta_3 + \delta(d_1, d_x), \\ \quad \text{for } n < \delta_{12} - \delta_2 \leq 2n \\ 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x), \\ \quad \text{for } \delta_{12} - \delta_2 > 2n \end{cases}$$

Proof: Suppose that for every $x = 1, 2, ..., 3n + 1$, $\text{inexp}(d_x, D^{(2)})$ is obtained using the (r_x, k_x)-walk. The proof is divided into three cases as follows.

Case 1. (for $\delta_{12} - \delta_2 \le n$)

First, it will be shown that $\text{inexp}(d_x, D^{(2)}) \ge 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x)$. We examine the paths P_{d_a,d_x} and P_{d_{c+1},d_x} and define $g_1 = q(L_2)p(P_{d_a,d_x}) - p(L_2)q(P_{d_a,d_x})$ and $g_2 = p(L_1)q(P_{d_{c+1},d_x}) - q(L_1)p(P_{d_{c+1},d_x})$. Four subcases must be examined.

The node d_x is located on the path $d_1 \to d_a$. Utilizing path P_{d_a,d_x}, that is, the $(3, \delta_{12} - 2 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 6 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(0, \delta_3 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \ge M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n - 3\delta_{12} + 3\delta_3 \\ 9n^2 + 3n(\delta_3 - \delta_{12}) - 9n - 2\delta_3 + 3\delta_{12} + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \ge 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x) \quad (24)$$

for every node d_x located on the path $d_1 \to d_a$.

The node d_x is located on the path $d_{a+1} \to d_b$. Utilizing path P_{d_a,d_x}, that is, the $(1, \delta_{12} - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 12n - 3\delta_{12} + 1 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(1, \delta_3 - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \ge M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_3 - 3\delta_{12} + 1 \\ 9n^2 + 3n(\delta_3 - \delta_{12}) - 9n - 2\delta_3 + 3\delta_{12} - 1 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \ge 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x) \quad (25)$$

for every node d_x located on the path $d_{a+1} \to d_b$.

The node d_x is located on the path $d_{b+1} \to d_c$. Utilizing P_{d_a,d_x}, that is, the $(2, \delta_{12} - 3n - 2 + \delta(d_1, d_x))$-path, we obtain $g_1 = 15n - 3\delta_{12} + 2 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(2, \delta_3 - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 2n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \ge M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_3 - 3\delta_{12} + 2 \\ 9n^2 + 3n(\delta_3 - \delta_{12}) - 9n - 2\delta_3 + 3\delta_{12} - 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \ge 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x) \quad (26)$$

for every node d_x located on the path $d_{b+1} \to d_c$.

The node d_x is located on the path $d_{c+1} \to d_{3n+1}$. Utilizing path P_{d_a,d_x}, that is, the $(3, \delta_{12} - 3n - 3 + \delta(d_1, d_x))$-path, we obtain $g_1 = 18n - 3\delta_{12} + 3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 3n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \ge M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3\delta_3 - 3\delta_{12} \\ 9n^2 + 3n(\delta_3 - \delta_{12}) - 12n - 2\delta_3 + 3\delta_{12} - 1 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 9n + 3\delta_3 - 3\delta_{12}$ and $m_2 = 9n^2 + 3n(\delta_3 - \delta_{12}) - 12n - 2\delta_3 + 3\delta_{12} - 1 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{c+1} to d_x. Note that the path is P_{d_{c+1},d_x}, that is, the $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path. Furthermore, the completion of the system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{c+1},d_x}) \\ q(P_{d_{c+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 9n + 3\delta_3 - 3\delta_{12}$ and $v_2 = 0$. The path P_{d_{c+1},d_x} located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{c+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{c+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\ge m_1 + m_2 + p(L_2) + q(L_2) \\ &= 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x) \quad (27) \end{aligned}$$

for every node d_x located on the path $d_{c+1} \to d_{3n+1}$.

From (24), (25), (26) and (27), it can be concluded that $\text{inexp}(d_x, D^{(2)}) \ge 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\text{inexp}(d_x, D^{(2)}) \le 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_1 + \delta(d_3, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\text{inexp}(d_1, D^{(2)}) = 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3$ and then use Lemma III.1 to guarantee that $\text{inexp}(d_x, D^{(2)}) \le 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (24), we obtain $\text{inexp}(d_1, D^{(2)}) \ge 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3$. Furthermore, it is enough to show that $\text{inexp}(d_1, D^{(2)}) \le 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} = \begin{bmatrix} 9n + 3\delta_3 - 3\delta_{12} \\ 9n^2 + 3n(\delta_3 - \delta_{12}) - 9n - 2\delta_3 + 3\delta_{12} \end{bmatrix} \quad (28)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (28) is $v_1 = 9n - 3\delta_{12} - (3n - 2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_3 - (1 - n)p(P_{d_u,d_1}) - q(P_{d_u,d_1})$.

If d_u is located on the $d_1 \to d_a$ path, then there is a $(3, 3n - 2 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 9n - 3(\delta_{12} + \delta(d_1, d_u)) \ge 0$ since $\delta_{12} + \delta(d_1, d_u) \le 3n$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 1 \ge 2$ since $\delta_3 + \delta(d_1, d_u) \ge n + 1$ with $n \ge 2$. If d_u is located on the $d_{c+1} \to d_{3n+1}$ path, then there is a $(0, 3n + 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 18n + 3 - 3(\delta_{12} + \delta(d_1, d_u)) \ge 3$ since $\delta_{12} + \delta(d_1, d_u) \le 6n$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 3n - 1 \ge 0$ since $\delta_3 + \delta(d_1, d_u) \ge 3n + 1$ with $n \ge 2$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (28) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n + 1$, there is $d_u \overset{(r,k)}{\to} d_1$ walk with $r = 9n + 3\delta_3 - 3\delta_{12}$ and $k = 9n^2 + 3n(\delta_3 - \delta_{12}) - 9n - 2\delta_3 + 3\delta_{12}$. Consequently, $\text{inexp}(d_1, D^{(2)}) \le 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3$. So, $\text{inexp}(d_1, D^{(2)}) = 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3$. By Lemma III.1, we can conclude that $\text{inexp}(d_x, D^{(2)}) \le 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x)$ for every

$x = 1, 2, ..., 3n + 1$.

Case 2. (for $n < \delta_{12} - \delta_2 \leq 2n$)
First, it will be shown that $\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$. We examine the paths P_{d_b,d_x} and P_{d_{c+1},d_x} and define $g_1 = b(L_2)r(P_{d_b,d_x}) - r(L_2)b(P_{d_b,d_x})$ and $g_2 = r(L_1)b(P_{d_{c+1},d_x}) - b(L_1)r(P_{d_{c+1},d_x})$. Four subcases must be examined.

The node d_x is located on the path $d_1 \to d_a$. Utilizing path P_{d_b,d_x}, that is, the $(2, \delta_3 + \delta(d_1, d_x))$-path, we obtain $g_1 = 6n - 4 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(0, \delta_3 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 4 \\ 6n^2 - 10n + \delta_3 + 4 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \quad (29)$$

for every node d_x located on the path $d_1 \to d_a$.

The node d_x is located on the path $d_{a+1} \to d_b$. Utilizing path P_{d_b,d_x}, that is, the $(3, \delta_3 - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 3 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(1, \delta_3 - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - n + \delta(d_1, d_x)$. By Lemma III.2, we have

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 3 \\ 6n^2 - 10n + \delta_3 + 3 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \quad (30)$$

for every node d_x located on the $d_{a+1} \to d_b$ path.

The node d_x is located on $d_{b+1} \to d_c$ path. Utilizing path P_{d_b,d_x}, that is, the $(1, \delta_3 - 3n + \delta(d_1, d_x))$-path, we obtain $g_1 = 12n - 2 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, the $(2, \delta_3 - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 2n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 2 \\ 6n^2 - 10n + \delta_3 + 2 + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \quad (31)$$

for every node d_x located on the $d_{b+1} \to d_c$ path.

The node d_x is located on the path $d_{c+1} \to d_{3n+1}$. Utilizing P_{d_b,d_x}, that is, the $(2, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 15n - 1 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{c+1},d_x}, that is, $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_3 - 3n - 1 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 6n - 4 \\ 6n^2 - 13n + \delta_3 + 3 + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 6n - 4$ and $m_2 = 6n^2 - 13n + \delta_3 + 3 + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{c+1} to d_x. Note that the path is P_{d_{c+1},d_x}, that is, the $(0, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path. Furthermore, the completion of system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{c+1},d_x}) \\ q(P_{d_{c+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 6n - 4$ and $v_2 = 0$. The path P_{d_{c+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{c+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{c+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= 6n^2 - 4n + \delta_3 + \delta(d_1, d_x) \quad (32) \end{aligned}$$

for every node d_x located on the path $d_{c+1} \to d_{3n+1}$.

From (29), (30), (31) and (32), it can be concluded that $\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\text{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + \delta_3$ and then use Lemma III.1 to guarantee that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (29), we obtained $\text{inexp}(d_1, D^{(2)}) \geq 6n^2 - 4n + \delta_3$. Furthermore, it is enough to show that $\text{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + \delta_3$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{z} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} = \begin{bmatrix} 6n - 4 \\ 6n^2 - 10n + \delta_3 + 4 \end{bmatrix} \quad (33)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (10) is $v_1 = 6n - 4 - 3\delta_3 - (3n - 2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_3 - (1 - n)p(P_{d_u,d_1}) - q(P_{d_u,d_1})$.

If d_u is located on the $d_1 \to d_a$ path, then there is a $(3, 3n - 2 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 6n - 4 - 3(\delta_3 + \delta(d_1, d_u)) \geq 2$ since $\delta_3 + \delta(d_1, d_u) \leq 2n - 2$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 1 \geq 0$ since $\delta_3 + \delta(d_1, d_u) \geq 1$. If d_u is located on the $d_{c+1} \to d_{3n+1}$ path, then there is a $(0, 3n + 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 15n - 1 - 3(\delta_3 + \delta(d_1, d_u)) \geq 5$ since $\delta_3 + \delta(d_1, d_u) \leq 5n - 2$ and $v_2 = \delta_3 + \delta(d_1, d_u) - 3n - 1 \geq 0$ since $\delta_3 + \delta(d_1, d_u) \geq 3n + 1$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (33) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n + 1$, there is a $d_u \overset{(r,k)}{\to} d_1$ walk with $r = 6n - 4$ and $k = 6n^2 - 10n + \delta_3 + 4$. Consequently, $\text{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + \delta_3$. So, $\text{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + \delta_3$. By Lemma III.1, we can conclude that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 3. (for $\delta_{12} - \delta_2 > 2n$)
First, it will be shown that $\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x)$. We examine the paths P_{d_b,d_x} and P_{d_{a+1},d_x} and

define $g_1 = b(L_2)r(P_{d_b,d_x}) - r(L_2)b(P_{d_b,d_x})$ and $g_2 = r(L_1)b(P_{d_{a+1},d_x}) - b(L_1)r(P_{d_{a+1},d_x})$. Four subcases must be examined.

The node d_x is located on the path $d_1 \to d_a$. Utilizing path P_{d_b,d_x}, that is, the $(2, \delta_3 + \delta(d_1, d_x))$-path, we obtain $g_1 = 6n - 4 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(0, \delta_{11} + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_{11} + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 4 + 3\delta_{11} - 3\delta_3 \\ 6n^2 + 3n(\delta_{11} - \delta_3) - 10n + 4 + 3\delta_3 - 2\delta_{11} + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x) \tag{34}$$

for every node d_x located on the path $d_1 \to d_a$.

The node d_x is located on the path $d_{a+1} \to d_b$. Utilizing path P_{d_b,d_x}, that is, the $(3, \delta_3 - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 9n - 3 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(0, \delta_{11} - n + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_{11} - n + \delta(d_1, d_x)$. By Lemma III.2, we have

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 3 + 3\delta_{11} - 3\delta_3 \\ 6n^2 + 3n(\delta_{11} - \delta_3) - 10n + 3 + 3\delta_3 - 2\delta_{11} + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x) \tag{35}$$

for every node d_x located on the $d_{a+1} \to d_b$ path.

The node d_x is located on $d_{b+1} \to d_c$ path. Utilizing path P_{d_b,d_x}, that is, the $(1, \delta_3 - 3n + \delta(d_1, d_x))$-path, we obtain $g_1 = 12n - 2 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, the $(0, \delta_{11} - 2n - 2 + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_{11} - 2n - 2 + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 8 + 3\delta_{11} - 3\delta_3 \\ 6n^2 + 3n(\delta_{11} - \delta_3) - 16n + 6 + 3\delta_3 - 2\delta_{11} + \delta(d_1, d_x) \end{bmatrix}.$$

Let $m_1 = 6n - 8 + 3\delta_{11} - 3\delta_3$ and $m_2 = 6n^2 + 3n(\delta_{11} - \delta_3) - 16n + 6 + 3\delta_3 - 2\delta_{11} + \delta(d_1, d_x)$. We examine the (m_1, m_2)-walk from d_{a+1} to d_x. Note that the path is P_{d_{a+1},d_x}, that is, the $(0, \delta_{11} - 2n - 2 + \delta(d_1, d_x))$-path. Furthermore, the completion of system $M\mathbf{v} + \begin{bmatrix} p(P_{d_{a+1},d_x}) \\ q(P_{d_{a+1},d_x}) \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is $v_1 = 6n - 8 + 3\delta_{11} - 3\delta_3$ and $v_2 = 0$. The path P_{d_{a+1},d_x} is located entirely in cycle L_2, and there is no (m_1, m_2)-walk from d_{a+1} to d_x. Hence, $\text{inexp}(d_x, D^{(2)}) > m_1 + m_2$. Note that the shortest walk from $d_{a+1} \to d_x$ that contains a minimum of m_1 red arcs and at least m_2 black arcs is the $(m_1 + p(L_2), m_2 + q(L_2))$-walk. Since $p(L_2) + q(L_2) = 3n + 1$, we have

$$\begin{aligned} \text{inexp}(d_x, D^{(2)}) &\geq m_1 + m_2 + p(L_2) + q(L_2) \\ &= 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x) \end{aligned} \tag{36}$$

for every node d_x located on the path $d_{b+1} \to d_c$.

The node d_x is located on the path $d_{c+1} \to d_{3n+1}$. Utilizing P_{d_b,d_x}, that is, the $(2, \delta_3 - 3n - 1 + \delta(d_1, d_x))$-path, we obtain $g_1 = 15n - 1 - 3\delta_3 - 3\delta(d_1, d_x)$. Utilizing path P_{d_{a+1},d_x}, that is, $(0, \delta_{11} - 3n + \delta(d_1, d_x))$-path, we obtain $g_2 = \delta_{11} - 3n + \delta(d_1, d_x)$. By Lemma III.2, we have that

$$\begin{bmatrix} r_x \\ k_x \end{bmatrix} \geq M \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 1 + 3\delta_{11} - 3\delta_3 \\ 6n^2 + 3n(\delta_{11} - \delta_3) - 10n + 1 + 3\delta_3 - 2\delta_{11} + \delta(d_1, d_x) \end{bmatrix}.$$

Hence,

$$\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x) \tag{37}$$

for every node d_x located on the $d_{c+1} \to d_{3n+1}$ path.

From (34), (35), (36) and (37), it can be concluded that $\text{inexp}(d_x, D^{(2)}) \geq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

Next, we will prove that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we will show that $\text{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11}$ and then use Lemma III.1 to guarantee that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (34), we obtained $\text{inexp}(d_1, D^{(2)}) \geq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11}$. Furthermore, it is enough to show that $\text{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11}$ for every d_u, $u = 1, 2, ..., 3n + 1$ when the system

$$M\mathbf{z} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 4 + 3\delta_{11} - 3\delta_3 \\ 6n^2 + 3n(\delta_{11} - \delta_3) - 10n + 4 - 2\delta_{11} + 3\delta_3 \end{bmatrix} \tag{38}$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. The completion of system (38) is $v_1 = 6n - 4 - 3\delta_3 - (3n - 2)p(P_{d_u,d_1}) + 3q(P_{d_u,d_1})$ and $v_2 = \delta_{11} - (1 - n)p(P_{d_u,d_1}) - q(P_{d_u,d_1})$.

If d_u is located on the $d_1 \to d_a$ path, then there is a $(1, n - 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 6n - 5 - 3(\delta_3 + \delta(d_1, d_u)) \geq 7$ since $\delta_3 + \delta(d_1, d_u) \leq n - 2$ with $n \geq 2$ and $v_2 = \delta_{11} + \delta(d_1, d_u) \geq 1$ since $\delta_{11} + \delta(d_1, d_u) \geq 1$. If d_u is located on the $d_{c+1} \to d_{3n+1}$ path, then there is a $(0, 3n + 1 - \delta(d_1, d_u))$-path from d_u to d_1. Utilizing this path, we obtain $v_1 = 15n - 1 - 3(\delta_3 + \delta(d_1, d_u)) \geq 5$ since $\delta_3 + \delta(d_1, d_u) \leq 3n - 2$ with $n \geq 2$ and $v_2 = \delta_{11} + \delta(d_1, d_u) - 3n - 1 \geq 1$ since $\delta_{11} + \delta(d_1, d_u) \geq 3n + 2$ with $n \geq 2$.

Therefore, for every $u = 1, 2, ..., 3n + 1$, the system of equations (38) has a nonnegative integer completion. Proposition III.1 guarantees that for every $u = 1, 2, ..., 3n + 1$, there is a $d_u \overset{(r,k)}{\to} d_1$ walk with $r = 6n - 4 - 3\delta_3 + 3\delta_{11}$ and $k = 6n^2 + 3n(\delta_{11} - \delta_3) - 10n + 4 + 3\delta_3 - 2\delta_{11}$. Consequently, $\text{inexp}(d_1, D^{(2)}) \leq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11}$. So, $\text{inexp}(d_1, D^{(2)}) = 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11}$. By Lemma III.1, we can conclude that $\text{inexp}(d_x, D^{(2)}) \leq 6n^2 - 4n + 3n(\delta_{11} - \delta_3) + \delta_{11} + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n + 1$. ■

Theorem IV.3. *Let a primitive bicolour digraph $D^{(2)}$ have two cycles of length n and $3n+1$. If $D^{(2)}$ has three red arcs with two arcs alternating with a difference of one at L_2, then for every $x = 1, 2, ..., 3n+1$,*
$\mathrm{inexp}(d_x, D^{(2)}) =$

$$\begin{cases} 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x), \\ \quad \text{for } \delta_{12} - \delta_2 \leq n \\ 6n^2 - n + 3n(\delta_3 - \delta_2) + \delta_3 + \delta(d_1, d_x), \\ \quad \text{for } n < \delta_{12} - \delta_2 < 2n \\ 6n^2 - n + 3n(\delta_{11} - \delta_2) + \delta_{11} + \delta(d_1, d_x), \\ \quad \text{for } \delta_{12} - \delta_2 \geq 2n \end{cases}$$

Proof: Proof of Theorem IV.3 given in the form of a proof sketch and uses the same arguments as Theorem IV.1 and Theorem IV.2.

Case 1. (for $\delta_{12} - \delta_2 \leq n$)

We will show that $\mathrm{inexp}(d_x, D^{(2)}) = 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n+1$. The lower bound obtained by constructing $g_1 = q(L_2)p(P_{d_a,d_x}) - p(L_2)q(P_{d_a,d_x})$ and $g_2 = p(L_1)q(P_{d_{c+1},d_x}) - q(L_1)p(P_{d_{c+1},d_x})$. The upper bound found by showing that for every d_u, $u = 1, 2, ..., 3n+1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} =$$

$$\begin{bmatrix} 9n - 3\delta_{12} + 3\delta_3 \\ 9n^2 - 9n + 3n(\delta_3 - \delta_{12}) - 2\delta_3 + 3\delta_{12} \end{bmatrix} \quad (39)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. This implies that $\mathrm{inexp}(d_1, D^{(2)}) = 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3$. By Lemma III.1, we can conclude that $\mathrm{inexp}(d_x, D^{(2)}) \leq 9n^2 + 3n(\delta_3 - \delta_{12}) + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n+1$ for every $x = 1, 2, ..., 3n+1$.

Case 2. (for $n < \delta_{12} - \delta_2 < 2n$)

We will show that $\mathrm{inexp}(d_x, D^{(2)}) = 6n^2 - n + 3n(\delta_3 - \delta_2) + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n+1$. The lower bound obtained by constructing $g_1 = q(L_2)p(P_{d_b,d_x}) - p(L_2)q(P_{d_b,d_x})$ and $g_2 = p(L_1)q(P_{d_{c+1},d_x}) - q(L_1)p(P_{d_{c+1},d_x})$. The upper bound found by showing that for every d_u, $u = 1, 2, ..., 3n+1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 1 - 3\delta_2 + 3\delta_3 \\ 6n^2 - 7n + 3n(\delta_3 - \delta_2) + 3\delta_2 - 2\delta_3 + 1 \end{bmatrix} \quad (40)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. This implies that $\mathrm{inexp}(d_1, D^{(2)}) = 6n^2 - n + 3n(\delta_3 - \delta_2) + \delta_3$. By Lemma III.1, we can conclude that $\mathrm{inexp}(d_x, D^{(2)}) \leq 6n^2 - n + 3n(\delta_3 - \delta_2) + \delta_3 + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n+1$ for every $x = 1, 2, ..., 3n+1$.

Case 3. (for $\delta_{12} - \delta_2 \geq 2n$)

We will show that $\mathrm{inexp}(d_x, D^{(2)}) = 6n^2 - n + 3n(\delta_{11} - \delta_2) + \delta_{11} + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n+1$. The lower bound obtained by constructing $g_1 = q(L_2)p(P_{d_b,d_x}) - p(L_2)q(P_{d_b,d_x})$ and $g_2 = p(L_1)q(P_{d_{a+1},d_x}) - q(L_1)p(P_{d_{a+1},d_x})$. The upper bound found by showing that for every d_u, $u = 1, 2, ..., 3n+1$ when the system

$$M\mathbf{v} + \begin{bmatrix} p(P_{d_u,d_1}) \\ q(P_{d_u,d_1}) \end{bmatrix} =$$

$$\begin{bmatrix} 6n - 1 - 3\delta_2 + 3\delta_{11} \\ 6n^2 - 7n + 1 + 3n(\delta_{11} - \delta_2) + 3\delta_2 - 2\delta_{11} \end{bmatrix} \quad (41)$$

has a nonnegative integer completion for some path P_{d_u,d_1} from d_u to d_1. This implies that $\mathrm{inexp}(d_1, D^{(2)}) = 6n^2 - n + 3n(\delta_{11} - \delta_2) + \delta_{11}$. By Lemma III.1, we can conclude that $\mathrm{inexp}(d_x, D^{(2)}) \leq 6n^2 - n + 3n(\delta_{11} - \delta_2) + \delta_{11} + \delta(d_1, d_x)$ for every $x = 1, 2, ..., 3n+1$ for every $x = 1, 2, ..., 3n+1$. ■

V. CONCLUSION

In general, the incoming local exponent of a two-cycle bicolour Hamiltonian digraph with a difference of $2n+1$ and four red arcs is $\mathrm{inexp}(d_x, D^{(2)}) = \mathrm{inexp}(d_1, D^{(2)}) + \delta(d_1, d_x)$. Research in this class can be continued for difference $kn+1$ with $k > 2$.

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