On the Generalized Migrativity of Nullnorms Over Overlap and Grouping Functions

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***Abstract*—In this paper, we study the generalized migrativity of nullnorms over any two fixed overlap functions or grouping functions. At first, we propose the concept of (α, O_1, O_2)-migrative nullnorms over any two fixed overlap functions O_1 and O_2. And then, we show equivalent characterizations of the (α, O_1, O_2)-migrativity equation when the nullnorm F becomes a t-norm T or a t-conorm S. Moreover, we give the solutions of the (α, O_1, O_2)-migrativity equations for nullnorms when $k \in]0,1[$. At last, we discuss the (α, G_1, G_2)-migrative nullnorms and obtain characterizations of the related (α, G_1, G_2)-migrativity equation.**

***Index Terms*—Migrativity; Nullnorms; Overlap functions; Grouping functions**

I. Introduction

A. A brief review of overlap and grouping functions

As two special cases of binary aggregation functions, overlap and grouping functions are introduced, respectively, by Bustince et al. [10], [11] in 2009 and 2012. Those two concepts, originate from some problems in image processing [9], classification [2], [24], and also in decision making [58]. In the past few years, overlap and grouping functions have had a rapid development both in theory and applications.

In theory, there exist many discussions involving various aspects of overlap and grouping functions, for example, the work related to some important properties [3], [14], [52], [53], [64], [66], [69], [71], [71], [74], the investigations of the corresponding implication [15], [16], [18], [19], [63], the study of the additive, multiplicative generators and interval functions [4], [20], [17], [54]. The research related to the concept extension [47], [31].

In applications, overlap and grouping functions play an important role in many aspects of real problems, for instance, in image processing [8], [33], classification [25], [26], [36], [37], [38], [39], [40], [41], [51], decision making [24] and fuzzy community detection problems [32].

B. A short introduction of migrativity

The α-migrativity of an aggregation function was introduced by Durante and Sarkoci [21] in order to express the fact that the effect of reducing one of its argument by a factor α is the same regardless of which argument is reduced. For any $\alpha \in [0,1]$ and a mapping $H : [0,1]^2 \to [0,1]$ is said to be α-migrative if, for any $x, y \in [0,1]$, it holds that

$$H(\alpha x, y) = H(x, \alpha y) \quad (1)$$

The interest of this property comes from its applications [6]. From then on, there are many researches which have pointed out that the study of α-migrativity for aggregation function or some special binary functions has an important meaning and value both in theory and applications. From an application point of view, Bustince et al. [7] said "migrativity is particularly interesting whenever one has to aggregate partial information coming from sources with meaningful difference (information about recent events or places close to one anther should in general not be treated similarly as information about events at distant moments in time or remote locations)" and they listed related applications in decision making [48], [49], [57] and image processing [8]. In theory, Mas et al. [43] said "One of the main topics in the study of aggregation function from the theoretical point of views is directed towards the characterization of those that verify certain properties that may be useful in each context. The study of these properties for certain aggregation functions usually involves the resolution of functional equations. One of these properties is α-migrativity." In addition, they pointed out that α-migrativity is interesting from the theoretical point of view because of its relationship in the construction of new t-norms through convex combinations of two given ones. In recent years, α-migrativity has been investigated for t-norms in [28], [29], [30], [50], for t-subnorms in [65], for semicopulas, quasi-copulas and copulas in [5], [22], [23], [27], [46], for uninorms in [62], for nullnorms in [76] and for aggregation functions in general in [7], [6], [35], [56]. In addition, the generalization of the migrative for t-norms has been studied in [29], [30]. In [44], Mas et al. gave a similar definition for t-conorms. Moreover, this study has been extended to uninorms with the same neutral in [45]. In [43], Mas et al. investigated the α-migrativity of uninorms and nullnorms over t-norms and t-conorms. As an addendum to [43], Zong et al. [75] studied the α-migrativity of t-norms and t-conorms over uninorms and nullnorms. Su et al. [60] studied migrativity property for uninorms and semi t-operators, In addition, Su et al. [59], [61], [62] considered the migrativity equation for two uninorms with different neutral elements. In 2018, Qiao and Hu [55] studied the α-migrativity of uninorms and nullnorms over overlap functions and grouping functions. As an addendum to [55], Zhu and Hu [68] considered the α-migrativity of overlap functions and grouping functions over uninorms and nullnorms. Recently, Zhou and Yan [67] investigated migrativity properties of overlap functions over uninorms. In addition, they showed equivalent characterizations of the migrativity equation when the uninorms belong to one of the usual classes (e.g., $\mathcal{U}_{\min}$,

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$\mathcal{U}_{\max}, \mathcal{U}_{id}, \mathcal{U}_{rep}$ or uninorms continuous on $]0,1[^2$). In 2020, Zhu et al. [70] obtained some new results on the migrativity properties of uninorms and nullnorms over t-norms and t-conorms. In the same year, by means of the ordinal sum of overlap and grouping functions, Zhu et al. [73] gave the necessary and sufficient conditions for the solutions of the (α, O)-migrativity equation when the uninorm U becomes a t-norm or a conjunctive uninorm locally internal on the boundary and the (α, G)–migrativity equation when the uninorm U becomes a t-conorm or a disjunctive uninorm locally internal on the boundary, respectively.

C. The motivation of our research

On the one hand, we have stated in Subsection 1.1 that overlap and grouping functions, as two new classes of special binary aggregation functions, have had a fast development both in applications and theory. In particular, there are many researches proposing some related properties for overlap and grouping functions, such as homogeneity, Archimedean and so on from the theoretical point of view and those two binary aggregation functions can be regarded, respectively. On the other hand, it has been stated in Subsection 1.2 that the α-migrativity among some peculiar classes of binary aggregation functions, as a meaningful and hot research area in the topic of the α-migrativity of two operations, have been continuously studied in many recent literature.

However, there are no corresponding researches for the generalized α-migrativity of nullnorms over overlap and grouping functions, although Qiao and Hu [52] have discussed the generalized α-migrativity for overlap functions. Therefore, as a supplement of this topic from the theoretical point of view, in this paper, we consider the generalized α-migrativity of nullnorms over overlap and grouping functions. More precisely, for all $x, y \in [0,1]$ and a given α in $[0,1]$, we propose the following two migrativity equations

$$F(O_1(\alpha, x), y) = F(x, O_2(\alpha, y)) \tag{2}$$

and

$$F(G_1(\alpha, x), y) = F(x, G_2(\alpha, y)) \tag{3}$$

where O_1, O_2 are two overlap functions, G_1, G_2 are two grouping functions and F is a given nullnorm.

The rest of this paper is organized as follows. In Section II, we review some notions and results about t-norms and t-conorms, overlap functions and grouping functions and nullnorms, which shall be used throughout this paper. In Section III, we discuss the generalized α-migrativity property of a nullnorm F over any two fixed overlap functions O_1 and O_2. In Section IV, we study the generalized α-migrativity property of a nullnorm F over any two fixed grouping functions G_1 and G_2. Section IV is conclusion and further work.

II. Preliminaries

In this section, we recall some basic notions and definitions which shall be needed in the sequel. Firstly, we give the definitions of t-norms and t-conorms as follows.

Definition 2.1: [34] A bivariate function $T : [0,1]^2 \longrightarrow [0,1]$ is said to be a t-norm if, for all $x, y, z \in [0,1]$, it satisfies the following conditions:

(T1) Commutativity: $T(x,y) = T(y,x)$;
(T2) Associativity: $T(T(x,y),z) = T(x,T(y,z))$;
(T3) Monotonicity: $T(x,y) \leq T(x,z)$ whenever $y \leq z$;
(T4) Boundary condition: $T(x,1) = x$.
Moreover, a t-norm T is said to be
(T5) continuous if it is continuous in both arguments at the same time.

Definition 2.2: [34] A bivariate function $S : [0,1]^2 \longrightarrow [0,1]$ is said to be a t-conorm if, for all $x, y, z \in [0,1]$, it satisfies the following conditions:

(S1) Commutativity: $S(x,y) = S(y,x)$;
(S2) Associativity: $S(S(x,y),z) = S(x,S(y,z))$;
(S3) Monotonicity: $S(x,y) \leq S(x,z)$ whenever $y \leq z$;
(S4) Boundary condition: $S(x,0) = x$.
Moreover, a t-conorm S is said to be
(S5) continuous if it is continuous in both arguments at the same time.

In the following, we introduce the concepts of overlap and grouping functions [10], [11].

Definition 2.3: [10] A bivariate function $O : [0,1]^2 \longrightarrow [0,1]$ is said to be an overlap function if it satisfies the following conditions:

(O1) O is commutative;
(O2) $O(x,y) = 0$ iff $xy = 0$;
(O3) $O(x,y) = 1$ iff $xy = 1$;
(O4) O is incerasing;
(O5) O is continuous.

In the following, we list some common overlap functions, most of which are from [3], [14], [15].

Example 2.4: (1) Any continuous t-norm with non-trivial zero divisors is an overlap function.

(2) The function $O_M : [0,1]^2 \longrightarrow [0,1]$ given by

$$O_M(x,y) = \min(x,y)$$

is an overlap function.

(3) For any $p > 0$, the function $O_p : [0,1]^2 \longrightarrow [0,1]$ given by

$$O_p(x,y) = x^p y^p$$

is an overlap function. Moreover, for any $p \neq 1$, $O_p(x,y)$ is neither associative nor has 1 as neutral element. Thus, it is not a t-norm.

(4) The function $O_{Mid} : [0,1]^2 \longrightarrow [0,1]$ given by

$$O_{Mid}(x,y) = xy\frac{x+y}{2}$$

is an overlap function.

(5) The function $O_{DB} : [0,1]^2 \longrightarrow [0,1]$ given by

$$O_{DB}(x,y) = \begin{cases} \frac{2xy}{x+y}, & \text{if } x+y \neq 0, \\ 0, & \text{if } x+y = 0 \end{cases}$$

is an overlap function.

Definition 2.5: [11] A bivariate function $G : [0,1]^2 \longrightarrow [0,1]$ is said to be a grouping function if it satisfies the following conditions:

(G1) G is commutative;
(G2) $G(x,y) = 0$ iff $x = y = 0$;
(G3) $G(x,y) = 1$ iff $x = 1$ or $y = 1$;
(G4) G is incerasing;

(G5) G is continuous.

In the following, we list some common grouping functions, most of which are from [3], [15].

Example 2.6: (1) Any continuous t-conorm with non-trivial one divisors is a grouping function.

(2) The function $G_M : [0,1]^2 \longrightarrow [0,1]$ given by

$$G_M(x,y) = \max(x,y)$$

is a grouping function.

(3) For any $p > 0$, the function $G_p : [0,1]^2 \longrightarrow [0,1]$ given by

$$G_p(x,y) = 1-(1-x)^p(1-y)^p$$

is a grouping function.

(4) For any $p > 0$, the function $G_{M_p} : [0,1]^2 \longrightarrow [0,1]$ given by

$$G_{M_p}(x,y) = \max(x^p, y^p)$$

is a grouping function.

In the following, we recall the definition of a nullnorm which is also called a t-operator in some literature, e.g. [42].

Definition 2.7: [12] [42] A bivariate function $F : [0,1]^2 \longrightarrow [0,1]$ is said to be a nullnorm if, for any $x,y,z \in [0,1]$, it satisfies the following conditions:

(F1) $F(x,y) = F(y,x)$;

(F2) $F(F(x,y),z) = F(x,F(y,z))$;

(F3) F is non-decreasing in each place;

(F4) There has an absorbing element $k \in [0,1]$, i.e., $F(k,x) = k$ and the following statements hold.

(1) $F(0,x) = x$ for all $x \leq k$.

(2) $F(1,x) = x$ for all $x \geq k$.

It follows from Definition 2.7 that if $k = 0$, then a nullnorm F becomes a t-norm T and if $k = 1$, then a nullnorm F becomes a t-conorm S.

In general, k is always given by $F(1,0)$.

In what follows, we present the structure of nullnorms, which shall be used in Sections III and IV.

Lemma 2.8: [42] Let $F : [0,1]^2 \longrightarrow [0,1]$ be a nullnorm with $k = F(1,0) \notin \{0,1\}$. Then

$$F(x,y) = \begin{cases} kS(\frac{x}{k}, \frac{y}{k}), & \text{if } x,y \in [0,k], \\ k+(1-k)T(\frac{x-k}{1-k}, \frac{y-k}{1-k}), & \text{if } x,y \in [k,1], \\ k, & \text{otherwise.} \end{cases}$$

where S is a t-conorm and T is a t-norm. And in such case, F is denoted by $F = \langle S,k,T \rangle$.

III. Generalized migrativity property of nullnorms over overlap functions

In this section, at first, we propose the notion of the generalized α-migrativity property of a nullnorm F over any two fixed overlap functions O_1 and O_2. In particular, we consider the situations when the nullnorm F becomes a t-norm T or a t-conorm S. In addition, we discuss the solutions of the (α, O_1, O_2)-migrativity equations for nullnorms when $k \in]0,1[$.

Definition 3.1: Consider $\alpha \in [0,1]$. Let O_1 and O_2 be two fixed overlap functions. A nullnorm $F : [0,1]^2 \longrightarrow [0,1]$ is said to be α-migrative with respect to O_1 and O_2 ((α, O_1, O_2)-migrative, for short) if

$$F(O_1(\alpha,x),y) = F(x,O_2(\alpha,y)) \tag{4}$$

for any $x,y \in [0,1]$.

Remark 3.2: (1) In Definition 3.1, if $O_1 = O_2$, then Eq. (4) is a special case of the α-migrativity equation $A(B(\alpha,x),y) = A(x,B(\alpha,y))$ for aggregation functions A and B investigated by Bustince et al. [7], and the functional equation $A(B(\alpha,x),y) = C(x,B(\alpha,y))$ discussed by Cutello and Montero in [13] to character the recursiveness of connective rules.

(2) From the viewpoint of functional equation, Eq. (4) is a particular case of the general associativity equation $A(B(\alpha,x),y) = C(x,D(\alpha,y))$ studied by Aczél et al. in [1].

Now, we discuss the properties of Eq. (4). At first, it follows from Definition 3.1 that we have the following trivial conclusion.

Proposition 3.3: For two given overlap functions O_1 and O_2, a nullnorm F is (α, O_1, O_2)-migrative iff F is (α, O_2, O_1)-migrative.

Proof. It is straightforward. □

Notice that when $k = 0$ the nullnorm F becomes a t-norm and when $k = 1$ the nullnorm F becomes a t-conorm. For the beginning, we consider the case for $k = 0$. And, in such case F becomes a t-norm T and Eq. (4) becomes the following form

$$T(O_1(\alpha,x),y) = T(x,O_2(\alpha,y)) \tag{5}$$

for all $x,y \in [0,1]$.

At first, for $\alpha = 1$, we obtain the following conclusion.

Proposition 3.4: Let O_1 and O_2 be two fixed overlap functions and T be a t-norm. Then the following statements are equivalent:

(1) T is $(1, O_1, O_2)$-migrative;

(2) $O_1(1,x) = x$ and $O_2(1,x) = x$ for any $x \in [0,1]$.

Proof. (1) $\Rightarrow$ (2). Since T is $(1, O_1, O_2)$-migrative, one has that

$$\begin{aligned} O_1(1,x) &= T(O_1(1,x),1) \\ &= T(x,O_2(1,1)) \\ &= T(x,1) \\ &= x. \end{aligned}$$

In a similar way, we obtain $O_2(1,x) = x$.

(2) $\Rightarrow$ (1). It is straightforward. □

For $\alpha = 0$, we also obtain the following conclusion.

Proposition 3.5: Let O_1 and O_2 be two fixed overlap functions and T be a t-norm. Then T is $(0, O_1, O_2)$-migrative.

Next, we consider $k = 1$. And, in such case F becomes a t-conorm S and Eq. (4) becomes the following form

$$S(O_1(\alpha,x),y) = S(x,O_2(\alpha,y)) \tag{6}$$

for all $x,y \in [0,1]$.

For $\alpha = 1$, we obtain the following conclusion.

Proposition 3.6: Let O_1 and O_2 be two fixed overlap functions and S be a t-conorm. Then the following statements are equivalent:

(1) S is $(1, O_1, O_2)$-migrative;
(2) $O_1(1, x) = x$ and $O_2(1, x) = x$ for any $x \in [0, 1]$.

Proof. (1) $\Rightarrow$ (2). Since S is $(1, O_1, O_2)$-migrative, one has that

$$\begin{aligned} O_1(1, x) &= S(O_1(1, x), 0) \\ &= S(x, O_2(1, 0)) \\ &= S(x, 0) \\ &= x. \end{aligned}$$

In a similar way, we obtain $O_2(1, x) = x$.

(2) $\Rightarrow$ (1). It is straightforward. $\square$

For $\alpha \in [0, 1[$, S is not (α, O_1, O_2)-migrative.

Proposition 3.7: Consider $\alpha \in [0, 1[$. Let O_1 and O_2 be two fixed overlap functions and S be a t-conorm. Then S is not (α, O_1, O_2)-migrative.

Proof. Suppose that S is (α, O_1, O_2)-migrative. Take $x = 0$ and $y = 1$ in Eq. (6). Then, one has that

$$\begin{aligned} 1 &= S(0, 1) \\ &= S(O_1(\alpha, 0), 1) \\ &= S(0, O_2(\alpha, 1)) \\ &= O_2(\alpha, 1). \end{aligned}$$

Thus, $O_2(\alpha, 1) = 1$. On the other hand, it follows from item (O3) of Definition 2.3 that $\alpha = 1$, which is contradiction. Therefore, S is not (α, O_1, O_2)-migrative. $\square$

As a consequence of Propositions 3.4, 3.5, 3.6 and 3.7, in the following, we only consider $\alpha \in]0, 1[$.

Proposition 3.8: Consider $\alpha \in]0, 1[$. Let O_1 and O_2 be two fixed overlap functions and T be a t-norm. If T is (α, O_1, O_2)-migrative, then $O_2(\alpha, x) = T(O_1(\alpha, 1), x)$ for all $x \in [0, 1]$.

Proof. For any $x \in [0, 1]$, we have

$$\begin{aligned} O_2(\alpha, x) &= T(1, O_2(\alpha, x)) \\ &= T(O_1(\alpha, 1), x). \end{aligned}$$

$\square$

In what follows, we study the generalized migrativity property for continuous t-norms over any two fixed overlap functions O_1 and O_2. For the convenience expression, we denote $\beta = O_1(\alpha, 1)$. Moreover, It follows follows from Proposition 3.8 that for any (α, O_1, O_2)-migrative t-norm T, it holds that $O_2(\alpha, x) = T(\beta, x) \leq \min(\beta, x)$ for any $x \in [0, 1]$.

Theorem 3.9: Consider $\alpha \in]0, 1[$. Let O_1 and O_2 be two fixed overlap functions and T be a continuous t-norm. Then the following statements hold.

(1) Let $O_2(\alpha, \beta) = \beta$. If T is (α, O_1, O_2)-migrative, then T is an ordinal sum of two continuous Archimedean t-norms T_1 and T_2, i.e., $T = (\langle 0, \beta, T_1 \rangle, \langle \beta, 1, T_2 \rangle)$ and $O_2(\alpha, x)$ has the following form

$$O_2(\alpha, x) = \begin{cases} x, & \text{if } x \in [0, \beta], \\ \beta, & \text{if } x \in [\beta, 1]. \end{cases}$$

(2) Let $O_2(\alpha, \beta) < \beta$. If T is (α, O_1, O_2)-migrative, then T is an ordinal sum of the form $T = (..., \langle \eta, \theta, T_0 \rangle, ...)$, where T_0 is a continuous Archimedean t-norm and $O_2(\alpha, x)$ has the following form

$$O_2(\alpha, x) = \begin{cases} x, & \text{if } x \in [0, \eta[, \\ \eta + (\theta - \eta) T_0(\frac{\beta - \eta}{\theta - \eta}, \frac{x - \eta}{\theta - \eta}) & \text{if } x \in [\eta, \theta], \\ \beta, & \text{if } x \in]\theta, 1]. \end{cases}$$

Proof. (1) It follows from Proposition 3.8 that $T(\beta, \beta) = O(\alpha, \beta) = \beta$, which implies that β is an idempotent element of T. Further, since T is continuous, there exist two continuous Archimedean t-norms T_1 and T_2 such that $T = (\langle 0, \beta, T_1 \rangle, \langle \beta, 1, T_2 \rangle)$.

In the following, we verify $O_2(\alpha, x)$ has the following form

$$O_2(\alpha, x) = \begin{cases} x, & \text{if } x \in [0, \beta], \\ \beta, & \text{if } x \in [\beta, 1]. \end{cases}$$

For any $x \in [0, \beta]$, it follows from Proposition 3.8 that

$$\begin{aligned} O_2(\alpha, x) &= T(\beta, x) \\ &= \beta T_1(1, \tfrac{x}{\beta}) \\ &= x. \end{aligned}$$

Moreover, for any $x \in [\beta, 1]$, we have

$$\begin{aligned} O_2(\alpha, x) &= T(\beta, x) \\ &= \beta + (1 - \beta) T_1(0, \tfrac{x - \beta}{1 - \beta}) \\ &= \beta. \end{aligned}$$

(2) It follows from the proof of (1) that $T(\beta, \beta) < \beta$. In addition, since T is continuous, there exists a continuous Archimedean t-norm T_0 such that $T = (..., \langle \eta, \theta, T_0 \rangle, ...)$ and $\beta \in]\eta, \theta[$. Now, we prove $O_2(\alpha, x)$ has the following form

$$O_2(\alpha, x) = \begin{cases} x, & \text{if } x \in [0, \eta[, \\ \eta + (\theta - \eta) T_0(\frac{\beta - \eta}{\theta - \eta}, \frac{x - \eta}{\theta - \eta}) & \text{if } x \in [\eta, \theta], \\ \beta, & \text{if } x \in]\theta, 1]. \end{cases}$$

Case (A) Let $x \in [0, \eta[$. On one hand, $O_2(\alpha, x) = T(\beta, x) \leq x$. Further, since T is continuous and $T(\eta, \eta) = \eta$, we have

$$\begin{aligned} O_2(\alpha, x) &= T(\beta, x) \\ &\geq T(\eta, x) \\ &= \min(\eta, x) \\ &= x. \end{aligned}$$

Therefore, $O_2(\alpha, x) = x$.

Case (B) Let $x \in [\eta, \theta[$. Then we have

$$\begin{aligned} O_2(\alpha, x) &= T(\beta, x) \\ &= \eta + (\theta - \eta) T_0(\tfrac{\beta - \eta}{\theta - \eta}, \tfrac{x - \eta}{\theta - \eta}) \end{aligned}$$

Case (C) Let $x \in]\theta, 1[$. On one hand, $O_2(\alpha, x) = T(\beta, x) \leq \beta$. On the one hand,

$$\begin{aligned} O_2(\alpha, x) &= T(\beta, x) \\ &\geq T(\beta, \theta) \\ &= \eta + (\theta - \eta) T_0(\tfrac{\beta - \eta}{\theta - \eta}, 1) \\ &= \beta. \end{aligned}$$

Therefore, $O_2(\alpha, x) = \beta$. $\square$

Now, in the sequel, we consider the generalized migrativity property of nullnorms for $k \in]0, 1[$. For $\alpha = 0$, we have the following conclusion.

Proposition 3.10: Let O_1 and O_2 be two fixed overlap functions and F be a nullnorm with absorbing element $k \in]0, 1[$. Then F is not (α, O_1, O_2)-migrative.

Proof. It is straightforward. $\square$

It follows from Proposition 3.10 that we only need to consider $\alpha \in]0,1]$. First, we discuss the (α, O_1, O_2)-migrativity property of nullnorms for $\alpha = 1$.

Proposition 3.11: Let O_1 and O_2 be two fixed overlap functions and F be a nullnorm with absorbing element $k \in]0,1[$. Then the following statements are equivalent:

(1) F is $(1, O_1, O_2)$-migrative.

(2) $O_2(1,x) = x$ and $O_1(1,x) = x$ for any $x \in [0,1]$.

Proof. (1) $\Rightarrow$ (2) First of all, we show that $O_2(1,k) = k$ for any $k \in]0,1[$. Assume that $O_2(1,k) \neq k$. Then we have the following two cases:

Case (I) $O_2(1,k) > k$.

Take $x = 1$ and $y = k$ in Eq. (4). Then we have

$$\begin{aligned} O_2(1,k) &= F(1, O_2(1,k)) \\ &= F(O_1(1,1), k) \\ &= F(1,k) \\ &= k, \end{aligned}$$

which is a contradiction.

Case (II) $O_2(1,k) < k$.

Take $x = 0$ and $y = k$ in Eq. (4). Then we have

$$\begin{aligned} k &= F(0,k) \\ &= F(O_1(1,0), k) \\ &= F(0, O_2(1,k)) \\ &= O_2(1,k), \end{aligned}$$

which is a contradiction.

Next, we show that for any $m < k$, $O_2(1,m) = m$.

Let $m < k$. Then $O_2(1,m) \leq O_2(1,k) = k$. Moreover,

$$\begin{aligned} O_2(1,m) &= F(0, O_2(1,m)) \\ &= F(O_1(1,0), m) \\ &= F(0,m) \\ &= m. \end{aligned}$$

Last, we show that for any $m > k$, $O_2(1,m) = m$.

Let $m > k$. Then $O_2(1,m) \geq O_2(1,k) = k$. Moreover,

$$\begin{aligned} m &= F(1,m) \\ &= F(O_1(1,1), m) \\ &= F(1, O_2(1,m)) \\ &= O_2(1,m). \end{aligned}$$

Thus, for any $x \in [0,1]$, $O_2(1,x) = x$. In a similar way, we conclude that $O_1(1,x) = x$ for any $x \in [0,1]$.

(2) $\Rightarrow$ (1) It is straightforward. □

It follows from Propositions 3.10 and 3.11. In the following, we only consider the (α, O_1, O_2)-migrativity property of nullnorms for $\alpha \in]0,1[$.

Theorem 3.12: Consider $\alpha \in]0,1[$. Let O_1 and O_2 be two fixed overlap functions, F be a nullnorm with absorbing element $k \in]0,1[$ and $\beta = k$. If F is (α, O_1, O_2)-migrative, then $O_2(\alpha, x)$ has the following form

$$O_2(\alpha,x) = \begin{cases} x, & \text{if } x \in [0,k], \\ k, & \text{if } x \in]k,1]. \end{cases}$$

Proof. Firstly, we prove that $O_2(\alpha,k) = k$ for any $k \in]0,1[$. Assume that $O_2(\alpha,k) \neq k$. Then we have the following two cases:

Case (I) $O_2(\alpha,k) > k$.

Take $x = \alpha$ and $y = k$ in Eq. (4). Then we have

$$\begin{aligned} O_2(\alpha,k) &= F(1, O_2(\alpha,k)) \\ &= F(O_1(\alpha,1), k) \\ &\leq F(O_1(1,1), k) \\ &= F(1,k)) \\ &= k, \end{aligned}$$

which is a contradiction.

Case (II) $O_2(\alpha,k) < k$.

Take $x = \alpha$ and $y = k$ in Eq. (4). Then we have

$$\begin{aligned} k &= F(0,k)) \\ &= F(O_1(\alpha,0), k) \\ &= F(0, O_2(\alpha,k)) \\ &= O_2(\alpha,k), \end{aligned}$$

which is a contradiction.

Next, we show that for any $n < k$, $O_2(\alpha,n) = n$.

Let $n < k$. Then $O_2(\alpha,n) \leq O_2(\alpha,k) = k$. Moreover,

$$\begin{aligned} O_2(\alpha,n) &= F(0, O_2(\alpha,n)) \\ &= F(O_1(\alpha,0), n) \\ &= F(0,n) \\ &= n. \end{aligned}$$

Thus, for any $x \leq k$, $O_2(\alpha,x) = x$.

Moreover, for any $x > k$, we have $O_2(\alpha,x) \geq O_2(\alpha,k) = k$. Hence,

$$\begin{aligned} O_2(\alpha,x) &= F(1, O_2(\alpha,x)) \\ &= F(O_1(\alpha,1), x) \\ &= F(k,x) \\ &= k. \end{aligned}$$

Therefore, we conclude that

$$O_2(\alpha,x) = \begin{cases} x, & \text{if } x \in [0,k], \\ k, & \text{if } x \in]k,1]. \end{cases}$$

□

Theorem 3.13: Consider $\alpha \in]0,1[$. Let O_1 and O_2 be two fixed overlap functions and $F = \langle S,k,T \rangle$ be a nullnorm with absorbing element $k = F(0,1) \in]0,1[$. Then the following statements hold.

(1) Let $\beta < k$. Then F is not (α, O_1, O_2)-migrative.

(2) Let $\beta > k$ and T is continuous. Then the following two items hold.

(a) Let $O_1(\alpha,\beta) = \beta$. If F is (α, O_1, O_2)-migrative, then T is an ordinal sum of two continuous Archimedean t-norms T_1 and T_2, i.e., $T = (\langle 0, \frac{\beta-k}{1-k}, T_1 \rangle, \langle \frac{\beta-k}{1-k}, 1, T_2 \rangle)$ and $O_2(\alpha,x)$ has the following form

$$O_2(\alpha,x) = \begin{cases} x, & \text{if } x \in [0,\beta], \\ \beta, & \text{if } x \in]\beta,1]. \end{cases}$$

(b) Let $O_1(\alpha,\beta) < k$. If F is (α, O_1, O_2)-migrative, then T is an ordinal sum of the form $T = (..., \langle \theta_1, \theta_2, T^\gamma \rangle, ...)$, where $\frac{\beta-k}{1-k} \in]\theta_1, \theta_2[$ and T^γ is a continuous Archimedean t-norm and if $x \in [0, k + (1-k)\theta_1[, O_2(\alpha,x) = x, x \in [k + (1-k)\theta_1, k + (1-k)\theta_2], O_2(\alpha,x) = k + (1-k)(\theta_1 + (\theta_2 - \theta_1)T^\gamma(\frac{\beta-k-(1-k)\theta_1}{(1-k)(\theta_2-\theta_1)}, \frac{x-k-(1-k)\theta_1}{(1-k)(\theta_2-\theta_1)})), x \in]k + (1-k)\theta_2, 1], O_2(\alpha,x) = \beta$.

Proof.

(1) Let F be (α, O_1, O_2)-migrative.

Take $x = 0$ and $y = 1$ in Eq. (4). Then we have

$$\begin{aligned} O_1(1,\alpha) &= F(O_1(1,\alpha),0) \\ &= F(1,O_2(\alpha,0)) \\ &= F(1,0) \\ &= k, \end{aligned}$$

which is a contradiction with $O_1(1,\alpha) < k$. Thus, F is not (α, O_1, O_2)-migrative.

(2) Let $x, y \in [0,1]$. Then it follows from the proof of Proposition 3.11 that for all $x \le k$, we have $O_2(\alpha, x) = x$.

Moreover, for any $x > k$, we have $O_2(\alpha,x) \ge O_2(\alpha,k) = k$. Hence,

$$\begin{aligned} O_2(\alpha,x) &= F(1,O_2(\alpha,x)) \\ &= F(O_1(\alpha,1),x) \\ &= F(\beta,x) \end{aligned}$$

Since $\beta > k$, it is easy to know that $O_2(\alpha,\beta) = F(\beta,\beta) \le \beta$.

Next, for all $x > k$, we show the conclusions of items (2a) and (2b).

(a) Let $O_1(\alpha,\beta) = \beta$. Then we have

$$\begin{aligned} \beta &= F(\beta,\beta) \\ &= k + (1-k)T(\tfrac{\beta-k}{1-k}, \tfrac{\beta-k}{1-k}), \end{aligned}$$

which implies that $\frac{\beta-k}{1-k}$ is an idempotent element of T. Further, since T is continuous, one has that there exist two continuous Archimedean t-norms T_1 and T_2 such that $T = (\langle 0, \frac{\beta-k}{1-k}, T_1\rangle, \langle \frac{\beta-k}{1-k}, 1, T_2\rangle)$.

In the following, we verify $O_2(\alpha,x)$ has the above form

Case (A). If $x \in]k, \beta]$, then we have

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &= k + (1-k)T(\tfrac{\beta-k}{1-k}, \tfrac{\beta-k}{1-k}) \\ &= k + (1-k)(\tfrac{\beta-k}{1-k}T_1(1, \tfrac{\beta-k}{1-k})) \\ &= x. \end{aligned}$$

Case (B). If $x \in]\beta, 1]$, then we have

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &= k + (1-k)T(\tfrac{\beta-k}{1-k}, \tfrac{\beta-k}{1-k}) \\ &= k + (1-k)(\tfrac{\beta-k}{1-k} + (1-\tfrac{\beta-k}{1-k})T_2(0, \tfrac{x-\beta}{1-\beta}) \\ &= \beta. \end{aligned}$$

(b) Let $O_1(\alpha,\beta) < \beta$. Then it follows from the proof of item (a) that

$$T(\frac{\beta-k}{1-k}, \frac{\beta-k}{1-k}) < \frac{\beta-k}{1-k}.$$

Since T is continuous, there exists a continuous Archimedean t-norm T^γ such that T is an ordinal sum of the form $T = (..., \langle \theta_1, \theta_2, T^\gamma\rangle, ...)$, where $\frac{\beta-k}{1-k} \in]\theta_1, \theta_2[$.

In the following, we verify $O_2(\alpha,x)$ has the above form

Case (A). Let $x \in]k, k+(1-k)\theta_1]$. On one hand,

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &\le F(1,x) \\ &= x. \end{aligned}$$

On the other hand, since $T(\theta_1,\theta_1) = \theta_1$ and T is continuous, we have

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &= k + (1-k)T(\tfrac{\beta-k}{1-k}, \tfrac{x-k}{1-k}) \\ &\ge k + (1-k)T(\theta_1, \tfrac{x-k}{1-k}) \\ &= k + (1-k)\min(\theta_1, \tfrac{x-k}{1-k}) \\ &= x. \end{aligned}$$

Therefore, $O_2(\alpha,x) = x$.

In the following, we verify $O_2(\alpha,x)$ has the above form

Case (A). Let $x \in]k, k+(1-k)\theta_1]$. On one hand,

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &\le F(1,x) \\ &= x. \end{aligned}$$

On the other hand, since $T(\theta_1,\theta_1) = \theta_1$ and T is continuous, we have

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &= k + (1-k)T(\tfrac{\beta-k}{1-k}, \tfrac{x-k}{1-k}) \\ &\ge k + (1-k)T(\theta_1, \tfrac{x-k}{1-k}) \\ &= k + (1-k)\min(\theta_1, \tfrac{x-k}{1-k}) \\ &= x. \end{aligned}$$

Therefore, $O_2(\alpha,x) = x$.

Case (B). If $x \in]k+(1-k)\theta_1, k+(1-k)\theta_2]$, then we have

$$\begin{aligned} &O_2(\alpha,x) \\ &= F(\beta,x) \\ &= k + (1-k)T(\tfrac{\beta-k}{1-k}, \tfrac{x-k}{1-k}) \\ &= k + (1-k)(\theta_1 + (\theta_2-\theta_1)T^\gamma(\tfrac{\frac{\beta-k}{1-k}-\theta_1}{\theta_2-\theta_1}, \tfrac{\frac{x-k}{1-k}-\theta_1}{\theta_2-\theta_1}) \\ &= k + (1-k)(\theta_1 + (\theta_2-\theta_1)T^\gamma(\tfrac{\beta-k-(1-k)\theta_1}{(1-k)(\theta_2-\theta_1)}, \tfrac{x-k-(1-k)\theta_1}{(1-k)(\theta_2-\theta_1)})) \end{aligned}$$

Case (C). Let $x \in]k+(1-k)\theta_2, 1]$. On one hand,

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &\le F(\beta,1) \\ &= \beta. \end{aligned}$$

On the other hand, since $T(\theta_2,\theta_2) = \theta_2$ and T is continuous, we have

$$\begin{aligned} O_2(\alpha,x) &= F(\beta,x) \\ &= k + (1-k)T(\tfrac{\beta-k}{1-k}, \tfrac{x-k}{1-k}) \\ &\ge k + (1-k)T(\tfrac{x-k}{1-k}, \theta_2) \\ &= k + (1-k)\min(\tfrac{\beta-k}{1-k}, \theta_2) \\ &= \beta. \end{aligned}$$

Therefore, $O_2(\alpha,x) = \beta$. □

IV. Generalized migrativity property of nullnorms over grouping functions

In this section, firstly, we introduce the definition of the generalized α-migrativity of a nullnorm F over two fixed grouping functions G_1 and G_2. And we study the α-migrativity property of nullnorms over two fixed grouping functions in a similar way.

Definition 4.1: Consider $\alpha \in [0,1]$. Let G_1 and G_2 be two fixed grouping functions. A nullnorm $F : [0,1]^2 \longrightarrow [0,1]$ is said to be α-migrative respect to G_1 and G_2 ((α, G_1, G_2)-migrative, for short) if

$$F(G_1(\alpha,x),y) = F(x, G_2(\alpha,y)) \tag{7}$$

for any $x, y \in [0, 1]$.

It follows from Definition 4.1 that we have the following trivial conclusion.

Proposition 4.2: For two given grouping grouping G_1 and G_2, a nullnorm F is (α, G_1, G_2)-migrative iff F is (α, G_2, G_1)-migrative.

Proof. It is straightforward. □

Notice that when $k = 0$ the nullnorm F becomes a t-norm and when $k = 1$ the nullnorm F becomes a t-conorm. For the beginning, we consider the case for $k = 1$. And, in such case F becomes a t-conorm S and Eq. (7) becomes the following form

$$S(G_1(\alpha, x), y) = S(x, G_2(\alpha, y)) \tag{8}$$

for all $x, y \in [0, 1]$.

At first, for $\alpha = 0$, we obtain the following conclusion.

Proposition 4.3: Let G_1 and G_2 be two fixed grouping functions and S be a t-conorm. Then the following statements are equivalent:

(1) S is $(0, O_1, O_2)$-migrative;

(2) $G_1(0, x) = x$ and $G_2(0, x) = x$ for any $x \in [0, 1]$.

Proof. $(1) \Rightarrow (2)$. Since S is $(0, G_1, G_2)$-migrative, one has that

$$\begin{aligned} G_1(0, x) &= S(G_1(0, x), 0) \\ &= S(x, G_2(0, 0)) \\ &= S(x, 0) \\ &= x. \end{aligned}$$

$(2) \Rightarrow (1)$. It is straightforward. □

For $\alpha = 1$, we also obtain the following conclusion.

Proposition 4.4: Let G_1 and G_2 be two fixed grouping functions and S be a t-conorm. Then S is $(1, O_1, O_2)$-migrative.

Next, we consider $k = 0$. And, in such case F becomes a t-norm T and Eq. (7) becomes the following form

$$T(G_1(\alpha, x), y) = T(x, G_2(\alpha, y)) \tag{9}$$

for all $x, y \in [0, 1]$.

For $\alpha = 0$, we obtain the following conclusion.

Proposition 4.5: Let G_1 and G_2 be two fixed grouping functions and T be a t-norm. Then the following statements are equivalent:

(1) T is $(0, G_1, G_2)$-migrative;

(2) $G_1(0, x) = x$ and $G_2(, x) = x$ for any $x \in [0, 1]$.

For $\alpha \in]0, 1]$, T is not (α, G_1, G_2)-migrative;

Proposition 4.6: Consider $\alpha \in [0, 1[$. Let G_1 and G_2 be two fixed grouping functions and T be a t-norm. Then T is not (α, G_1, G_2)-migrative.

Proof. Suppose that T is (α, T_1, T_2)-migrative. Take $x = 1$ and $y = 0$ in Eq. (9). Then, one has that

$$\begin{aligned} 0 &= T(1, 0) \\ &= T(G_1(\alpha, 1), 0) \\ &= T(1, G_2(\alpha, 0)) \\ &= G_2(\alpha, 0). \end{aligned}$$

Thus, $G_2(\alpha, 0) = 0$. On the other hand, it follows from item (G3) of Definition 2.5 that $\alpha = 0$, which is contradiction. Thus, T is not (α, G_1, G_2)-migrative. □

As a consequence of Propositions 4.3, 4.4, 4.5 and 4.6, in the following, we only consider $\alpha \in]0, 1[$.

Proposition 4.7: Consider $\alpha \in]0, 1[$. Let G_1 and G_2 be two fixed grouping functions and S be a t-conorm. If S is (α, G_1, G_2)-migrative, then $G_2(\alpha, x) = S(G_1(\alpha, 0), x)$ for all $x \in [0, 1]$.

Proof. For any $x \in [0, 1]$, we have

$$\begin{aligned} G_2(\alpha, x) &= S(0, O_2(\alpha, x)) \\ &= S(G_1(\alpha, 0), x). \end{aligned}$$

□

In what follows, we study the generalized migrativity property for continuous t-norms over any two fixed grouping functions. For the convenience expression, we denote $\gamma = G_1(0, \alpha)$. Moreover, it follows from Proposition 4.7 that, for any (α, G_1, G_2)-migrative t-conorm S, it holds that $G_2(\alpha, x) = S(\gamma, x) \geq \max(\gamma, x)$ for any $x \in [0, 1]$.

Theorem 4.8: Consider $\alpha \in]0, 1[$. Let G_1 and G_2 be two fixed grouping functions and S be a continuous t-conorm. Then the following statements hold.

(1) Let $G_2(\alpha, \gamma) = \gamma$. If S is (α, G_1, G_2)-migrative, then S is an ordinal sum of two continuous Archimedean t-conorms S_1 and S_2, i.e., $S = (\langle 0, \gamma, S_1 \rangle, \langle \gamma, 1, S_2 \rangle)$ and $G_2(\alpha, x)$ has the following form

$$O_2(\alpha, x) = \begin{cases} \gamma, & \text{if } x \in [0, \gamma], \\ x, & \text{if } x \in]\gamma, 1]. \end{cases}$$

(2) Let $G_2(\alpha, \gamma) > \gamma$. If S is (α, G_1, G_2)-migrative, then S is an ordinal sum of the form $S = (..., \langle \eta, \theta, S_0 \rangle, ...)$, where S_0 is a continuous Archimedean t-conorm and $G_2(\alpha, x)$ has the following form

$$G_2(\alpha, x) = \begin{cases} \gamma, & \text{if } x \in [0, \eta_0[, \\ \eta_0 + (\theta_0 - \eta_0) S_0(\frac{\gamma - \eta_0}{\theta_0 - \eta_0}, \frac{x - \eta_0}{\theta_0 - \eta_0}) & \text{if } x \in [\eta_0, \theta_0], \\ x, & \text{if } x \in]\theta_0, 1]. \end{cases}$$

Proof.

(1) It follows from Proposition 4.7 that $S(\gamma, \gamma) = O(\alpha, \gamma) = \gamma$, which implies that γ is an idempotent element of S. Further, since S is continuous, there exist two continuous Archimedean t-conorms S_1 and S_2 such that $S = (\langle 0, \gamma, S_1 \rangle, \langle \gamma, 1, S_2 \rangle)$.

In the following, we verify $G_2(\alpha, x)$ has the following form

$$G_2(\alpha, x) = \begin{cases} \gamma, & \text{if } x \in [0, \gamma], \\ x, & \text{if } x \in]\gamma, 1]. \end{cases}$$

For any $x \in [0, \gamma]$, it follows from Proposition 4.7 that

$$\begin{aligned} G_2(\alpha, x) &= S(\gamma, x) \\ &= \gamma S_1(1, \tfrac{x}{\gamma}) \\ &= \gamma. \end{aligned}$$

Moreover, for any $x \in [\gamma, 1]$, we have

$$\begin{aligned} G_2(\alpha, x) &= S(\gamma, x) \\ &= \gamma + (1 - \gamma) S_1(0, \tfrac{x - \gamma}{1 - \gamma}) \\ &= x. \end{aligned}$$

(2) It follows from the proof of (1) that $S(\gamma, \gamma) > \gamma$. In addition, since S is continuous, there exists a continuous Archimedean t-conorm S_0 such that $S = (..., \langle \eta_0, \theta_0, S_0 \rangle, ...)$

and $\gamma \in]\eta_0, \theta_0[$. Now, we prove $G_2(\alpha, x)$ has the following form

$$G_2(\alpha,x)=\begin{cases}\gamma, & \text{if } x\in[0,\eta_0[,\\ \eta_0+(\theta_0-\eta_0)S_0(\frac{\gamma-\eta_0}{\theta_0-\eta_0},\frac{x-\eta_0}{\theta_0-\eta_0}) & \text{if } x\in[\eta_0,\theta_0]\\ x, & \text{if } x\in]\theta_0,1].\end{cases}$$

Case (A) Let $x \in [0, \eta_0[$. On the one hand, $G_2(\alpha, x) = T(\gamma, x) \geq \gamma$. Further, since S is continuous and $S(\eta_0, \eta_0) = \eta_0$, we have

$$\begin{aligned} G_2(\alpha,x) &= G(\gamma,x)\\ &\leq S(\gamma,\eta_0)\\ &= \max(\gamma,\eta_0)\\ &= \gamma. \end{aligned}$$

Therefore, $O_2(\alpha, x) = \gamma$.

Case (B) Let $x \in [\eta_0, \theta_0]$. Then we have

$$\begin{aligned} G_2(\alpha,x) &= S(\gamma,x)\\ &= \eta_0+(\theta_0-\eta_0)S_0(\tfrac{\gamma-\eta_0}{\theta_0-\eta_0},\tfrac{x-\eta_0}{\theta_0-\eta_0}) \end{aligned}$$

Case (C) Let $x \in]\theta_0, 1]$. On one hand, $G_2(\alpha, x) = G(\gamma, x) \geq x$. On the one hand, since S is continuous and $S(\theta_0, \theta_0) = \theta_0$, we have

$$\begin{aligned} G_2(\alpha,x) &= G(\gamma,x)\\ &\leq S(\beta_0,x)\\ &= \max(\beta_0,x)\\ &= x. \end{aligned}$$

Therefore, $O_2(\alpha, x) = x$.

□

Now, in the sequel, we consider the generalized migrativity property of nullnorms for $k \in]0, 1[$.

For $\alpha = 1$, we have the following conclusion.

Proposition 4.9: Let G_1 and G_2 be two fixed grouping functions and F be a nullnorm with absorbing element $k \in]0, 1[$. Then F is not $(1, G_1, G_2)$-migrative.

Proof. It is straightforward. □ It follows from Proposition 4.9 that we only need to consider $\alpha \in [0, 1[$. First, we introduce the (α, O_1, O_2)-migrative for $\alpha = 0$.

Proposition 4.10: Let G_1 and G_2 be two fixed grouping functions and F be a nullnorm with absorbing element $k \in]0, 1[$. Then the following statements are equivalent:

(1) F is $(0, G_1, G_2)$-migrative.

(2) $G_1(0, x) = x$ and $G_2(0, x) = x$ for any $x \in [0, 1]$.

Proof. The proof is similar to the one of Proposition 3.11. □

Theorem 4.11: Consider $\alpha \in]0, 1[$. Let G_1 and G_2 be two fixed grouping functions, F be a nullnorm with absorbing element $k \in]0, 1[$ and $\gamma = k$. If F is (α, G_1, G_2)-migrative, then $G_2(\alpha, x)$ has the following form

$$G_2(\alpha,x)=\begin{cases}k, & \text{if } x\in[0,k[,\\ x, & \text{if } x\in[k,1].\end{cases} \tag{10}$$

Proof. The proof is similar to the one of Theorem 3.12. □

Theorem 4.12: Consider $\alpha \in]0, 1[$. Let G_1 and G_2 be two fixed grouping functions and $F = \langle S, k, T\rangle$ be a nullnorm with absorbing element $k = F(0, 1) \in]0, 1[$. Then the following statements hold.

(1) Let $\gamma > k$. Then F is not (α, G_1, G_2)-migrative.

(2) Let $\gamma < k$ and S is continuous. Then the following two items hold.

(a) Let $G_1(\alpha, \gamma) = \gamma$. If F is (α, G_1, G_2)-migrative, then S is an ordinal sum of two continuous Archimedean t-conorms S_1 and S_2, i.e., $S = (\langle 0, \frac{\gamma}{k}, S_1\rangle, \langle \frac{\gamma}{k}, 1, S_2\rangle)$ and $G_2(\alpha, x)$ has the following form

$$G_2(\alpha,x)=\begin{cases}\gamma, & \text{if } x\in[0,\gamma[,\\ x, & \text{if } x\in[\gamma,1].\end{cases}$$

(b) Let $O_1(\alpha, \gamma) > \gamma$. If F is (α, G_1, G_2)-migrative, then S is an ordinal sum of the form $S = (..., \langle\theta_3, \theta_4, S^\xi\rangle, ...)$, where $\frac{\gamma}{k} \in]\theta_3, \theta_4[$ and S^ξ is a continuous Archimedean t-conorm and if $x \in [0, k\theta_3[, G_2(\alpha, x) = \gamma, x \in [k\theta_3, k\theta_4], G_2(\alpha, x) = k(\theta_3 + (\theta_4 - \theta_3)S^\xi(\frac{\gamma-k\theta_3}{k(\theta_4-\theta_3)}, \frac{x-k\theta_3}{k(\theta_4-\theta_3)})), x \in]k\theta_4, 1], G_2(\alpha, x) = x$.

Proof. The proof is similar to the one of Theorem 3.13. □

V. CONCLUSIONS

In this paper, we introduce the notion of (α, O_1, O_2)-migrativity of nullnorms, where O_1 and O_2 are any two fixed overlap functions. We also show some equivalent characterizations of the (α, O_1, O_2)-migrativity equation when the nullnorm F becomes a t-norm or a t-conorm. In addition, we give the notion of (α, G_1, G_2)-migrativity of nullnorms over any two fixed grouping functions G_1 and G_2 and propose (α, G_1, G_2)-migrativitity equation using an analogous method. The main conclusions are listed as follows.

(1) We generalize α-migrativity of any nullnorm F from the usual formula $F(\alpha x, y) = F(x, \alpha y)$ to the so-called (α, O_1, O_2)-migrativity $F(O_1(\alpha, x), y) = F(x, O_2(\alpha, y))$, where O_1 and O_2 are two fixed overlap functions.

(2) We discuss the (α, O_1, O_2)-migrativity of a nullnorm F by taking F as a t-norm T or a t-conorm S and gave an equivalent characterization of it.

(3) We propose the solutions of the (α, O_1, O_2)-migrativity and (α, G_1, G_2)-migrativity equations for nullnorms. In addition, it has been showed that, for $\alpha = 0$, no nullnorm satisfies the (α, O_1, O_2)-migrativity equation and for $\alpha = 1$, no nullnorm satisfies the (α, G_1, G_2)-migrativity equation. Also, we obtain some characterizations of the (α, O_1, O_2)-migrativity equation for any nullnorm by considering $\alpha \in]0, 1]$. And the case for grouping functions are analogous.

As further works, we intend to study the generalized α-migrativity property of overlap functions and grouping functions by the following formulas $O_1(O_2(\alpha, x), y) = O_1(x, T(\alpha, y))$, $G_1(G_2(\alpha, x), y) = G_1(x, S(\alpha, y))$ for $\alpha \in [0, 1]$ and $x, y \in [0, 1]$, where O_1, O_2 are two overlap functions, G_1, G_2 are two grouping functions, T is a t-norm and S is a t-conorm.

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