Antichain Graphs

SHAHISTHA, K ARATHI BHAT* and SUDHAKARA G

***Abstract*—A vertex $u \in V_1$ in a bipartite graph $G(V_1 \cup V_2, E)$ is redundant if all the vertices of V_2 that are adjacent to u are also adjacent to a vertex w ($\neq u$) in V_1. In other words, $N_G(u) \subseteq N_G(w)$. Such vertices increase the cost of the network (when it is a communication network) or increase the unnecessary membership of the network (when it is a social network). An ideal cost effective network is the one where there is no redundant vertex. In this article, we model the above type of networks using graphs and call them antichain graphs. We characterize such graphs and study their properties. We show that if G and H are antichain graphs then so is their cartesian product $G \times H$. We design few more methods to construct new antichain graphs from the existing ones. We also present generating procedures, which generate some regular and biregular antichain graphs. We define a critical edge with reference to the antichain property. We also characterize the critical edge.**

***Index Terms*—Bipartite graphs, Cartesian product, Adjacency matrix, $k-$complement, Antichain.**

I. Introduction

CONSIDER a network which is a bipartite graph with the bipartition $\{V_1, V_2\}$. Two vertices u and v in G can directly communicate with each other only if (u, v) is an edge in G. Note that no two vertices in the same partite set can directly communicate with each other. A vertex u belonging to V_1 (say) is called redundant if all the vertices of V_2 with which u has direct communication, also have direct communication with some vertex w ($\neq u$) in V_1. In other words, $N_G(u) \subseteq N_G(w)$. Such vertices increase the cost of the network (when it is a communication network) or increase the unnecessary membership of the network (when it is a social network). An ideal cost effective network is the one where there is no redundant vertex. In this article, we model the above type of networks using graphs and call them antichain graphs.

The following are necessary definitions and notations ([1], [2]) used in the later part of the article. We write $u \sim v$ if the vertices u and v are adjacent, $u \nsim v$ if they are not. The neighborhood of a vertex $u \in V(G)$ is the set $N_G(u)$ consisting of all the vertices v which are adjacent to u in G. The closed neighborhood is $N_G[u] = N_G(u) \cup \{u\}$. A bigraph or a bipartite graph is a graph G whose vertex set $V(G)$ can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. We denote a bipartite graph with the bipartition $V(G) = V_1 \cup V_2$ by $G(V_1 \cup V_2, E)$. If G contains every edge joining V_1 and V_2, then it is a complete bipartite graph. Graphs are completely determined either by their adjacencies or by their incidences which can be easily represented in matrix form. Indeed, with a given labeled graph, there are several matrices associated. It is often possible to make use of these matrices to identify certain graph properties. Also, it is in this form are the graphs commonly stored in computers. The adjacency matrix is one such matrix. Let G be a labeled graph with n vertices. The adjacency matrix $A = [a_{ij}]$ is the $n \times n$ matrix in which $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. For a bipartite graph, the adjacency matrix can be written as $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, where B is called the biadjacency matrix. There are several operations defined on graphs. The cartesian product of two graphs G and H, written $G \times H$ is a graph whose vertex set is $V(G) \times V(H)$ where the two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 \sim v_2$ or $v_1 = v_2$ and $u_1 \sim u_2$. Given a graph G on n vertices ($n \geq 2$) with vertex set $V(G)$ and the partition of $P = \{V_1, V_2, ..., V_k\}$ of V of order k, the k-complement G_k^P of G is defined as follows: For all V_r and V_s in P, $r \neq s$, remove the edges between V_r and V_s and add the missing edges between them. Specifically when $k = 2$, it is the 2-complement G_2^P.

A. *Chain graphs*

A class of sets $S = \{S_1, S_2, ..., S_n\}$ is called a chain with respect to the operation of set inclusion if for every $S_i, S_j \in S$, either $S_i \subseteq S_j$ or $S_j \subseteq S_i$. Similarly, a class of sets $S = \{S_1, S_2, ..., S_n\}$ is called an antichain with respect to the operation of set inclusion if for every $S_i, S_j \in S$, neither $S_i \subseteq S_j$ nor $S_j \subseteq S_i$. A graph is called a chain graph if it is bipartite and the neighborhoods of the vertices in each partite set form a chain with respect to set inclusion. In other words, for every two vertices u and v in the same partite set and their neighborhoods $N_G(u)$ and $N_G(v)$, either $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$.

Chain graphs are the maximizers for the largest eigenvalue of the graphs among bipartite graphs (connected) of fixed size and order [3]. It is interesting to note that chain graphs have no eigenvalues in the interval $(0, \frac{1}{2})$ and all their non-zero eigenvalues are simple [4]. Other than these interesting features, few other facts concerned with chain graphs, readers are referred to [5], [6], [7], [8], [9], [10], [11] and [12].

Let $G(V_1 \cup V_2, E)$ be a bipartite graph. Note that, if for every two vertices in the same partite set, the property that the neighborhood of one of them contains the neighborhood of the other is true, then the same property is true for every two vertices in the other partite set also. There may exist pairs of vertices say, u and v in the same partite set such that $N_G(u) \nsubseteq N_G(v)$ and $N_G(v) \nsubseteq N_G(u)$ even when $N_G(u) \cap N_G(v) \neq \phi$. We consider the class of bipartite graphs in which, neither $N_G(u)$ is completely contained in $N_G(v)$ nor $N_G(v)$ is completely contained in $N_G(u)$ for

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every $u, v \in V(G)$. In this article, we study such a graph, in which neighborhoods of vertices form an antichain with respect to the operation of set inclusion. Quite naturally, this graph is called an antichain graph and is formally defined as below.

B. Antichain graphs

Definition 1.1: A bipartite graph $G(V_1 \cup V_2, E)$ is called an antichain graph if neighborhoods of vertices of G form an antichain with respect to the operation of set inclusion.
In the following example, a bipartite graph G with bipartition $\{V_1, V_2\}$ and neighborhoods of vertices of G are given. It is easy to observe that G is an antichain graph.

Example 1.1: Consider the graph $G(V_1 \cup V_2, E)$ shown in Fig. 1 where $V_1 = \{v_1, v_3, v_5, v_7\}$ and $V_2 = \{v_2, v_4, v_6, v_8, v_9\}$ and the neighborhood of vertices given by $N_G(v_1) = \{v_2, v_8\}, N_G(v_2) = \{v_1, v_3\}, N_G(v_3) = \{v_2, v_4, v_9\}, N_G(v_4) = \{v_3, v_5\}, N_G(v_5) = \{v_4, v_6\}, N_G(v_6) = \{v_5, v_7\}, N_G(v_7) = \{v_6, v_8, v_9\}, N_G(v_8) = \{v_1, v_7\}, N_G(v_9) = \{v_3, v_7\}$.

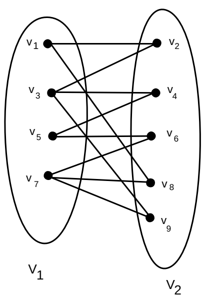

Fig. 1. An antichain graph on 9 vertices

It is interesting to note that the complete graph K_2 is the connected antichain graph with a minimum number of vertices and a minimum number of edges. The even cycle C_{2n} is an antichain graph for every n except $n = 2$. When $n = 2$, the cycle C_4 has two pairs of vertices having the same neighborhood and hence not an antichain graph.

II. Properties of Antichain graphs

In this section, we derive some properties of antichain graphs. First, we make the following notes.

Remark 2.1: Since the neighborhood of an isolated vertex is the empty set and empty set is contained in every set, an antichain graph does not have isolated vertices.

Remark 2.2: A disconnected graph G is an antichain graph if and only if every component of G is an antichain graph.

Remark 2.3: Let $G(V_1 \cup V_2, E)$ be an antichain graph and $u \in V_1$. Unless $|V_2| = 1$, u is not adjacent to all the vertices of V_2. If $|V_2| = 1$, then $|V_1| = 1$ and the graph is K_2.

Theorem 2.1: Let $G(V_1 \cup V_2, E)$ be a connected antichain graph where $G \neq K_2$. Then G neither contains a pendant vertex nor contains a vertex which is adjacent with all the vertices in the other partite set.

Proof: Suppose $G(\neq K_2)$ contains a pendant vertex u which is in V_1 (say). Let $w \in V_2$ be adjacent to u. Since $G \neq K_2$ and is connected, there is a vertex v in V_1 such that $v \sim w$ which implies that $N_G(u) \subseteq N_G(v)$, a contradiction. The proof of the other part of the theorem is obvious. ∎

Corollary 2.2: The graph K_2 is the only tree which is an antichain graph.

Corollary 2.3: If $G \neq K_2$ is a connected antichain graph, then G has at least one even cycle.

Theorem 2.4: Out of all the graphs with less than or equal to seven vertices, the complete graph K_2, the cycle graph C_6 and their possible vertex disjoint unions are the only antichain graphs. Moreover, C_6 is the only antichain graph in which one of the partite sets contains exactly three vertices.

Proof: Let $G(V_1 \cup V_2, E)$ be an antichain graph. If $|V_1| = 1$, then, since G has no isolated vertices, $|V_2| = 1$ and the only possible graph is K_2. If $|V_1| = 2$ and G is connected, then every vertex of V_2 is of degree one or two. In either case, G is not an antichain graph irrespective of number of vertices in V_2 by Theorem 2.1. But if G is disconnected, the only possibility is $|V_1| = |V_2| = 2$ and G is the union of two K_2s. Thus there is no connected antichain graph on $n < 6$ vertices. If $|V_1| = 3$ and G is connected, then all the vertices of V_2 are of degree two. Inorder that G is an antichain graph, the neighborhood of each vertex of V_2 is obtained by selecting two distinct vertices of V_1 out of three, which can be done in $\binom{3}{2}$ ways. Hence $|V_2| \leq \binom{3}{2} = 3$. Thus the only possibility is $|V_1| = |V_2| = 3$ and G is the cycle C_6. But if G is disconnected, the only possibility is $|V_1| = |V_2| = 3$ and G is union of three K_2s. Hence the graph C_6 is the only connected antichain graph on $n \leq 6$ vertices. ∎

Theorem 2.5: For a bipartite graph G with bipartition $\{V_1, V_2\}$, G_2^P is an antichain graph with respect to the same bipartition $\{V_1, V_2\}$ if and only if G itself is an antichain graph.

Proof: Let $G(V_1 \cup V_2, E)$ be an antichain graph. Note that, for any vertex u in V_i, $N_{G_2^P}(u) = V_j \setminus N_G(u)$ where $1 \leq i, j \leq 2$ and $i \neq j$. And also note that, for any two vertices u and v in V_i, $N_{G_2^P}(u) \subseteq N_{G_2^P}(v)$ if and only if $N_G(v) \subseteq N_G(u)$. Hence the result follows. The converse part follows from the fact $\left(G_2^P\right)_2^P \cong G$. ∎
We obtain bounds for number of edges separately for antichain graphs with even and odd number of vertices in the following theorems.

Theorem 2.6: Let $G(V_1 \cup V_2, E)$ be a connected antichain graph on $2n (n \neq 2)$ vertices having m edges. Then $2n \leq m \leq n(n-1)$.

Proof: Let G be a connected antichain graph on $2n(n \neq 2)$ vertices. Since G is not a tree, it has at least one even cycle and has at least $2n$ edges. We know that C_{2n} $(n \neq 2)$ is an antichain graph. Thus $m \geq 2n$.
Maximum number of edges in a bipartite graph with $|V_1| = m$ and $|V_2| = n$ happens when every vertex of V_1 is adjacent to every vertex of V_2, in such a case, the maximum is equal to mn. This product takes the maximum value when $m = n$. Since an antichain graph does not have a vertex with full degree, the maximum number of edges corresponds to the antichain graph obtained by removing a one factor from $K_{n,n}$, in which every vertex is of degree $n-1$. Thus $m \leq n(n-1)$. ∎

Theorem 2.7: Let $G(V_1 \cup V_2, E)$ be a connected antichain graph on $(2n+1), n > 3$ vertices having m edges. Then $m \geq 2n+2$.

Proof: Since G is connected antichain graph, $m \geq 2n+2$. We observe that the following graph on $2n+1$ vertices and $2n+2$ edges is an antichain graph with respect to the bipartition $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{v_{n+1}, v_{n+2}, ..., v_{2n}, v_{2n+1}\}$. The neighborhoods of the vertices are given by $N_G(v_{n+i}) = \{v_i, v_{i+1}\}$ for $i = 1, 2, ..., n-1$, $N_G(v_{2n}) = \{v_1, v_n\}$ and $N_G(v_{2n+1}) = \{v_1, v_3\}$. This has $2n-1$ vertices of degree two and two vertices of degree three (Fig. 2). Thus $m \geq 2n+2$. ■

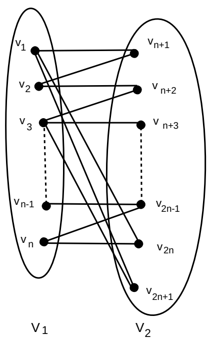

Fig. 2. An antichain graph on $2n+1$ vertices with minimum number of edges

Lemma 2.8: Let G be a disconnected antichain graph on $(2n+1), n > 4$ vertices. The graph G has minimum number of edges if and only if G is a graph whose $n-4$ components are K_2s and one component is G_1, where G_1 is a connected antichain graph on nine vertices and ten edges given by Fig. 3.

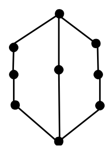

Fig. 3. The Graph G_1

Proof: Let G be an antichain graph on $2n+1$ vertices whose $n-4$ components are K_2s and one component is G_1. Observe that either when the edge from K_2 is removed (which results in two isolated vertices) or when an edge from G_1 is removed (which results in at least one pendant vertex), the resulting graph is not an antichain graph. Hence it is true that G has minimum number of edges.
The graph G has minimum number of edges when every component except one component is K_2. Let C be the component which is not K_2. Since C is an antichain graph, number of vertices in C is at least nine (Theorem 2.4). The minimum number of edges in the graph results when the component C has minimum number of edges. This is possible when G has nine vertices and ten edges (Theorem 2.7) and the graph is G_1 as shown above (Fig. 3). Thus there are $\dfrac{(2n-8)}{2} = n-4$ components which are K_2s and one component is G_1 which is shown in Fig. 3. ■

Remark 2.4: Let $G(V_1 \cup V_2, E)$ be a connected antichain graph. Then the number of edges in G is maximum if and only if G_2^P with respect to the same bipartition $\{V_1, V_2\}$ has minimum number of edges.

Theorem 2.9: Let G be a connected antichain graph on $(2n+1), n > 3$ vertices having m edges. Then $m \leq n^2 - 6$.

Proof: Let $G(V_1 \cup V_2, E)$ be an antichain graph on $2n+1$ vertices. The graph G has maximum number of edges if and only if G_2^P with respect to the same bipartition $\{V_1, V_2\}$ has minimum number of edges. From Lemma 2.8, G has maximum number of edges when G_2^P is a graph whose $n-4$ components are K_2s and one component is G_1, where G_1 is shown in Fig. 3. We note that the number of edges in G_2^P is $n+6$. Since the maximum number of edges in any bipartite graph on $2n+1$ vertices is $n(n+1)$, the maximum number of edges possible is given by $n(n+1) - (n+6) = n^2 - 6$. ■

III. Characterization of Antichain graphs using its Adjacency matrix

The adjacency matrix of a bipartite graph is well studied. It is interesting to observe the additional structure in the adjacency matrix of a bipartite graph, when it is an antichain graph. In the following theorem, we state some equivalent conditions in a graph to be an antichain graph, in terms of its adjacency matrix.

Theorem 3.1: Let $G(V_1 \cup V_2, E)$ be a bipartite graph on n vertices with the adjacency matrix A. The following are equivalent.

(i) G is an antichain graph.
(ii) For every two vertices v_i and v_j, the number of paths of length two between v_i and v_j in G is less than $min\{deg(v_i), deg(v_j)\}$.
(iii) For every i and j, $i \neq j, 1 \leq i, j \leq n$, the $(i,j)^{th}$ entry of A^2 is less than $min\{(i,i)^{th} entry, (j,j)^{th} entry\}$ of A^2.

Proof: $(i) \implies (ii)$:
Since G is an antichain graph, for every two vertices v_i and v_j in G, there exists a vertex w in $N_G(v_i) \setminus N_G(v_j)$ and a vertex x in $N_G(v_j) \setminus N_G(v_i)$, $1 \leq i, j \leq n, i \neq j$. Also, note that the number of paths of length two between v_i and v_j is equal to the number of vertices in $N_G(v_j) \cap N_G(v_i)$. Hence, the number of paths of length two between v_i and v_j is less than $deg(v_i)$ and $deg(v_j)$.
$(ii) \implies (iii)$:
This follows by noting that the diagonal entries of A^2 is the respective degree of the vertex and $(i,j)^{th}$ entry, for $i \neq j$, of A^2 is the number of distinct paths of length two in G, between v_i and v_j.
$(iii) \implies (i)$:
By the assumption of (3), it follows that, there is a vertex in $N_G(v_i)$ which is not in $N_G(v_i) \cap N_G(v_j)$ and there is a vertex in $N_G(v_j)$, not in $N_G(v_i) \cap N_G(v_j)$. Hence, $N_G(v_i) \nsubseteq N_G(v_j)$ and $N_G(v_j) \nsubseteq N_G(v_i)$, $1 \leq i, j \leq n, i \neq j$. It follows that graph G is an antichain graph. ■
We use the above theorem to prove that the derived graphs obtained from the existing antichain graph is also an antichain graph.

Theorem 3.2: Let G and H be two antichain graphs where $G, H \neq K_2$. Then the resulting graph obtained by identifying the vertices u_1 of G and $u_1^{'}$ of H is an antichain graph.

Proof: Let $G(U \cup V, E_1)$ and $H(U^{'} \cup V^{'}, E_2)$ be two antichain graphs. Let Γ be the resulting graph obtained by identifying the vertices u_1 of G and $u_1^{'}$ of H. Without loss of generality, let $u_1 \in U$, $u_1^{'} \in U^{'}$ and x be the new vertex. Then the graph Γ is bipartite graph and has bipartition given by $W_1 = (U \setminus \{u_1\}) \bigcup \left(U' \setminus \{u_1'\}\right) \bigcup \{x\}$ and $W_2 = V \cup V'$. Let B_1 and B_2 be the biadjacency matrices of G and H respectively. Then adjacency matrix A of the graph Γ is given by,

$$A = \begin{array}{c} \\ x \\ U \\ U' \\ V \\ V' \end{array} \begin{array}{c} \begin{array}{ccccc} x & U & U' & V & V' \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & X_1 & X_2 \\ 0 & 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & 0 & C_2 \\ X_1^T & C_1^T & 0 & 0 & 0 \\ X_2^T & 0 & C_2^T & 0 & 0 \end{pmatrix} \end{array}$$

where C_1, C_2 are the matrices obtained by deleting the first row of B_1 and B_2 respectively. Also, X_1, X_2 are the row vectors which are the first rows of B_1 and B_2 respectively and X_i^T represents the transpose of X_i. Consider the matrix A^2, the diagonal entry corresponding to the vertex x is the sum of degrees of vertices u_1 and u_1'. All the other entries of A^2 remains the same as that of $[A(G)]^2$ and $[A(H)]^2$ ($A(G)$, $A(H)$ being the adjacency matrices of G and H respectively) except the entries corresponding to the vertices in the neighborhood of u_1 and u_1'. That is, for all the vertices $v_k \in V$ such that $v_k \sim u_1$ and $v_l' \in V'$ such that $v_l' \sim u_1'$, the corresponding entries in A^2 are one. Also, the diagonal entries of A^2 are at least two except when either of G or H is K_2, as G and H are antichain graphs, and all the vertices are of degree at least two. Thus $\forall\, v_k, v_l' \in V(\Gamma)$, the entries in A^2 is less than the diagonal entries corresponding to the vertices v_k and v_l'. Hence Γ is an antichain graph when neither of G, H is K_2. ■

An illustration for the above theorem is given in Fig. 4.

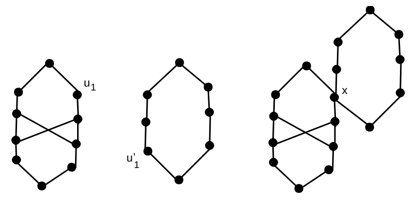

Fig. 4. An antichain graph obtained by identifying the vertices

Theorem 3.3: Let G and H be two antichain graphs. Let $e_1 = (u_1, v_1) \in E(G)$ and $e_2 = (u_1', v_1') \in E(H)$. The graph obtained by identifying the edges e_1 and e_2 is also an antichain graph.

Proof: Let $G(U \cup V, E_1)$ and $H(U' \cup V', E_2)$ be two antichain graphs. Let Γ be the resulting graph obtained by identifying the edges e_1 of G and e_2 of H. Without loss of generality, let $u_1 \in U$, $v_1 \in V$ and $u_1' \in U'$, $v_1' \in V'$. Let x, y be the new vertices obtained by identifying u_1, u_1' and v_1, v_1' respectively. The graph Γ has bipartition given by $W_1 = (U \setminus \{u_1\}) \bigcup \left(U' \setminus \{u_1'\}\right) \bigcup \{x\}$ and $W_2 = (V \setminus \{v_1\}) \bigcup \left(V' \setminus \{v_1'\}\right) \bigcup \{y\}$. Let B_1, B_2 be the biadjacency matrices of G and H respectively. Then adjacency matrix A of the graph Γ is given by,

$$A = \begin{array}{c} \\ x \\ U \\ U' \\ y \\ V \\ V' \end{array} \begin{array}{c} \begin{array}{cccccc} x & U & U' & y & V & V' \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & X_1 & X_2 \\ 0 & 0 & 0 & Y_1 & C_1 & 0 \\ 0 & 0 & 0 & Y_2 & 0 & C_2 \\ 1 & Y_1^T & Y_2^T & 0 & 0 & 0 \\ X_1^T & C_1^T & 0 & 0 & 0 & 0 \\ X_2^T & 0 & C_2^T & 0 & 0 & 0 \end{pmatrix} \end{array}$$

where C_1, C_2 are the matrices obtained from B_1, B_2 respectively by deleting the first rows. The row vectors X_1, X_2 are the first rows of B_1 and B_2 respectively in which the first entries are deleted. The column vectors Y_1, Y_2 are the first columns of B_1 and B_2 respectively in which the first entries are deleted. Consider the matrix A^2, the diagonal entry corresponding to the vertex x is the sum of degrees of vertices u_1 and u_1'. And entry corresponding to the vertex y is the sum of degrees of vertices v_1 and v_1'. All the other entries of A^2 remains same as that of $[A(G)]^2$ and $[A(H)]^2$, ($A(G), A(H)$ being the adjacency matrices of G and H respectively) except the entries corresponding to the vertices in the neighborhood of u_1 and u_1' as well as v_1 and v_1'. That is, for all the vertices $u_k \in U$ such that $u_k \sim v_1$ and $u_l' \in U'$ such that $u_l' \sim v_1'$, the corresponding entries in A^2 are one. Also, the diagonal entries of A^2 are at least two except when either of G, H is K_2, as G and H are antichain graphs and all the vertices are of degree at least two. Thus, for all $u_k, u_l' \in V(\Gamma)$, the entries in A^2 is less than the diagonal entries corresponding to the vertices u_k and u_l'. Similarly, for all the vertices $v_i \in V$ such that $v_i \sim u_1$ and $v_j' \in V'$ such that $v_j' \sim u_1'$, the corresponding entries in A^2 are less than the diagonal entries corresponding to the vertices v_i and v_j'. Hence Γ is an antichain graph when neither of G, H is K_2. ■

An illustration for the above theorem is given below in Fig. 5.

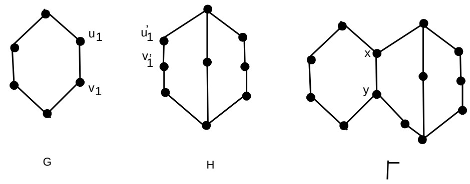

Fig. 5. An antichain graph obtained by identifying the edges

IV. New antichain graphs obtained from the existing ones

In the previous section, we have seen some new antichain graphs constructed using the ones which are known. In proving them to be antichain graphs, we made use of characterization of adjacency matrices of an antichain graph (Theorem 3.1). In this section, we get few more classes of antichain graphs and the techniques used to prove them to be antichain graphs are different. We make use of the definition and properties of antichain graphs. As the first result in this method, we prove that the cartesian product of two antichain graphs is again an antichain graph.

Theorem 4.1: Let G and H be two antichain graphs. Then their cartesian product $G \times H$, is also an antichain graph, except when $G = H = K_2$.

Proof: Let $V(G) = \{u_1, u_2, \ldots, u_{p_1}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{p_2}\}$. Note that, since G and H are bipartite, so is $G \times H$. To prove that $G \times H$ is an antichain graph, we prove that, for every two vertices $X = (u_i, v_j)$ and $Y = (u_k, v_l)$, $1 \le i, k \le p_1$ and $1 \le j, l \le p_2$, there exists vertices X' and Y' of $G \times H$ such that $X' \in N_{G\times H}(X) \setminus N_{G\times H}(Y)$ and $Y' \in N_{G\times H}(Y) \setminus N_{G\times H}(X)$.

When X and Y are in two different partite sets of $G \times H$, their neighborhoods are disjoint and hence the theorem follows.

Suppose that X and Y are in the same partite set. Then the following cases are to be considered.

1) $u_i \sim u_k$, $v_j \neq v_l$ and $v_j \nsim v_l$.
2) $u_i \sim u_k$, $v_j \neq v_l$ and $v_j \sim v_l$.
3) $u_i \nsim u_k$, $v_j \neq v_l$ and $v_j \nsim v_l$.

Case 1: In this case, the vertex $Y' = (u_i, v_l)$ is such that $Y' \in N_{G\times H}(Y) \setminus N_{G\times H}(X)$ and $X' = (u_k, v_j)$ is such that $X' \in N_{G\times H}(X) \setminus N_{G\times H}(Y)$. Hence $G \times H$ is an antichain graph.

Case 2: Suppose that u_i and u_k are vertices of degree one in G and vertices v_j and v_l are of degree one in H. This is possible only when $G = H = K_2$, in which case, $G \times H = K_2 \times K_2 = C_4$, which is not an antichain graph. Now suppose that a vertex, say u_i, is of degree greater than one. Let the vertices u_r and u_k be adjacent to u_i. Then (u_i, v_l), (u_r, v_j) and (u_k, v_j) are all adjacent to (u_i, v_j) in $G \times H$. Then the vertex $X' = (u_r, v_j)$ is such that $X' \in N_{G\times H}(X) \setminus N_{G\times H}(Y)$. Since G is an antichain graph, u_k is not of degree one in G. Hence there is a vertex u_t $(\neq u_i)$ such that $u_t \sim u_k$. This implies that the vertex $Y' = (u_t, v_l)$ is such that $Y' \in N_{G\times H}(Y) \setminus N_{G\times H}(X)$. Hence $G \times H$ is an antichain graph.

Case 3: We consider both the possibilities that $u_i = u_k$ and $u_i \neq u_k$. Suppose $u_i = u_k$. Then $X = (u_i, v_j)$ and $Y = (u_i, v_l)$. Since G is an antichain graph, there exists a vertex u_r in G such that $u_i \sim u_r$. Now the vertices $X' = (u_r, v_j)$ and $Y' = (u_r, v_l)$ are such that $X' \in N_{G\times H}(X) \setminus N_{G\times H}(Y)$ and $Y' \in N_{G\times H}(Y) \setminus N_{G\times H}(X)$ and $G \times H$ is an antichain graph. Finally, suppose $u_i \neq u_k$. Since H is an antichain graph, there is a vertex v_s in H with $v_l \sim v_s$. Then $X' = (u_i, v_r)$ and $Y' = (u_k, v_s)$ are such that $X' \in N_{G\times H}(X) \setminus N_{G\times H}(Y)$ and $Y' \in N_{G\times H}(Y) \setminus N_{G\times H}(X)$. Thus $G \times H$ is an antichain graph. ■

Corollary 4.2: The n-cube Q_n $(n \neq 2)$ is an antichain graph for every $n > 2$.

Proof: This follows by the observation that, when $n = 2$, $Q_2 = K_2 \times K_2$ which is not an antichain graph and when $n > 2$, $Q_n = Q_{n-1} \times Q_2$ is an antichain graph by Theorem 4.1. ■

Theorem 4.3: Consider the even cycle C_{2n}. Let $V(C_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$ with $v_i \sim v_{i+1}$ for $1 \le i \le 2n-1$ and $v_{2n} \sim v_1$. Let G be the graph obtained by joining two vertices, say v_i and v_j $(v_i \neq v_j)$ of C_{2n}, by a path P_k given by $P_k : (v_i = w_1, w_2, \ldots, w_{k-1}, w_k = v_j)$ in such a way that both of the two new cycles induced say C_{n_1} and C_{n_2} are even and are different from C_4. Then G is an antichain graph.

Proof: Let G be as shown in Fig. 6.

If either C_{n_1} or C_{n_2} is an odd cycle, then G is not bipartite and hence is not an antichain graph. We note that, if the distance between v_i and v_j, $d(v_i, v_j) = 2$ and the path P_k

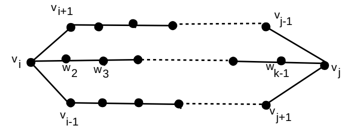

Fig. 6. Graph obtained by joining v_i and v_j in C_{2n}

is of length two, one of cycles induced is C_4 with vertices $(v_i = w_1, w_2, w_3 = v_j$ and $w_4 = v_{i-1} = v_{j+1})$ as in Fig. 7. Then $N_G(w_2) = N_G(w_4)$ and hence the graph is not an antichain graph.

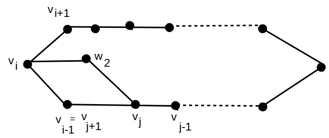

Fig. 7. Graph G having induced C_4

Similarly, when $d(v_i, v_j) = 3$ and the path P_k is just an edge joining v_i and v_j, the vertices $v_i, v_{i+1}, v_{i+2} = v_{j-1}, v_j$ induce a C_4 with $N_G(v_{i+1}) \subseteq N_G(v_j)$ (see Fig. 8).

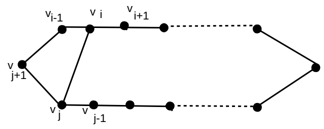

Fig. 8. Graph G having induced C_4

When $d(v_i, v_j) = 1$ and the the path P_k is of length three, one of cycles induced is C_4 with vertices $(v_i = w_1, w_2, w_3, w_4 = v_j)$ as in Fig. 9. Then $N_G(v_i) = N_G(w_2)$ and hence the graph is not an antichain graph.

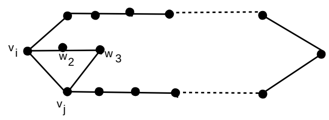

Fig. 9. Graph G having induced C_4

Now suppose that n_1 and n_2 are even integers and $d(v_i, v_j) \neq 2$. Let u and v be two vertices in G which are not in $N_G[v_i] \cup N_G[v_j]$. Then, in all the three of the following cases that either both of them are on the cycle C_{2n}, both of them are on the path P_k or one on C_{2n} and the other on P_k, neither $N_G(u) \subseteq N_G(v)$, nor $N_G(v) \subseteq N_G(u)$. So, to prove that G is an antichain graph, we need to consider only the vertices in $N_G[v_i] \cup N_G[v_j]$. Consider two vertices u and v in $N_G[v_i] \cup N_G[v_j]$. The following cases are to be considered.

1) They are adjacent to each other.
2) They are non adjacent to each other and both are in the neighborhood of exactly one of v_i or v_j.
3) One in the neighborhood of v_i and the other one is in the neighborhood of v_j.

When they are adjacent to each other as in case (1), they are in two different partite sets of the bipartition of G and hence the neighborhoods are disjoint.

In cases (2) and (3) one can easily show that there exists a vertex $x \in N_G(v) \setminus N_G(u)$. Thus G is an antichain graph. ■

Theorem 4.4: Let $G \neq K_2$ and $H \neq K_2$ be any two antichain graphs and let u and v be any two vertices with one in $V(G)$ and one in $V(H)$. Then the graph Γ obtained by joining u and v by a path $P_k : (u = w_1, w_2, \ldots, w_{k-1}, w_k = v)$, is also an antichain graph.

Proof: Let $u \in V(G)$ and $v \in V(H)$. To verify the antichain property of neighborhoods of vertices of Γ, we need to consider only those vertices in $N_\Gamma[u] \cup N_\Gamma[v]$. For other vertices, neighborhoods remain the same as their neighborhoods either in G or in H and since G and H are antichain graphs neighborhoods form an antichain. For any two vertices in $N_\Gamma[u] \cup N_\Gamma[v]$, following steps similar to the steps in the proof of Theorem 4.3, one can show that, the neighborhoods follow the antichain property. ■

V. Generating regular and biregular antichain graphs

All antichain graphs are subgraphs of complete bipartite graphs $K_{n,m}$. Many of them are regular (like cycles) and some of them are biregular in the sense that every vertex has degree either k or l. In this section, we generate regular and biregular antichain graphs. We first prove the following two lemmas.

Lemma 5.1: For every r with $2 \leq r \leq n-1$, there exists at least one connected r-regular antichain graph on $2n$ vertices.

Proof: Let $V_1 = \{u_1, u_2, ..., u_n\}$ and $V_2 = \{v_1, v_2, ..., v_n\}$. The graph $G(V_1 \cup V_2, E)$ such that $|V_1| = |V_2|$ and the neighborhoods of vertices given by $N_G(u_i) = \{v_i, v_{i+1}, ..., v_{r+i-1}\}$ $(1 \leq i \leq n)$ taking $v_{n+i} = v_i$ is r-regular. Since $N_G(u_i) \neq N_G(u_j)$ and $N_G(v_i) \neq N_G(v_j)$ for all $i \neq j$ and $r < n$, G is antichain graph with regularity r. Also if $r = 1$, then G is not connected. Thus for all $2 \leq r \leq n-1$, there exists a connected r-regular antichain graph. ■

Lemma 5.2: For every $r \geq 3$, there exists at least one biregular antichain graph with $k + \binom{k}{r}$ vertices. It has the bipartition $\{V_1, V_2\}$ where $|V_1| = k, |V_2| = \binom{k}{r}$ and every vertex of V_1 is of degree $\binom{k-1}{r-1}$ and every vertex of V_2 is of degree r, $2 \leq r \leq k-1$.

Proof: Let $V_1 = \{u_1, u_2, \ldots, u_k\}$ and $V_2 = \{v_1, v_2, \ldots, v_{\binom{k}{r}}\}$. There $\binom{k}{r}$ distinct subsets each of size r of the set V_1. Each v_i in V_2 is made adjacent to vertices of one subset with r elements. We note that it is an antichain graph. ■

Theorem 5.3: Let H be an $(n-2)$-regular subgraph of $K_{n,n}$. Then H is an antichain graph if and only if the 2-complement of H, H_2^P with respect to the bipartition of $K_{n,n}$ has no cycle of length four.

Proof: The 2-complement H_2^P of H with respect to the bipartition of $K_{n,n}$ is a 2-regular subgraph of $K_{n,n}$. Each component of H_2^P is a cycle and we know that C_4 is the only even cycle which is not an antichain graph. We also know that the union of antichain graphs is again an antichain graph. Hence, if H_2^P has no C_4, then it is an antichain graph. Now, since 2-complement of any antichain graph is an antichain graph and 2-complement of H_2^P is H, the graph H is an antichain graph.

Conversely, if an $(n-2)$-regular subgraph H of $K_{n,n}$ is an antichain graph, then H_2^P is also an antichain graph. Hence, every component of H_2^P is an antichain graph. Since each component of H_2^P is a cycle and C_4 is not an antichain graph, no component of H_2^P is C_4. ■

From the above theorem, one can conclude that, if H is an $(n-2)$-regular subgraph of $K_{n,n}$, then by observing the number of vertices in each component of H_2^P with respect to the bipartition of $K_{n,n}$, one can get a partition of the integer $2n$ into even parts each of which is greater than or equal to six and conversely. This gives the procedure to generate all $(n-2)$-regular antichain graphs on $2n$ vertices. The procedure is explained as follows.

Consider all the possible partitions of the integer $2n$ into even parts with each part greater than or equal to six. Let $(n_1, n_2, \ldots, n_k)$ be one such partition with $\sum_{i=1}^{k} n_i = 2n$, each $n_i \geq 6$ and each n_i is even. Corresponding to each n_i, consider cycle graph C_{n_i}, $1 \leq i \leq k$. That gives a collection of cycle graphs $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ each of which is an antichain graph. The 2-complement of $H = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$ is an $(n-2)$-regular antichain graph.

Similarly, we have the following result.

Theorem 5.4: Let H be biregular subgraph with regularities $n-1$ and $n-2$ of $K_{n,n}$. Then H is an antichain graph if and only if its 2-complement H_2^P with respect to the partition P of $K_{n,n}$ has each of its component as either K_2 or C_n, n is even and $n \geq 6$.

Proof: The 2-complement H_2^P of H with respect to the bipartition of $K_{n,n}$ is a biregular subgraph of $K_{n,n}$ with regularities one and two. Each component of H_2^P is either a cycle or a K_2. We know that the union of antichain graphs is again an antichain graph. Hence, if H_2^P has even cycles $C_n (n \neq 4)$ and K_2, then it is an antichain graph. Now, since 2-complement of any antichain graph is an antichain graph and 2-complement of H_2^P is H, the graph H is an antichain graph.

Conversely, if a biregular subgraph with regularities $n-1$ and $n-2$ of $K_{n,n}$ is an antichain graph, then H_2^P is also an antichain graph. Hence, every component of H_2^P is an antichain graph. Since each component of H_2^P is either a cycle C_n or a K_2 and C_4 is not an antichain graph, the theorem follows. ■

Similarly, if H is biregular subgraphs with regularities $n-1$ and $n-2$ of $K_{n,n}$, then by observing the number of vertices in each component of H_2^P with respect to the bipartition of $K_{n,n}$, one can get a partition of the integer $2n$ into even parts greater than or equal to six, at least one of the part being two and conversely. This gives the procedure to generate all biregular antichain graphs with regularities $n-1$ and $n-2$ on $2n$ vertices. The procedure is explained as follows.

Consider all the possible partitions of the integer $2n$ into even parts such that no part is four and at least one of the part is two. Let $(n_1, n_2, \ldots, n_k)$ be one such partition with $\sum_{i=1}^{k} n_i = 2n$, each n_i is an even number greater than or equal to six and at least one of the n_i is two. Let r be the multiplicity of two as a part in the partition and $n_i = 2$ for $i \leq r$ for some $1 < r < n_k$. Corresponding to each $n_i = 2$, consider the graph K_2 and corresponding to each $n_i \neq 2$, consider cycle graph C_{n_i}. That gives a collection of r complete graphs each of which is K_2 and $k-r$ cycle graphs given by $C_{n_{r+1}}, C_{n_{r+2}}, \ldots, C_{n_k}$. The union of these collection is an antichain graph as every component is an antichain graph. The 2-complement of $H = rK_2 \cup C_{n_{r+1}} \cup C_{n_{r+2}} \cup \cdots \cup C_{n_k}$ is a biregular antichain graph with regularities $n-1$ and $n-2$.

VI. Critical edges

Criticality is an important concept associated with most of the graph parameters. In this section, we study the critical edges with reference to the concept of antichain graphs.

Definition 6.1: Let $G(V_1 \cup V_2, E)$ be an antichain graph. An edge $e \in E$ is said to be critical if the graph $H = G \setminus e$ is not an antichain graph.

For example, for a cycle graph $C_{2n}, n \geq 3$, every edge is critical. Since $C_{2n} \setminus e$ is P_{2n}, which is not an antichain graph. Consider two copies of the antichain graph C_{2n}. Join any vertex of one copy with any vertex of the other copy by an edge, say e. Then every edge on the cycle C_{2n} is critical, but the edge e is not critical. We characterize an edge in antichain graph which is not critical.

Theorem 6.1: Let $G(V_1 \cup V_2, E)$ be an antichain graph. An edge $e = (u, v)$ with $u \in V_1$ and $v \in V_2$ is not a critical edge if and only if the following are true.

(i) For every $w \in V_1,\ N_G(u) \setminus N_G(w) \neq \{v\}$

(ii) For every $x \in V_2,\ N_G(v) \setminus N_G(x) \neq \{u\}$

Proof: Let G be an antichain graph having an edge $e = (u, v)$ which is not a critical edge. That is, $G \setminus e$ is antichain graph. We note that $N_{G\setminus e}(u) = N_G(u) \setminus \{v\}$, $N_{G\setminus e}(v) = N_G(v) \setminus \{u\}$ and $N_{G\setminus e}(w) = N_G(w), \forall w \in V(G)$. We follow the method of contradiction to prove this part. Suppose if any of the above two statements are not true, say for some $w \in V_1,\ N_G(u) \setminus N_G(w) = \{v\}$. Then $N_{G\setminus e}(u) \setminus N_{G\setminus e}(w) = \phi$ which implies $N_{G\setminus e}(u) \subseteq N_{G\setminus e}(w)$, a contradiction to the fact that $G \setminus e$ is an antichain graph. Thus (1) is true. Similarly we can prove (2) is true.

Converse: Let the statements (1) and (2) are true. From (1), $N_G(u)$ contains at least one vertex $y \in V_2$ such that $y \notin N_G(w)$. That is, $N_G(u) \setminus \{v\}$ contains at least one vertex $y \in V_2$ such that $y \notin N_G(w)$. This implies $N_{G\setminus e}(u)$ contains at least one vertex $y \in V_2$ such that $y \notin N_{G\setminus e}(w)$. Thus, for all $w \in V_1$, it is true that $N_{G\setminus e}(u) \nsubseteq N_{G\setminus e}(w)$. Similarly from (2), we can prove that for $N_{G\setminus e}(v) \nsubseteq N_{G\setminus e}(x)$ for all $x \in V_2$. And the cases $N_{G\setminus e}(w) \nsubseteq N_{G\setminus e}(u), \text{for all } w \in V_1$ and $N_{G\setminus e}(x) \nsubseteq N_{G\setminus e}(v), \text{for all } x \in V_2$ are obvious since $N_G(w) \nsubseteq N_G(u), \text{for all } w \in V_1$ and $N_G(x) \nsubseteq N_G(v), \text{for all } x \in V_2$. Thus $G \setminus e$ is an antichain graph which means e is not a critical edge. ■

The following theorem follows from Theorems 3.1 and 6.1.

Theorem 6.2: Let $G(V_1 \cup V_2, E)$ be an antichain graph with bipartition $V_1 = \{v_1, v_2, \ldots, v_m\}$, $V_2 = \{v_{m+1}, v_{m+2}, \ldots, v_{m+n}\}$ and $e = (v_i, v_j)$ be an edge of G with $1 \leq i \leq m$ and $m+1 \leq j \leq m+n$. Let $A = A(G)$ be the adjacency matrix of G. Then, e is a critical edge if and only if one of the following is true.

(i) There exists a k, $1 \leq k \leq m$ such that $a_{ii}^{(2)} - a_{ik}^{(2)} = 1$.

(ii) There exists an l, $m+1 \leq l \leq m+n$ such that $a_{jj}^{(2)} - a_{jl}^{(2)} = 1$

where $a_{rs}^{(2)}$ represents the $(r,s)^{th}$ entry of A^2.

VII. Concluding remarks

Networks, in the form of bipartite graphs, are abundant in existence. Irredundancy of any type is the most studied property of a network because it makes the network cost effective. We define a particular type of redundancy and define a graph (network) which is irredundant. We study the properties, give procedure for the construction of such networks and characterize the edges which are critical with respect to this irredundancy. Construction of new antichain graphs from the existing ones gives new ways to expand the existing irredundant network in to a bigger network, preserving the irredundancy. We propose the following open problem.

Problem: Let $G(V_1 \cup V_2, E)$ be an antichain graph with $|V_1| = |V_2|$. There are at least two perfect matchings in G.

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