Normalized Laplacian Spectra of Two Subdivision-coronae of Three Regular Graphs

Fei Wen, You Zhang, Wei Wang

***Abstract*—In this paper, we first introduce two new graph operations called *the subdivision vertex-edge neighbourhood vertex-corona* and *the subdivision vertex-edge neighbourhood edge-corona* for three graphs G_1, G_2 and G_3, and the resulting graphs are respectively denoted by $G_1^S \bowtie (G_2^V \cup G_3^E)$ and $G_1^S \diamondsuit (G_2^V \cup G_3^E)$, and then, their normalized Laplacian spectra are determined in terms of the corresponding normalized Laplacian spectra of the connected regular graphs G_1, G_2 and G_3, which extend the corresponding results of Das and Panigrahi [19]. As applications, these results enable us to construct infinitely many pairs of *normalized Laplacian cospectral graphs*. Moreover, we also give the number of the *spanning trees*, the *multiplicative degree-Kirchhoff index* and *Kemeny's constant* of $G_1^S \bowtie (G_2^V \cup G_3^E)$ (resp. $G_1^S \diamondsuit (G_2^V \cup G_3^E)$).**

***Index Terms*—subdivision vertex-edge neighbourhood vertex-corona, subdivision vertex-edge neighbourhood edge-corona, normalized Laplacian spectrum, cospectral graphs.**

I. Introduction

THROUGHOUT this paper, we are concerned only with simple connected graphs (loops and multiple edges are not allowed). Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$ where $|V(G)| = n$ and $|E(G)| = m$. The *line graph* $\ell(G)$ of G is a graph whose vertices corresponding the edges of G, and where two vertices are adjacent iff the corresponding edges of G are adjacent. We denote the *complete graph* and the *cycle* of order n by K_n and $C_n (n \geq 3)$, respectively. A *graph matrix* $M = M(G)$ is defined to be a symmetric matrix with respect to adjacency matrix $A(G)$ of G. The *M-characteristic polynomial* of G is defined as $\Phi_M(x) = \det(xI - M)$, where I is the identity matrix. The *M-eigenvalues* of G are the roots of its M-characteristic polynomial. The M-spectrum, denoted by $Spec_M(G)$, of G is a multiset consisting of the M-eigenvalues. And two graphs G and H are M-cospectral if $\Phi_{M(G)}(x) = \Phi_{M(H)}(x)$.

Let $D(G) = diag(d(v_1), d(v_2), \dots, d(v_n))$ be the degree diagonal matrix of G. The graph matrix $M = M(G)$ is respectively called *adjacency matrix*, *signless Laplacian matrix* and *normalized Laplacian matrix* of G if M equals $A(G)$, $Q(G) = D(G) + A(G)$ and $\mathcal{L}(G) = D(G)^{-1/2}(D(G) - A(G))D(G)^{-1/2} = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$, respectively. Conventionally, the *adjacency eigenvalues* and *normalized Laplacian eigenvalues* of graph G are ordered respectively in non-increased sequence as follows: $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. In fact, Chung [1] has proved that $\lambda_i \leq 2$ for all i.

As far as we know, many graph operations such as the *disjoint union*, the *corona*, the *edge corona* and the *neighborhood corona* etc. were introduced in [2]–[9] to determine their spectra. Note that the *subdivision graph* $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. Based on the *subdivision*, some new graph operations such as *subdivision-vertex join* and *subdivision-edge join* were defined in [10], and their A-spectrum were also investigated. Further works on their L-spectrum were considered in [11]. Recently, Wen et al. in [12] introduced another operation called *subdivision-vertex-edge join* for three regular graphs, and then A, L, Q-spectra of the graph were calculated. By the way, they constructed many infinite families of pairs of cospectral graphs, which generalized those results of [10] and [11].

In addition, Lu and Miao [14] determined the A-spectra of graphs called *subdivision-vertex corona* and *subdivision-edge corona*, respectively. As a further extension, Liu and Lu [13] respectively considered the A-spectra of *subdivision-vertex neighborhood corona* and *subdivision-edge neighborhood corona*. Subsequently, Song and Huang [15] obtained the A-spectrum and L-spectrum of *subdivision vertex-edge corona* $G_1^S \circ (G_2^V \cup G_3^E)$ (see $P_4^S \circ (P_2^V \cup P_1^E)$, shown in Fig.1 for instance).

In this section, it was motivated by literatures [13] and [14] that two new graph operations are introduced below: Let G_i be a graph with order n_i and size m_i, where $i = 1, 2, 3$. Let $S(G_1)$ be the subdividing graph of G_1 whose vertex set has two parts: one the original vertices $V(G_1)$, another, denoted by $I(G_1)$, the inserting vertices corresponding to the edges of G_1. Suppose that G_2 and G_3 are two disjoint graphs. Then we have the following definitions.

Definition I.1. Subdivision vertex-edge neighbourhood vertex-corona *(short for SVEV-corona) of G_1 with G_2 and G_3, denoted by $G_1^S \bowtie (G_2^V \cup G_3^E)$, is the graph consisting of $S(G_1)$, $|V(G_1)|$ copies of G_2 and $|I(G_1)|$ copies of G_3, all vertex-disjoint, and joining the neighbours of the* i*-th vertex of $V(G_1)$ to every vertex in the* i*-th copy of G_2 and* i*-th vertex of $I(G_1)$ to each vertex in the* i*-th copy of G_3.*

For simplicity, we depict $P_4^S \bowtie (P_2^V \cup P_1^E)$ in Fig.1. By the Definition I.1, $G_1^S \bowtie (G_2^V \cup G_3^E)$ has $n = n_1 + m_1 + n_1n_2 + m_1n_3$ vertices and $m = 2m_1 + n_1m_2 + m_1m_3 + 2m_1n_2 + m_1n_3$ edges. We see that $G_1^S \bowtie (G_2^V \cup G_3^E)$ will be a *subdivision-vertex neighbourhood corona* (see [13]) if G_3 is null, and will be a *subdivision-edge corona* (see [14])

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if G_2 is null. Thus *subdivision vertex-edge neighbourhood vertex-corona* can be viewed as the generalizations of both *subdivision-vertex neighbourhood corona* (denoted by $G_1 \boxdot G_2$) and *subdivision-edge corona* (denoted by $G_1 \ominus G_3$).

Definition I.2. Subdivision vertex-edge neighbourhood edge-corona *(short for SVEE-corona) of G_1 with G_2 and G_3, denoted by $G_1^S \diamondsuit (G_2^V \cup G_3^E)$, is the graph consisting of $S(G_1)$, $|V(G_1)|$ copies of G_3 and $|I(G_1)|$ copies of G_2, all vertex-disjoint, joining the neighbours of the* i*-th vertex of $I(G_1)$ to every vertex in the* i*-th copy of G_2 and* i*-th vertex of $V(G_1)$ to each vertex in the* i*-th copy of G_3.*

As an illustration, we depict $P_4^S \diamondsuit (P_2^V \cup P_1^E)$ in Fig.1. By the Definition I.2, $G_1^S \diamondsuit (G_2^V \cup G_3^E)$ has $n = n_1 + m_1 + m_1 n_2 + n_1 n_3$ vertices and $m = 2m_1 + m_1 m_2 + n_1 m_3 + n_1 n_3 + 2m_1 n_2$ edges. We see that $G_1^S \diamondsuit (G_2^V \cup G_3^E)$ will be a *subdivision-edge neighbourhood corona* (see [13]) if G_3 is null, and will be a *subdivision-vertex corona* (see [14]) if G_2 is null. Thus *subdivision vertex-edge neighbourhood edge-corona* can be viewed as the generalizations of both *subdivision-edge neighbourhood corona* (denoted by $G_1 \boxminus G_2$) and *subdivision-vertex corona* (denoted by $G_1 \odot G_3$).

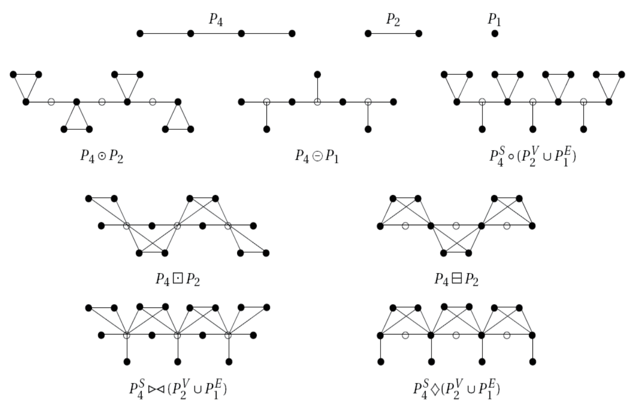

Fig. 1. Some related graphs

The *normalized Laplacian matrix* of a graph was introduced in [1], it was a rather new but important tool popularized by Chung in 1990s. The normalized Laplacian eigenvalues of a graph have a good relationship with other graph invariants for general graphs in a way that other eigenvalues of matrices (such as adjacency, signless Laplacian etc.) fail to do. Thus, for a given graph, calculating its normalized Laplacian spectrum as well as formulating the normalized Laplacian characteristic polynomial is a fundamental and very meaningful work in spectral graph theory. In recent years, several graph operations (see [12], [17], [18], [20]) their normalized Laplacian spectra were computed taking different approaches. In 2017, Das and Panigrahi in [19] have determined the normalized Laplacian spectra of *subdivision-vertex(edge)coronas* [14] and *subdivision-vertex (edge) neighbourhood coronas* [13].

In this paper, we focus on determining the normalized Laplacian spectra of $G_1^S \bowtie (G_2^V \cup G_3^E)$ and $G_1^S \diamondsuit (G_2^V \cup G_3^E)$ in terms of the corresponding normalized Laplacian spectra of three connected regular graphs G_1, G_2 and G_3, which extends the corresponding results of [19]. As applications, these results enable us to construct infinitely many pairs of $\mathcal{L}$-cospectral graphs. Moreover, we also give the number of the spanning trees, the multiplicative degree-Kirchhoff index and Kemeny's constant of $G_1^S \bowtie (G_2^V \cup G_3^E)$ (resp. $G_1^S \diamondsuit (G_2^V \cup G_3^E)$).

II. Preliminaries

In this section, we first list some known results for latter use.

Lemma II.1 ([2]). *For a graph G, let $R(G)$ and $\ell(G)$ be the incidence matrix of G and the line graph of G, respectively. Then*

$$R(G)^T R(G) = 2I_m + A(\ell(G)) \tag{1}$$

where m is the number of edges of G.

Note that

$$R(G)R(G)^T = D(G) + A(G) = Q(G). \tag{2}$$

Since non-zero eigenvalues of both $R(G)R(G)^T$ and $R(G)^T R(G)$ are the same, from the relations (1) and (2) one can obtain

$$\Phi_{A(\ell(G))}(x) = (x+2)^{m-n} \Phi_{Q(G)}(x+2). \tag{3}$$

In particular, if G is r-regular graph, then by Lemma II.1, we immediately have the following corollary.

Corollary II.1. *Let G be an r–regular graph of order n. Then*

$$\begin{aligned} \Phi_{A(\ell(G))}(x) &= (x+2)^{m-n} \cdot \prod_{i=1}^{n} (x - (r-2) - \nu_i(G)), \\ \Phi_{A(\ell(G))}(x) &= (x+2)^{m-n} \cdot \prod_{i=1}^{n} (x - (2r-2) + r\lambda_i(G)) \end{aligned}$$

where $\nu_i(G)$ and $\lambda_i(G)$ are the eigenvalues of $A(G)$ and $\mathcal{L}(G)$ for $i = 1, 2, \ldots, n$, respectively.

As usual, we denote by $\mathbf{1}_n$ and $\mathbf{0}_n$ the column vector of size n with all the entries equal one and all the entries equal 0, respectively. For a graph matrix M of order n, *M-coronal* $\Gamma_M(\lambda)$ is defined, in [3] and [6], to be the sum of the entries of the matrix $(\lambda I - M)^{-1}$, i.e.,

$$\Gamma_M(\lambda) = \mathbf{1}_n^T (\lambda I - M)^{-1} \mathbf{1}_n. \tag{4}$$

If M has constant row sum t, it is easy to verify that

$$\Gamma_M(\lambda) = \frac{\mathbf{1}_n^T \mathbf{1}_n}{\lambda - t} = \frac{n}{\lambda - t}. \tag{5}$$

It is well-known for invertible matrix M_1 and M_4 that

$$\begin{aligned} \det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} &= \det(M_4) \cdot \det(M_1 - M_2 M_4^{-1} M_3) \\ &= \det(M_1) \cdot \det(M_4 - M_3 M_1^{-1} M_2). \end{aligned} \tag{6}$$

where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the *Schur complements* [16] of M_4 and M_1, respectively.

For two matrices $A = (a_{ij})$ and $B = (b_{ij})$, of same size $m \times n$, the *Hadamard product* $A \bullet B = (c_{ij})$ of A and B is a matrix of the same size $m \times n$ with entries given by $c_{ij} = a_{ij} \times b_{ij}$(entrywise multiplication). The *Kronecker product* $A \otimes B$ of two matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This is an associative operation with the property that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist. The latter implies $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for nonsingular matrices A and B. Moreover, if A and B are $n \times n$ and $p \times p$ matrices, then $\det(A \otimes B) = \det(A)^p \det(B)^n$. The reader is referred to [7] for other properties of the Kronecker product not mentioned here.

III. The $\mathcal{L}$-spectra of SVEV-corona and SVEE-corona

In this section, we mainly determine the normalized Laplacian spectra of *SVEV-corona* and *SVEE-corona*, respectively. For the sake of convenience, we write $\mathcal{G}$ as $G_1^S \bowtie (G_2^V \cup G_3^E)$, and $\mathcal{H}$ as $G_1^S \diamondsuit (G_2^V \cup G_3^E)$. And we respectively denote the eigenvalues of $\mathcal{L}(G_1)$, $\mathcal{L}(G_2)$ and $\mathcal{L}(G_3)$ by θ_i $(i = 1, 2, \ldots, n_1)$, μ_j $(j = 1, 2, \ldots, n_2)$ and η_k $(k = 1, 2, \ldots, n_3)$. Those symbols will be persisted in what follows.

Theorem III.1. *Let G_i be an r_i-regular graph with n_i vertices and m_i edges, where $i = 1, 2, 3$. Then*

$$\mathcal{L}(\mathcal{G}) = \begin{pmatrix} I_{n_1} & -aR(G_1) & O_{n_1 \times n_1 n_2} & O_{n_1 \times m_1 n_3} \\ -aR(G_1)^T & I_{m_1} & \boldsymbol{g}_{11} & -I_{m_1} \otimes c_{n_3}^T \\ O_{n_1 n_2 \times n_1} & \boldsymbol{g}_{12} & \boldsymbol{g}_{13} & \boldsymbol{g}_{14} \\ O_{m_1 n_3 \times n_1} & -I_{m_1} \otimes c_{n_3} & \boldsymbol{g}_{15} & \boldsymbol{g}_{16} \end{pmatrix} \quad (7)$$

where $\boldsymbol{g}_{11} = -R(G_1)^T \otimes b_{n_2}^T$, $\boldsymbol{g}_{12} = -R(G_1) \otimes b_{n_2}$, $\boldsymbol{g}_{13} = I_{n_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2))$, $\boldsymbol{g}_{14} = O_{n_1 n_2 \times m_1 n_3}$, $\boldsymbol{g}_{15} = O_{m_1 n_3 \times n_1 n_2}$, $\boldsymbol{g}_{16} = I_{m_1} \otimes (\mathcal{L}(G_3) \bullet B(G_3))$, $b_{n_2} = \frac{1}{\sqrt{(r_1+r_2)(2n_2+n_3+2)}} \mathbf{1}_{n_2}$, $B(G_2) = \frac{r_2}{r_1+r_2} J_{n_2} + \frac{r_1}{r_1+r_2} I_{n_2}$, $c_{n_3} = \frac{1}{\sqrt{(r_3+1)(2n_2+n_3+2)}} \mathbf{1}_{n_3}$, $B(G_3) = \frac{r_3}{r_3+1} J_{n_3} + \frac{1}{r_3+1} I_{n_3}$. *Moreover, J_n is a all-1 matrix of order n, and* $a = \frac{1}{\sqrt{r_1(2n_2+n_3+2)}}$ *is a constant.*

$$\mathcal{L}(\mathcal{H}) = \begin{pmatrix} I_{n_1} & -aR(G_1) & \boldsymbol{h}_{11} & -I_{n_1} \otimes c_{n_3}^T \\ -aR(G_1)^T & I_{m_1} & O_{m_1 \times m_1 n_2} & O_{m_1 \times n_1 n_3} \\ \boldsymbol{h}_{12} & O_{m_1 n_2 \times m_1} & \boldsymbol{h}_{13} & \boldsymbol{h}_{14} \\ -I_{n_1} \otimes c_{n_3} & O_{n_1 n_3 \times m_1} & \boldsymbol{h}_{15} & \boldsymbol{h}_{16} \end{pmatrix} \quad (8)$$

where $\boldsymbol{h}_{11} = -R(G_1) \otimes b_{n_2}^T$, $\boldsymbol{h}_{12} = -R(G_1)^T \otimes b_{n_2}$, $\boldsymbol{h}_{13} = I_{m_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2))$, $\boldsymbol{h}_{14} = O_{m_1 n_2 \times n_1 n_3}$, $\boldsymbol{h}_{15} = O_{n_1 n_3 \times m_1 n_2}$, $\boldsymbol{h}_{16} = I_{n_1} \otimes (\mathcal{L}(G_3) \bullet B(G_3))$, $b_{n_2} = \frac{1}{\sqrt{(r_2+2)(r_1 n_2+r_1+n_3)}} \mathbf{1}_{n_2}$, $B(G_2) = \frac{r_2}{r_2+2} J_{n_2} + \frac{2}{r_2+2} I_{n_2}$, $c_{n_3} = \frac{1}{\sqrt{(r_3+1)(r_1 n_2+r_1+n_3)}} \mathbf{1}_{n_3}$, $B(G_3) = \frac{r_3}{r_3+1} J_{n_3} + \frac{1}{r_3+1} I_{n_3}$ *and* $a = \frac{1}{\sqrt{2(r_1 n_2+r_1+n_3)}}$ *is a constant.*

Proof: The proof of Eq.(8) is similar to the Eq.(10), so it's only necessary to show the result given in Eq.(10) as follows.

We first label the vertices of $\mathcal{G}$: $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$, $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ and $V(G_3) = \{w_1, w_2, \ldots, w_{n_3}\}$. For $i = 1, 2, \ldots, n_1$, let $U_i = \{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ denote the vertices of the i-th copy of G_2 in $\mathcal{G}$, and $W_j = \{w_1^j, w_2^j, \ldots, w_{n_3}^j\}$ $(j = 1, 2, \ldots, m_1)$ the j-th copy of G_3 in $\mathcal{G}$. Then the vertices of $\mathcal{G}$ are partitioned by

$$V(G_1) \cup I(G_1) \cup (U_1 \cup U_2 \cup \ldots \cup U_{n_1}) \cup (W_1 \cup W_2 \cup \ldots \cup W_{m_1}). \quad (9)$$

By Definition I.1 we see that

$$\begin{cases} d_{\mathcal{G}}(v_i) = d_{G_1}(v_i) = r_1, & i = 1, 2, \ldots, n_1; \\ d_{\mathcal{G}}(e_i) = 2n_2 + n_3 + 2, & i = 1, 2, \ldots, m_1; \\ d_{\mathcal{G}}(u_j^i) = d_{G_2}(u_j) + d_{G_1}(v_i) & \\ \qquad = r_2 + r_1, & j = 1, \ldots, n_2, i = 1, \ldots, n_1; \\ d_{\mathcal{G}}(w_j^i) = d_{G_3}(w_j) + 1 & \\ \qquad = r_3 + 1, & j = 1, 2, \ldots, n_3, i = 1, 2, \ldots, m_1. \end{cases}$$

So one can get

$$D(\mathcal{G}) = \begin{pmatrix} r_1 I_{n_1} & & & \\ & (2n_2 + n_3 + 2) I_{m_1} & & \\ & & (r_1 + r_2) I_{n_1 n_2} & \\ & & & (r_3 + 1) I_{m_1 n_3} \end{pmatrix}.$$

Let $R(G_1)$ be the vertex-edge incidence matrix of G_1. Then, according to the ordering of (9), the adjacency matrix of $\mathcal{G}$ can be represented in the form of block-matrix below

$$A(\mathcal{G}) = \begin{pmatrix} O_{n_1 \times n_1} & R(G_1) & O_{n_1 \times n_1 n_2} & O_{n_1 \times m_1 n_3} \\ R(G_1)^T & O_{m_1 \times m_1} & R(G_1)^T \otimes \mathbf{1}_{n_2}^T & I_{m_1} \otimes \mathbf{1}_{n_3}^T \\ O_{n_1 n_2 \times n_1} & R(G_1) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes A(G_2) & \boldsymbol{g}_{14} \\ O_{m_1 n_3 \times n_1} & I_{m_1} \otimes \mathbf{1}_{n_3} & \boldsymbol{g}_{15} & I_{m_1} \otimes A(G_3) \end{pmatrix}$$

where $A(G_2)$ and $A(G_3)$ represent the adjacency matrices of G_2 and G_3, respectively.

Since G_2 is an r_2-regular graph we have $\mathcal{L}(G_2) = I_{n_2} - \frac{1}{r_2} A(G_2)$. Thus,

$$\mathcal{L}(G_2) \bullet B(G_2) = (I_{n_2} - \frac{1}{r_2} A(G_2)) \bullet B(G_2) = I_{n_2} - \frac{1}{r_1 + r_2} A(G_2).$$

By direct computation, one can obtain that

$$I_{n_1 n_2} - \frac{1}{r_1 + r_2} I_{n_1} \otimes A(G_2) = I_{n_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2)).$$

Furthermore, we can obtain that

$$I_{m_1 n_3} - \frac{1}{r_3 + 1} I_{m_1} \otimes A(G_3) = I_{m_1} \otimes (\mathcal{L}(G_3) \bullet B(G_3)).$$

By $\mathcal{L}(\mathcal{G}) = I - D(\mathcal{G})^{-1/2} A(\mathcal{G}) D(\mathcal{G})^{-1/2}$, the required normalized Laplacian matrix is given in the following:

$$\mathcal{L}(\mathcal{G}) = \begin{pmatrix} I_{n_1} & -aR(G_1) & O_{n_1 \times n_1 n_2} & O_{n_1 \times m_1 n_3} \\ -aR(G_1)^T & I_{m_1} & \boldsymbol{g}_{11} & -I_{m_1} \otimes c_{n_3}^T \\ O_{n_1 n_2 \times n_1} & \boldsymbol{g}_{12} & \boldsymbol{g}_{13} & \boldsymbol{g}_{14} \\ O_{m_1 n_3 \times n_1} & -I_{m_1} \otimes c_{n_3} & \boldsymbol{g}_{15} & \boldsymbol{g}_{16} \end{pmatrix} \quad (10)$$

The proof is completed. ■

Remark III.1. *Obviously, using the same partition of* (9) *to the graph $\mathcal{H}$, one can obtain*

$$\begin{cases} d_{\mathcal{H}}(e_i) = 2, & i = 1, 2, \ldots, m_1; \\ d_{\mathcal{H}}(u_j^i) = d_{G_2}(u_j) + 2 & \\ \qquad = r_2 + 2, & j = 1, 2, \ldots, n_2, i = 1, 2, \ldots, m_1; \\ d_{\mathcal{H}}(w_j^i) = d_{G_3}(w_j) + 1 & \\ \qquad = r_3 + 1, & j = 1, 2, \ldots, n_3, i = 1, 2, \ldots, n_1; \\ d_{\mathcal{H}}(v_i) = (n_2 + 1) d_{G_1}(v_i) + n_3 & \\ \qquad = (n_2 + 1) r_1 + n_3, & i = 1, 2, \ldots, n_1; \end{cases}$$

and

$$A(\mathcal{H}) = \begin{pmatrix} O_{n_1 \times n_1} & R(G_1) & \boldsymbol{h}_{21} & I_{n_1} \otimes \boldsymbol{1}_{n_3}^T \\ R^T(G_1) & O_{m_1 \times m_1} & I_{m_1} \otimes \boldsymbol{0}_{n_2}^T & R^T(G_1) \otimes \boldsymbol{0}_{n_3}^T \\ \boldsymbol{h}_{22} & I_{m_1} \otimes \boldsymbol{0}_{n_2} & I_{m_1} \otimes A(G_2) & \boldsymbol{h}_{23} \\ I_{n_1} \otimes \boldsymbol{1}_{n_3} & R(G_1) \otimes \boldsymbol{0}_{n_3} & \boldsymbol{h}_{24} & I_{n_1} \otimes A(G_3) \end{pmatrix}.$$

where $\boldsymbol{h}_{21} = R(G_1) \otimes \boldsymbol{1}_{n_2}^T$, $\boldsymbol{h}_{22} = R^T(G_1) \otimes \boldsymbol{1}_{n_2}$, $\boldsymbol{h}_{23} = R^T(G_1) \otimes O_{n_2 \times n_3}^T$, $\boldsymbol{h}_{24} = R(G_1) \otimes O_{n_3 \times n_2}$. *By immediate calculation, the $\mathcal{L}(\mathcal{H})$ follows.*

Theorem III.2. *Let G_i be an r_i-regular graph with n_i vertices and m_i edges, where $i = 1, 2, 3$. Then*

$$\begin{aligned}
\text{(i) } \Phi_{\mathcal{L}(\mathcal{G})}(\lambda) =& (\lambda-1-\frac{n_3}{(r_3+1)(2n_2+n_3+2)(\lambda-\frac{1}{r_3+1})})^{m_1-n_1} \\
&\times \prod_{j=1}^{n_2}(\lambda-\frac{r_1+r_2\mu_j}{r_1+r_2})^{n_1}\cdot\prod_{k=1}^{n_3}(\lambda-\frac{1+r_3\eta_k}{r_3+1})^{m_1} \\
&\times \prod_{i=1}^{n_1}\Big((\lambda-1)(\lambda-1-\frac{n_3}{(r_3+1)(2n_2+n_3+2)(\lambda-\frac{1}{r_3+1})}) \\
&-\frac{(2-\theta_i)((r_1+r_2+n_2r_1)\lambda-(1+n_2)r_1)}{(r_1+r_2)(2n_2+n_3+2)(\lambda-\frac{r_1}{r_1+r_2})}\Big);
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } \Phi_{\mathcal{L}(\mathcal{H})}(\lambda) =& (\lambda-1)^{m_1-n_1}\cdot\prod_{j=1}^{n_2}(\lambda-\frac{2+r_2\mu_j}{r_2+2})^{m_1} \\
&\times \prod_{k=1}^{n_3}(\lambda-\frac{1+r_3\eta_k}{r_3+1})^{n_1}\cdot\prod_{i=1}^{n_1}\Big((\lambda-1) \\
&\times(\lambda-1-\frac{n_3}{(r_3+1)(r_1n_2+n_3+r_1)(\lambda-\frac{1}{r_3+1})}) \\
&-\frac{r_1(2-\theta_i)((2n_2+r_2+2)\lambda-2n_2-2)}{2(r_2+2)(r_1n_2+n_3+r_1)(\lambda-\frac{2}{r_2+2})}\Big)
\end{aligned}$$

where θ_i, μ_j and η_k are the eigenvalues of $\mathcal{L}(G_1)$, $\mathcal{L}(G_2)$ and $\mathcal{L}(G_3)$, respectively.

Proof: According to Theorem III.1, the normalized Laplacian characteristic polynomial of $\mathcal{G}$ is given by

$$\Phi_{\mathcal{L}(\mathcal{G})}(\lambda) = \det(\lambda I_n - \mathcal{L}(\mathcal{G})) = \det(B_0)$$

where

$$B_0 = \begin{pmatrix} (\lambda-1)I_{n_1} & aR(G_1) & O & O \\ aR(G_1)^T & (\lambda-1)I_{m_1} & R(G_1)^T\otimes b_{n_2}^T & I_{m_1}\otimes c_{n_3}^T \\ O & R(G_1)\otimes b_{n_2} & \boldsymbol{g}_{21} & O \\ O & I_{m_1}\otimes c_{n_3} & O & \boldsymbol{g}_{22} \end{pmatrix},$$

$\boldsymbol{g}_{21} = I_{n_1}\otimes(\lambda I_{n_2} - \mathcal{L}(G_2)\bullet B(G_2))$ and $\boldsymbol{g}_{22} = I_{m_1}\otimes(\lambda I_{n_3} - \mathcal{L}(G_3)\bullet B(G_3))$.

We write P as the elementary block matrix below

$$P = \begin{pmatrix} I_{n_1} & O & O & O \\ O & I_{m_1} & \boldsymbol{g}_{31} & \boldsymbol{g}_{32} \\ O & O & I_{n_1}\otimes I_{n_2} & O \\ O & O & O & I_{m_1}\otimes I_{n_3} \end{pmatrix}$$

where $\boldsymbol{g}_{31} = -R(G_1)^T\otimes(b_{n_2}^T(\lambda I_{n_2} - \mathcal{L}(G_2)\bullet B(G_2))^{-1})$ and $\boldsymbol{g}_{32} = -I_{m_1}\otimes(c_{n_3}^T(\lambda I_{n_3} - \mathcal{L}(G_3)\bullet B(G_3))^{-1})$.

Let $B = PB_0$. Then

$$B = \begin{pmatrix} (\lambda-1)I_{n_1} & aR(G_1) & O & O \\ aR(G_1)^T & \boldsymbol{g}_{41} & O & O \\ O & R(G_1)\otimes b_{n_2} & \boldsymbol{g}_{42} & O \\ O & I_{m_1}\otimes c_{n_3} & O & \boldsymbol{g}_{43} \end{pmatrix}$$

where $\Gamma_2(\lambda) = b_{n_2}^T(\lambda I_{n_2} - \mathcal{L}(G_2)\bullet B(G_2))^{-1}b_{n_2}$, $\Gamma_3(\lambda) = c_{n_3}^T(\lambda I_{n_3} - \mathcal{L}(G_3)\bullet B(G_3))^{-1}c_{n_3}$, $\boldsymbol{g}_{41} = (\lambda-1-\Gamma_3(\lambda))I_{m_1} - \Gamma_2(\lambda)R(G_1)^TR(G_1)$, $\boldsymbol{g}_{42} = I_{n_1}\otimes(\lambda I_{n_2} - \mathcal{L}(G_2)\bullet B(G_2))$ and $\boldsymbol{g}_{43} = I_{m_1}\otimes(\lambda I_{n_3} - \mathcal{L}(G_3)\bullet B(G_3))$.

Note that $\det(P) = 1$. So we have

$$\Phi_{\mathcal{L}(\mathcal{G})}(\lambda) = \det(B_0) = \det(P^{-1})\det(B) = \det(B).$$

For the matrix B, one can get

$$\begin{aligned}
\det(B) =& \det\big(I_{n_1}\otimes(\lambda I_{n_2} - \mathcal{L}(G_2)\bullet B(G_2))\big) \\
&\times\det\big(I_{m_1}\otimes(\lambda I_{n_3} - \mathcal{L}(G_3)\bullet B(G_3))\big)\cdot\det(S_1)
\end{aligned}$$

where

$$S_1 = \begin{pmatrix} (\lambda-1)I_{n_1} & aR(G_1) \\ aR(G_1)^T & (\lambda-1-\Gamma_3(\lambda))I_{m_1} - \Gamma_2(\lambda)R(G_1)^TR(G_1) \end{pmatrix}.$$

Let θ_i, μ_j and η_k be the eigenvalues of $\mathcal{L}(G_1)$, $\mathcal{L}(G_2)$ and $\mathcal{L}(G_3)$, respectively. By applying Eq.(6), the following result follows from Corollary II.1 that

$$\begin{aligned}
\det(S_1) =& \begin{vmatrix} (\lambda-1)I_{n_1} & aR(G_1) \\ aR(G_1)^T & (\lambda-1-\Gamma_3(\lambda))I_{m_1} - \Gamma_2(\lambda)R(G_1)^TR(G_1) \end{vmatrix} \\
=& \det((\lambda-1)I_{n_1})\cdot\det((\lambda-1-\Gamma_3(\lambda))I_{m_1} \\
&-(\Gamma_2(\lambda)+\tfrac{a^2}{\lambda-1})\cdot R(G_1)^TR(G_1)) \\
=& (\lambda-1)^{n_1}\det\big((\lambda-1-\Gamma_3(\lambda))I_{m_1} \\
&-(\Gamma_2(\lambda)+\tfrac{a^2}{\lambda-1})(A(\ell(G_1))+2I_{m_1})\big) \\
=& (\lambda-1)^{n_1}\cdot\big((\lambda-1-\Gamma_3(\lambda)) \\
&-(\Gamma_2(\lambda)+\tfrac{a^2}{\lambda-1})\cdot(-2+2)\big)^{m_1-n_1} \\
&\times\det\big((\lambda-1-\Gamma_3(\lambda))I_{n_1} - (\Gamma_2(\lambda)+\tfrac{a^2}{\lambda-1}) \\
&\times(2r_1I_{n_1} - r_1\mathcal{L}(G_1))\big) \\
=& (\lambda-1-\Gamma_3(\lambda))^{m_1-n_1}\cdot\det\big((\lambda-1)(\lambda-1-\Gamma_3(\lambda))I_{n_1} \\
&-r_1((\lambda-1)\Gamma_2(\lambda)+a^2)(2I_{n_1}-\mathcal{L}(G_1))\big) \\
=& (\lambda-1-\Gamma_3(\lambda))^{m_1-n_1}\cdot\prod_{i=1}^{n_1}\big((\lambda-1)(\lambda-1-\Gamma_3(\lambda)) \\
&-r_1((\lambda-1)\Gamma_2(\lambda)+\tfrac{1}{r_1(2n_2+n_3+2)})(2-\theta_i)\big).
\end{aligned}$$

Since $\mathcal{L}(G_2)\bullet B(G_2) = I_{n_2} - \frac{1}{r_1+r_2}A(G_2)$ and $A(G_2) = r_2(I_{n_2} - \mathcal{L}(G_2))$ we get $\mathcal{L}(G_2)\bullet B(G_2) = \frac{1}{r_1+r_2}(r_1I_{n_2} + r_2\mathcal{L}(G_2))$. Similarly, $\mathcal{L}(G_3)\bullet B(G_3) = \frac{1}{r_3+1}(I_{n_3} + r_3\mathcal{L}(G_3))$.

Also since G_2 is r_2-regular, the sum of all entries on every row of its normalized Laplacian matrix is zero. In other words, $\mathcal{L}(G_2)b_{n_2} = (1-\frac{r_2}{r_2})b_{n_2} = 0\cdot b_{n_2} = \mathbf{0}$. Then $(\mathcal{L}(G_2)\bullet B(G_2))b_{n_2} = (1-\frac{r_2}{r_1+r_2})b_{n_2} = \frac{r_1}{r_1+r_2}b_{n_2}$ and $(\lambda I_{n_2} - (\mathcal{L}(G_2)\bullet B(G_2)))b_{n_2} = (\lambda - \frac{r_1}{r_1+r_2})b_{n_2}$. In addition, $b_{n_2}^Tb_{n_2} = \frac{n_2}{(r_1+r_2)(2n_2+n_3+2)}$. Thus

$$\begin{aligned}
\Gamma_2(\lambda) &= b_{n_2}^T(\lambda I_{n_2} - \mathcal{L}(G_2)\bullet B(G_2))^{-1}b_{n_2} = \frac{b_{n_2}^Tb_{n_2}}{\lambda-\frac{r_1}{r_1+r_2}} \\
&= \frac{n_2}{(r_1+r_2)(2n_2+n_3+2)(\lambda-\frac{r_1}{r_1+r_2})}
\end{aligned}$$

We notice that the value of $\Gamma_3(\lambda)$ is similar to that of $\Gamma_2(\lambda)$, and so,

$$\begin{aligned}
\Gamma_3(\lambda) &= c_{n_3}^T(\lambda I_{n_3} - \mathcal{L}(G_3)\bullet B(G_3))^{-1}c_{n_3} = \frac{c_{n_3}^Tc_{n_3}}{\lambda-\frac{1}{r_3+1}} \\
&= \frac{n_3}{(r_3+1)(2n_2+n_3+2)(\lambda-\frac{1}{r_3+1})}
\end{aligned}$$

In summary, the normalized Laplacian characteristic polynomial of $\mathcal{G}$ is

$$\begin{aligned}
\Phi_{\mathcal{L}(\mathcal{G})}(\lambda) =& \prod_{j=1}^{n_2}(\lambda-\tfrac{r_1+r_2\mu_j}{r_1+r_2})^{n_1}\cdot\prod_{k=1}^{n_3}(\lambda-\tfrac{1+r_3\eta_k}{r_3+1})^{m_1}\cdot\det(S_1) \\
=& (\lambda-1-\frac{n_3}{(r_3+1)(2n_2+n_3+2)(\lambda-\frac{1}{r_3+1})})^{m_1-n_1} \\
&\times\prod_{j=1}^{n_2}(\lambda-\tfrac{r_1+r_2\mu_j}{r_1+r_2})^{n_1}\cdot\prod_{k=1}^{n_3}(\lambda-\tfrac{1+r_3\eta_k}{r_3+1})^{m_1} \\
&\times\prod_{i=1}^{n_1}\big((\lambda-1)(\lambda-1-\tfrac{n_3}{(r_3+1)(2n_2+n_3+2)(\lambda-\frac{1}{r_3+1})}) \\
&-\frac{(2-\theta_i)((r_1+r_2+n_2r_1)\lambda-(1+n_2)r_1)}{(r_1+r_2)(2n_2+n_3+2)(\lambda-\frac{r_1}{r_1+r_2})}\big),
\end{aligned}$$

as required.

The proof of (ii) is similar to (i), and is omitted. ■

Remark III.2. *In Theorem III.2, if one of graphs G_2 and G_3 is null in $\mathcal{G}$ or $\mathcal{H}$, then one can directly deduce the main results(see Theorems 2.2-2.5) due to A. Das and P. Panigrahi in [19]. For the simplicity, we here omit these corollaries (see Theorems 2.2-2.5 in [19]).*

From Theorem III.2, one can easily obtain the following corollaries.

Corollary III.1. *Let G_i be an r_i-regular graph with n_i vertices and m_i edges for $i = 1, 2, 3$. Then the normalized Laplacian spectrum of $\mathcal{G}$ consist of:*

(a) $\frac{r_1+r_2\mu_j}{r_1+r_2}$ *repeats n_1 times for each eigenvalue μ_j of $\mathcal{L}(G_2)$, $j = 1, 2, \ldots, n_2 - 1$;*
(b) $\frac{1+r_3\eta_k}{r_3+1}$ *repeats m_1 times for each eigenvalue η_k of $\mathcal{L}(G_3)$, $k = 1, 2, \ldots, n_3 - 1$;*
(c) *two roots of the equation*

$$(2n_2r_3 + n_3r_3 + 2r_3 + 2n_2 + n_3 + 2)\lambda^2 - (2n_2r_3 + n_3r_3 + 2r_3 + 4n_2 + 2n_3 + 4)\lambda + 2n_2 + 2 = 0 \quad (11)$$

where each root repeats $m_1 - n_1$ times;
(d) *four roots of the equation*

$$(2n_2+n_3+2)\big((1+r_3)\lambda-1\big)\big((r_1+r_2)\lambda-r_1\big)(\lambda-1)^2 - n_3(\lambda-1)\big((r_1+r_2)\lambda-r_1\big) - (2-\theta_i)\big((r_1+r_2+n_2r_1)\lambda-(r_1+r_1n_2)\big)\big((1+r_3)\lambda-1\big) = 0 \quad (12)$$

where each eigenvalue θ_i of $\mathcal{L}(G_1)$, $i = 1, 2, \ldots, n_1$.

Corollary III.2. *If G_i is an r_i-regular graph with n_i vertices and m_i edges for $i = 1, 2, 3$, the normalized Laplacian spectrum of $\mathcal{H}$ consist of:*

(a) $\frac{2+r_2\mu_j}{r_2+2}$ *repeats m_1 times for each eigenvalue μ_j of $\mathcal{L}(G_2)$, $j = 1, 2, \ldots, n_2 - 1$;*
(b) $\frac{1+r_3\eta_k}{r_3+1}$ *repeats n_1 times for each eigenvalue η_k of $\mathcal{L}(G_3)$, $k = 1, 2, \ldots, n_3 - 1$;*
(c) 1 *repeats $m_1 - n_1$ times,* $\frac{2}{r_2+2}$ *repeats $m_1 - n_1$ times;*
(d) *four roots of the equation*

$$2(r_1n_2 + n_3 + r_1)((1 + r_3)\lambda - 1)((2 + r_2)\lambda - 2)(\lambda - 1)^2 - 2n_3(\lambda - 1)((2 + r_2)\lambda - 2) - r_1(2 - \theta_i)\big((2n_2 + r_2 + 2)\lambda - 2n_2 - 2\big)((1 + r_3)\lambda - 1) = 0,$$

where each eigenvalue θ_i of $\mathcal{L}(G_1)$, $i = 1, 2, \ldots, n_1$.

Example III.1. *Let $G = C_4^S \bowtie (K_2^V \cup K_2^E)$ and $H = C_4^S \diamondsuit (K_2^V \cup K_1^E)$ (shown in Fig.2). By simple computation, $Spec_{\mathcal{L}}(C_4) = \{0, 1, 1, 2\}$, $Spec_{\mathcal{L}}(K_2) = \{0, 2\}$ and $Spec_{\mathcal{L}}(K_1) = \{0\}$.*

From Corollary III.1, the normalized Laplacian spectrum of G consist of: $\frac{4}{3}$ (multiplicity 4), $\frac{3}{2}$ (multiplicity 4), four roots of the equation $24x^4 - 76x^3 + 71x^2 - 20x = 0$, four roots (multiplicity 2) of the equation $48x^4 - 152x^3 + 156x^2 - 59x + 6 = 0$, and four roots of the equation $48x^4 - 152x^3 + 170x^2 - 78x + 12 = 0$.

From Corollary III.2, the normalized Laplacian spectrum of H consist of: $\frac{4}{3}$ (multiplicity 4), four roots of the equation $42x^4 - 154x^3 + 176x^2 - 64x = 0$, four roots (multiplicity 2) of the equation $42x^4 - 154x^3 + 190x^2 - 90x + 12 = 0$, and four roots of the equation $42x^4 - 154x^3 + 204x^2 - 116x + 24 = 0$.

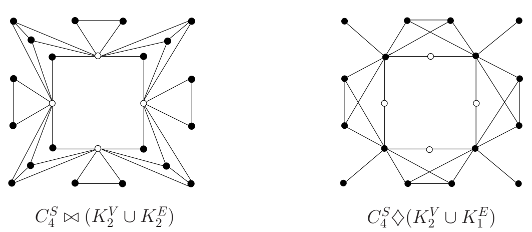

Fig. 2. $G = C_4^S \bowtie (K_2^V \cup K_2^E)$ and $H = C_4^S \diamondsuit (K_2^V \cup K_1^E)$

IV. Applications

In this section, we will give four distinct applications, such as construction for $\mathcal{L}$-cospectral graphs, computation for the number of *spanning trees*, *the multiplicative degree-Kirchhoff index* and *Kemeny's constant* on SVEV-corona and SVEE-corona respectively.

A. Construct $\mathcal{L}$-cospectral graphs

In [21], Dam and Haemers have proposed *'which graphs are determined by their spectra?'*. The question for the *normalized Laplacian* spectrum is also one of the outstanding unsolved problems in the theory of graph spectra. Thus, if one wish to settle the question for graphs in general, it is natural to look for constructing pairs of $\mathcal{L}$-cospectral graphs. In this section, we will respectively construct many infinite families of pairs of $\mathcal{L}$-cospectral graphs from *SVEV-corona* and *SVEE-corona*, which are generalized Theorem 2.7 due to Das and Panigrahi in [19].

Theorem IV.1. *Let G_i and H_i (not necessarily distinct isomorphic) are pairwise $\mathcal{L}$-cospectral regular graphs for $i = 1, 2, 3$. Then*

(1) $G_1^S \bowtie (G_2^V \cup G_3^E)$ *and* $H_1^S \bowtie (H_2^V \cup H_3^E)$ *are $\mathcal{L}$-cospectral graphs;*
(2) $G_1^S \diamondsuit (G_2^V \cup G_3^E)$ *and* $H_1^S \diamondsuit (H_2^V \cup H_3^E)$ *are $\mathcal{L}$-cospectral graphs.*

Proof: From Theorem III.2 we know that, the normalized Laplacian spectra of $G_1^S \bowtie (G_2^V \cup G_3^E)$ and $G_1^S \diamondsuit (G_2^V \cup G_3^E)$ are completely determined by the degrees of regularities, the number of vertices, the number of edges and the normalized Laplacian spectra of regular graphs G_i $(i = 1, 2, 3)$. So the conclusions follows. ■

Example IV.1. *Let G_1 and H_1 be two graphs shown in Fig.3. Then by Matlab 7.0 one can get $\Phi_{A(G_1)}(x) = \Phi_{A(H_1)}(x) = x^{14} - 21x^{12} - 2x^{11} + 164x^{10} + 22x^9 - 599x^8 - 88x^7 + 1047x^6 + 168x^5 - 800x^4 - 160x^3 + 216x^2 + 40x - 12$. It is easy to see that G_1 and H_1 are A-cospectral but not isomorphic with each other. Note that $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$. And so, two graphs are A-cospectral implies that they are $\mathcal{L}$-cospectral. Consequently, it follows from Theorem IV.1 that $G_1^S \bowtie (K_3^V \cup K_2^E)$ and $H_1^S \bowtie (K_3^V \cup K_2^E)$ are $\mathcal{L}$-cospectral graphs, so are $G_1^S \diamondsuit (K_3^V \cup K_2^E)$ and $H_1^S \diamondsuit (K_3^V \cup K_2^E)$, see Fig.4 and Fig.5 for instance.*

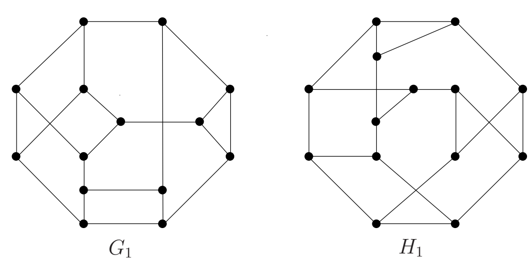

Fig. 3. G_1 and H_1

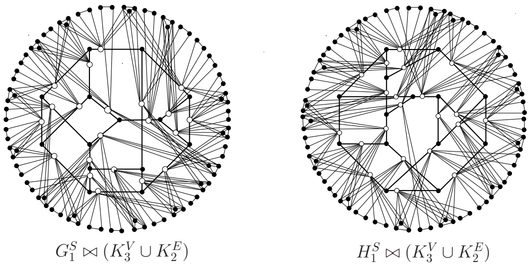

Fig. 4. $G_1^S \bowtie (K_3^V \cup K_2^E)$ and $H_1^S \bowtie (K_3^V \cup K_2^E)$

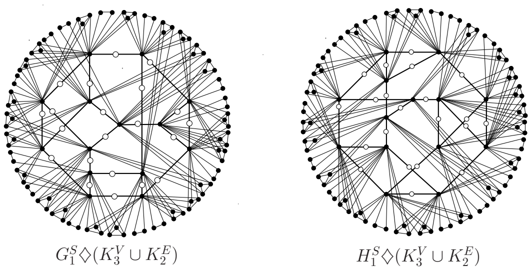

Fig. 5. $G_1^S \diamondsuit (K_3^V \cup K_2^E)$ and $H_1^S \diamondsuit (K_3^V \cup K_2^E)$

B. The number of spanning trees

Let G be a connected graph of order n. A *spanning tree* is a spanning subgraph of G that is a tree. A known result from Chung [1] allows the calculation of this number from the normalized Laplacian spectrum and the degrees of all the vertices, thus the number of spanning trees $\tau(G)$ of connected graph G is

$$\tau(G) = \frac{\Pi_{i=1}^{n} d_i \Pi_{i=1}^{n-1} \lambda_i}{\sum_{i=1}^{n} d_i}. \tag{13}$$

Theorem IV.2. *Let G_i be an r_i-regular graph with n_i vertices and m_i edges for $i = 1, 2, 3$. Then*

(1) $$\tau(\mathcal{G}) = \prod_{i=1}^{n_1-1} \theta_i \cdot \prod_{j=1}^{n_2-1} (r_1 + r_2\mu_j)^{n_1} \cdot \prod_{k=1}^{n_3-1} (1 + r_3\eta_k)^{m_1} \cdot r_1^{2n_1-1} \times (1+n_2)^{m_1-1} \times 2^{m_1-n_1-1} \times \frac{n_3r_1r_3 + 4n_2r_1 + 2n_2r_2 + 2n_3r_1 + 4r_1}{2m_1 + n_1m_2 + m_1m_3 + 2m_1n_2 + m_1n_3}.$$

(2) $$\tau(\mathcal{H}) = 2^{2m_1-n_1-1} \cdot \prod_{i=1}^{n_1-1} (r_1 + r_1n_2)\theta_i \cdot \prod_{j=1}^{n_2-1} (2 + r_2\mu_j)^{m_1} \times \prod_{k=1}^{n_3-1} (1 + r_3\eta_k)^{n_1} \times \frac{n_2r_1r_2 + 4n_2r_1 + 2n_3r_3 + 4n_3 + 4r_1}{2m_1 + m_1m_2 + n_1m_3 + n_1n_3 + 2m_1n_2}.$$

Proof: The proof of (2) is similar to the proof of (1), it is here need to prove (1). We first consider the normalized Laplacian eigenvalues of $\mathcal{G}$ in the following way:

In Corollary III.1 (c), one can by the well-known *Vieta Theorem* obtain the relation of the two roots α_1 and α_2 of Eq.(11) such that

$$\alpha_1\alpha_2 = \frac{2n_2 + 2}{(r_3 + 1)(2n_2 + n_3 + 2)}. \tag{14}$$

In Corollary III.1 (d), let $\beta_1, \beta_2, \beta_3$ and β_4 be the four roots of Eq.(12) for each $\theta_i, i = 1, 2, \ldots, n_1 - 1$. Then

$$\beta_1\beta_2\beta_3\beta_4 = \frac{r_1(1 + n_2)\theta_i}{(r_1 + r_2)(1 + r_3)(2n_2 + n_3 + 2)}. \tag{15}$$

For $i = n_1$, we notice that $\theta_{n_1} = 0$. By Eq.(12) one can get

$$\begin{aligned}&(2n_2r_1r_3 + 2n_2r_2r_3 + n_3r_1r_3 + n_3r_2r_3 + 2n_2r_1 + 2n_2r_2\\ &+n_3r_1 + n_3r_2 + 2r_1r_3 + 2r_2r_3 + 2r_1 + 2r_2)\lambda^4\\ &-(6n_2r_1r_3 + 4n_2r_2r_3 + 3n_3r_1r_3 + 2n_3r_2r_3 + 8n_2r_1\\ &+6n_2r_2 + 4n_3r_1 + 3n_3r_2 + 6r_1r_3 + 4r_2r_3 + 8r_1 + 6r_2)\lambda^3\\ &+(4n_2r_1r_3 + 2n_2r_2r_3 + 3n_3r_1r_3 + n_3r_2r_3 + 10n_2r_1\\ &+6n_2r_2 + 5n_3r_1 + 2n_3r_2 + 4r_1r_3 + 10r_1 + 4r_2)\lambda^2\\ &-(n_3r_1r_3 + 4n_2r_1 + 2n_2r_2 + 2n_3r_1 + 4r_1)\lambda = 0.\end{aligned} \tag{16}$$

Suppose that γ_1, γ_2 and γ_3 are three non-zero roots of Eq.(16). Then by *Vieta Theorem*,

$$\gamma_1\gamma_2\gamma_3 = \frac{n_3r_1r_3 + 4n_2r_1 + 2n_2r_2 + 2n_3r_1 + 4r_1}{(r_1 + r_2)(1 + r_3)(2n_2 + n_3 + 2)}. \tag{17}$$

In light of Corollary III.1, Eqs.(14), (15) and (17) we see that

$$\begin{aligned}\tau(\mathcal{G}) &= \frac{\prod_{i=1}^{n} d_i \prod_{i=1}^{n-1} \lambda_i}{\sum_{i=1}^{n} d_i}\\ &= \frac{(r_1)^{n_1}(2n_2 + n_3 + 2)^{m_1}(r_1 + r_2)^{n_1n_2}(r_3 + 1)^{m_1n_3}}{2(2m_1 + n_1m_2 + m_1m_3 + 2m_1n_2 + m_1n_3)}\\ &\times \prod_{k=1}^{n_3-1} (\frac{1 + r_3\eta_k}{r_3 + 1})^{m_1} \cdot \prod_{i=1}^{n_1-1} \frac{r_1(1 + n_2)\theta_i}{(r_1 + r_2)(1 + r_3)(2n_2 + n_3 + 2)}\\ &\times \prod_{j=1}^{n_2-1} (\frac{r_1 + r_2\mu_j}{r_1 + r_2})^{n_1} \cdot (\frac{2n_2 + 2}{(r_3 + 1)(2n_2 + n_3 + 2)})^{m_1-n_1}\\ &\times \frac{n_3r_1r_3 + 4n_2r_1 + 2n_2r_2 + 2n_3r_1 + 4r_1}{(r_1 + r_2)(r_3 + 1)(2n_2 + n_3 + 2)}\\ &= \prod_{i=1}^{n_1-1} \theta_i \cdot \prod_{j=1}^{n_2-1} (r_1 + r_2\mu_j)^{n_1} \cdot \prod_{k=1}^{n_3-1} (1 + r_3\eta_k)^{m_1}\\ &\times r_1^{2n_1-1} \cdot (1 + n_2)^{m_1-1} \cdot 2^{m_1-n_1-1}\\ &\times \frac{n_3r_1r_3 + 4n_2r_1 + 2n_2r_2 + 2n_3r_1 + 4r_1}{2m_1 + n_1m_2 + m_1m_3 + 2m_1n_2 + m_1n_3},\end{aligned}$$

as required. ■

Example IV.2. *Let $G = C_4^S \bowtie (K_2^V \cup K_2^E)$ and $H = C_4^S \diamondsuit (K_2^V \cup K_1^E)$ (shown in Fig.2). It is easy to see that $\prod_{i=1}^{n_1-1} \theta_i = 2$, $\prod_{j=1}^{n_2-1} (r_1 + r_2\mu_j)^{n_1} = 4^4 = 2^8$, $\prod_{k=1}^{n_3-1} (1 + r_3\eta_k)^{m_1} \cdot r_1^{2n_1-1} = 3^4 \cdot 2^7$, $(1 + n_2)^{m_1-1} \cdot 2^{m_1-n_1-1} = 3^3 \cdot 2^{-1}$, $n_3r_1r_3 + 4n_2r_1 + 2n_2r_2 + 2n_3r_1 + 4r_1 = 40$, $2m_1 + n_1m_2 + m_1m_3 + 2m_1n_2 + m_1n_3 = 40$. Thus, by Theorem IV.2 (1) we get $\tau(G) = 2^{15} \cdot 3^7$. On the other hand, combining with the Example III.1 and Eq.(13), one can easily obtain that $\tau(G) = 2^{15} \cdot 3^7$. Similarly, $\tau(H) = 2^{15} \cdot 3^3$.*

C. The multiplicative degree-Kirchhoff index

In [22], the *multiplicative degree-Kirchhoff index* of G is defined as

$$Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}$$

by Chen and Zhang, where r_{ij} is the resistance between i and j. This index is distinct the classical *Kirchhoff index* $Kf(G) = \sum_{i<j} r_{ij}$ since it takes into account the degree distribution of G. Meanwhile, they also have been proved that $Kf^*(G)$ can be obtained from the non-zero normalized Laplacian eigenvalues of G, i.e.,

$$Kf^*(G) = 2m \cdot \sum_{i=1}^{n-1} \frac{1}{\lambda_i}. \tag{18}$$

Theorem IV.3. *Let G_i be an r_i-regular graph with n_i vertices and m_i edges for $i = 1, 2, 3$. Then*

(1)
$$\begin{aligned} Kf^*(\mathcal{G}) =& 2(2m_1 + n_1m_2 + m_1m_3 + 2m_1n_2 + m_1n_3) \\ &\times\Big(\sum_{j=1}^{n_2-1}\frac{n_1(r_1+r_2)}{r_1+r_2\mu_j} + \sum_{k=1}^{n_3-1}\frac{m_1(r_3+1)}{1+r_3\eta_k} \\ &+\frac{(m_1-n_1)(r_3+2)(2n_2+n_3+2)}{2n_2+2} \\ &+\frac{(2n_2+n_3+2)(6r_1+3r_2+3r_1r_3+r_2r_3)-\boldsymbol{g}_{51}}{(4r_1+r_2+r_1r_3)(2n_2+n_3+2)-\boldsymbol{g}_{52}} \\ &+\sum_{i=1}^{n_1-1}\frac{(4r_1+r_2+r_1r_3)(2n_2+n_3+2)-n_3(2r_1+r_2)}{r_1(1+n_2)\theta_i} \\ &-\frac{(2-\theta_i)\big((1+n_2)(2r_1+r_1r_3)+r_2\big)}{r_1(1+n_2)\theta_i}\Big). \end{aligned}$$

where $\boldsymbol{g}_{51} = n_3(r_1 + r_2) + 2(r_3 + 1)(r_1 + r_2 + r_1n_2)$, $\boldsymbol{g}_{52} = (2r_1n_3 + r_2n_3 + 2r_2) + (2 + 2n_2)(2r_1 + r_1r_3)$,

(2)
$$\begin{aligned} Kf^*(\mathcal{H}) =& 2(2m_1 + m_1m_2 + n_1m_3 + n_1n_3 + 2m_1n_2) \\ &\times\Big(\sum_{j=1}^{n_2-1}\frac{m_1(r_2+2)}{2+r_2\mu_j} + \sum_{k=1}^{n_3-1}\frac{n_1(r_3+1)}{1+r_3\eta_k} + \frac{(r_2+4)(m_1-n_1)}{2} \\ &+\frac{2(r_1n_2+r_1+n_3)(r_2r_3+6r_3+3r_2+12)-n_3(2r_2+4)}{2n_2r_1r_2+8n_2r_1+4n_3r_3+8n_3+8r_1} \\ &-\frac{2r_1(r_3+1)(2n_2+r_2+2)}{2n_2r_1r_2+8n_2r_1+4n_3r_3+8n_3+8r_1} \\ &+\sum_{i=1}^{n_1-1}\Big(\frac{2(r_1n_2+r_1+n_3)(2r_3+r_2+8)-(2r_2n_3+8n_3)}{r_1(2+2n_2)\theta_i} \\ &-\frac{r_1(2-\theta_i)((2+2n_2)(2+r_3)+r_2)}{r_1(2+2n_2)\theta_i}\Big)\Big). \end{aligned}$$

Proof: From Eq.(18), $Kf^*(\mathcal{G})$ can be computed from the following way:

In Corollary III.1 (c), let α_1 and α_2 be the two eigenvalues of equation (11). Then by *Vieta Theorem*, we have

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2} = \frac{(r_3+2)(2n_2+n_3+2)}{2n_2+2}.$$

In Corollary III.1 (d), for each $\theta_i (i = 2, 3, \ldots, n_1)$, let β_1, β_2, β_3 and β_4 be the eigenvalues of Eq.(12). By *Vieta Theorem*, we have

$$\begin{aligned} &\frac{1}{\beta_1}+\frac{1}{\beta_2}+\frac{1}{\beta_3}+\frac{1}{\beta_4} \\ &= \frac{\beta_2\beta_3\beta_4+\beta_1\beta_3\beta_4+\beta_1\beta_2\beta_4+\beta_1\beta_2\beta_3}{\beta_1\beta_2\beta_3\beta_4} \\ &= \frac{(4r_1+r_2+r_1r_3)(2n_2+n_3+2)-n_3(2r_1+r_2)}{r_1(1+n_2)\theta_i} \\ &-\frac{(2-\theta_i)\big((1+n_2)(2r_1+r_1r_3)+r_2\big)}{r_1(1+n_2)\theta_i}. \end{aligned}$$

Note that $\theta_{n_1} = 0$. Then Eq.(12) is equal to Eq.(16). Let γ_1, γ_2 and γ_3 be the non-zero eigenvalues of Eq.(16). Then

$$\begin{aligned} &\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\frac{1}{\gamma_3} = \frac{\gamma_2\gamma_3+\gamma_1\gamma_3+\gamma_1\gamma_2}{\gamma_1\gamma_2\gamma_3} \\ &= \frac{(2n_2+n_3+2)(6r_1+3r_2+3r_1r_3+r_2r_3)-\boldsymbol{g}_{51}}{(4r_1+r_2+r_1r_3)(2n_2+n_3+2)-\boldsymbol{g}_{52}} \end{aligned}$$

In summary above, the result of (1) follows. Similarly, (2) can be obtained also. ∎

Example IV.3. *Let $G = C_4^S \bowtie (K_2^V \cup K_2^E)$ and $H = C_4^S \diamondsuit (K_2^V \cup K_1^E)$ (shown in Fig.2). By Theorem IV.3, $Kf^*(G) = \frac{2123\times80}{60} = \frac{8492}{3}$. On the other hand, combining with Example III.1 and Eq.(18), one can also obtain that $Kf^*(G) = \frac{8492}{3}$. Similarly, $Kf^*(H) = \frac{307\times64}{12} = \frac{4912}{3}$.*

D. Kemeny's constant

For a graph G, *Kemeny's constant* $K(G)$, also known as average hitting time, is the expected number of steps required for the transition from a starting vertex i to a destination vertex, which is chosen randomly according to a stationary distribution of unbiased random walks on G, see [23] for more details. From literature [24] we know that

$$K(G) = \sum_{i=1}^{n-1}\frac{1}{\lambda_i}.$$

Note that $Kf^*(G) = 2m \cdot K(G)$. Thus, the following result follows from Theorem IV.3 immediately.

Theorem IV.4. *Let G_i be an r_i-regular graph with n_i vertices and m_i edges, where $i = 1, 2, 3$. Then*

(1)
$$\begin{aligned} K(\mathcal{G}) =& \sum_{j=1}^{n_2-1}\frac{n_1(r_1+r_2)}{r_1+r_2\mu_j} + \sum_{i=1}^{n_1-1}\frac{(4r_1+r_2+r_1r_3)(2n_2+n_3+2)}{r_1(1+n_2)\theta_i} \\ &-\frac{n_3(2r_1+r_2)+(2-\theta_i)\big((1+n_2)(2r_1+r_1r_3)+r_2\big)}{r_1(1+n_2)\theta_i} \\ &+\frac{(m_1-n_1)(r_3+2)(2n_2+n_3+2)}{2n_2+2} \\ &+\frac{(2n_2+n_3+2)(6r_1+3r_2+3r_1r_3+r_2r_3)-\boldsymbol{g}_{61}}{(4r_1+r_2+r_1r_3)(2n_2+n_3+2)-\boldsymbol{g}_{62}} \\ &+\sum_{k=1}^{n_3-1}\frac{m_1(r_3+1)}{1+r_3\eta_k}. \end{aligned}$$

where $\boldsymbol{g}_{61} = n_3(r_1 + r_2) + 2(r_3 + 1)(r_1 + r_2 + r_1n_2)$ *and* $\boldsymbol{g}_{62} = (2r_1n_3 + r_2n_3 + 2r_2) + (2 + 2n_2)(2r_1 + r_1r_3)$.

(2)
$$\begin{aligned} K(\mathcal{H}) =& \sum_{i=1}^{n_1-1}\frac{2(r_1n_2+r_1+n_3)(2r_3+r_2+8)-(2r_2n_3+8n_3)}{r_1(2+2n_2)\theta_i} \\ &-\frac{r_1(2-\theta_i)((2+2n_2)(2+r_3)+r_2)}{r_1(2+2n_2)\theta_i} \\ &+\sum_{j=1}^{n_2-1}\frac{m_1(r_2+2)}{2+r_2\mu_j} + \frac{(r_2+4)(m_1-n_1)}{2} \\ &+\frac{2(r_1n_2+r_1+n_3)(r_2r_3+6r_3+3r_2+12)-n_3(2r_2+4)}{2n_2r_1r_2+8n_2r_1+4n_3r_3+8n_3+8r_1} \\ &-\frac{2r_1(r_3+1)(2n_2+r_2+2)}{2n_2r_1r_2+8n_2r_1+4n_3r_3+8n_3+8r_1} + \sum_{k=1}^{n_3-1}\frac{n_1(r_3+1)}{1+r_3\eta_k}. \end{aligned}$$

Example IV.4. *For the graphs $G = C_4^S \bowtie (K_2^V \cup K_2^E)$ and $H = C_4^S \diamondsuit (K_2^V \cup K_1^E)$ (shown in Fig.2), according to Theorem IV.4, one can get $K(G) = \frac{2123}{60}$, $K(H) = \frac{307}{12}$.*

References

[1] F.R.K. Chung, Spectral graph theory. CBMS. Regional conference series in mathematics. Vol.92, Providence (RI): AMS; 1997.

[2] D.M. Cvetković, P. Rowlinson, S.K. Simić, An Introduction to the Theory of Graph Spectra. Cambridge University Press, Cambridge, 2010.

[3] S.Y. Cui, G.X. Tian, The spectrum and the signless Laplacian spectrum of corona. Linear Algabra Appl., vol 437, pp. 1692–2703, 2012.

[4] I. Gopalapillai, The spectrum of neighborhood corona of graphs. Kragujevac J. Math., vol 35, pp. 493–500, 2011.

[5] Y.P. Hou, W.C. Shiu, The spectrum of edge corona two graphs. Electron. J. Linear Algebra., vol 20, pp. 586–594, 2010.

[6] C. McLeman, E. McNicholas, "Spectra of coronae", Linear Algebra Appl., vol 435, pp. 998–1007, 2011.

[7] S.L. Wang, B. ZhouThe signless Laplacian spectra of the corona and edge corona of two graphs. Linear and Multilinear Algeb., vol 61, pp. 197-204, 2013.

[8] Q. Liu, Resistance Distance and Kirchhoff Index of Two Edge-subdivision Corona Graphs. IAENG International Journal of Applied Mathematics., vol. 49, no.1, pp. 127-133, 2019.

[9] Q. Liu, Resistance Distance and Kirchhoff Index of the Diamond Hierarchical Graph and the Generalized Corona Graph. IAENG International Journal of Applied Mathematics., vol. 50, no.4, pp.878-882, 2020.

[10] G. Indulal, Spectrum of two new joins of graphs and infinite families of integral graphs. Kragujevac J. Math., vol 36, pp. 133–139, 2012.
[11] X.G. Liu, Z.H. Zhang, Spectra of subdivision-vertex and subdivision-edge joins of graphs. Mathematics. 2015.
[12] F. Wen, Y. Zhang, M.C. Li, Spectra of Subdivision Vertex-Edge Join of Three Graphs. Mathematics., vol 7, no.171, 2019.
[13] X.G. Liu, P.L. Lu, Spectra of the subdivision-vertex and subdivision-edge neighborhood corona. Linear Algebra and its Applications, 2013.
[14] P.L. Lu, Y.F. Miao, Spectra of the subdivision-vertex and subdivision-edge coronae. Linear Algebra and its Applications, vol 438, pp. 3547–3559, 2013.
[15] C.X. Song, Q.X. Huang, Spectra of subdivision vertex-edge coronae for graphs. Advances in Mathematics (China)., vol 45, pp. 38–47, 2016.
[16] F.Z. Zhang, The Schur Complement and its Applications. Springer US, 2005.
[17] P. Xie, Z. Zhang, F. Comellas, The normalized Laplacian spectrum of subdivisions of a graph. Appl. Math. Comput., vol 286, pp. 250–256, 2016.
[18] P.K. Yu, G.X. Tian, The normalized Laplacian spectra of the double corona based on R-graph. arxiv.org/ abs/1709.02687v1, 2017.
[19] A. Das, P. Panigrahi, Normalized Laplacian spectrum of some subdivision-coronas of two regular graphs. Linear Multilinear Algebra, vol 65, no.5, pp.962–972, 2017.
[20] M.C. Li, Y. Zhang, F. Wen, The Normalized Laplacian Spectrum of Subdivision Vertex-Edge Corona for Graphs. Journal of Mathematical Research with Applications, vol 39, no.3, pp. 221-232, 2019.
[21] E.R. van Dam and W.H. Haemers, Which graphs are determined by their spectrum? . Linear Algebra Appl., vol 373, pp. 241–272, 2003.
[22] H. Chen, F. Zhang, Resistance distance and the normalized Laplacian spectrum. Disc. Appl. Math., vol 155, pp. 654–661, 2003.
[23] J.J. Hunter, The role of Kemeny's constant in properties of Markov chains. Communication in Statistics, vol 43, no.7, pp. 1309–1321, 2014.
[24] S. Butler, Algebraic aspects of the normalized Laplacian, Recent Trends in Combinatorics. The IMA Volumes in Mathematics and its Applications, 2016.