A New Study on Soft Rough Hemirings (Ideals) of Hemirings

Xiangxiang Zeng, Kuanyun Zhu* and Jingru Wang

***Abstract*—Soft set and rough set theory, as two mathematical tools for dealing with uncertainties. Combining soft sets and rough sets, Feng first put forward the concept of soft rough sets. However, Shabir pointed out that an upper approximation of any non-empty set may be empty and upper approximation of a subset of a set may not contain the set. Based on the reason, Shabir modified this concept and put forward a revised soft rough set, which is called an MSR-set. In this paper, soft rough hemirings (ideals) of hemirings with respect to MS-approximation spaces are studied. And some new soft rough operations over hemirings are explored. In particular, soft rough hemirings (k-ideals, h-ideals and strong h-ideals) are also investigated.**

***Index Terms*—Hemiring; Soft rough set; Soft rough hemiring (ideal).**

I. Introduction

It is well known that, classical methods are not always successful in dealing with the problems in economy, engineering and social science, because of various types of uncertainties presented in these problems. As far as known that there are several theories to describe uncertainty, for example, fuzzy set theory [36], rough set theory [28] and other mathematical tools. However, the theories mentioned above have their own limitations. In 1999, Molodtsov [26] put forward soft set theory as a new mathematical tool for dealing with uncertainties. Nowadays, the research on soft sets is progressing rapidly. In 2003, Maji et al. [24] proposed some basic operations. Further, Ali et al. [2] revised some operations. In 2011, Ali [3] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications, such as [5], [6], [7], [25], [30]. In particular, Zhan and Zhu [38] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures [19], [20]. In 2019, Zhan and Alcantud [42] gave a survey of parameter reduction of soft sets and corresponding algorithms In 2020, Ma et al. [22] studied interval-valued intuitionistic fuzzy soft sets based decision making and parameter reduction. Wang et al. [33] studied hesitant bipolar-valued fuzzy soft sets and their application in decision making. In 2021, Zhang [43] investigated N-soft rough sets and its applications.

The concept of rough sets was first proposed by Pawlak [28] as an approach to deal with inexact and uncertain knowledge. It is well known that, an equivalence relation on set into disjoint classes and vice versa. The Pawlak approximation operators are defined by an equivalence relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applied areas. Hence, some more general models have been proposed, such as [44], [45], [47]. In 2010, Herawan et al. [16] studied a rough set approach for selecting clustering attribute. In 2013, Ali et al. [4] investigated some properties of generalized rough sets. Nowadays, this theory has been applied to many fields, such as patter recognition, intelligent systems, machine learning, image processing, cognitive science, signal analysis and so on. On the other hand, many researchers applied this theory to algebraic structures in many papers, such as [8], [9], [18].

As far as known that hemirings provides an algebraic framework for modeling and investigating the key factors in different areas of mathematics as functional analysis, graph theory, formal language theory and parallel computation systems and so on. We know that ideals of semirings play a central role in the structure theory and are useful for many purposes. Many results in rings apparently have no analogues in hemirings using only ideals. In order to overcome this insufficient, Henriksen [15] defined the k-ideals of hemirings. Further, a still more restricted, but very important, a class of ideals, called an h-ideal, has been given and investigated by Izuka [17] and La Torre [32]. Furthermore, Yin [35] give the concept of strong h-ideal of hemirings. In particular, Abdullah [1] studied (α, β)-intuitionistic fuzzy ideals in hemirings. In fact, the relationships among rough sets, fuzzy sets, soft sets and semirings (hemirings) have been considered by many scientists in many papers, such as [34], [39], [40], [41], [49], [50].

Soft set and rough set theory are all mathematical tools to deal with uncertainty. In 2010, Feng et al. [12] provided a framework to combine rough sets and soft sets, which gives rise to some interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. In 2014, Li and Xie [21] investigated the relationship among soft sets, soft rough sets and topologies. In 2015, Zhan et al. [39] applied rough soft set theory to algebraic structures, hemirings. In [23], Ma and Zhan put forth rough soft BCI-algebras by means of an ideal of BCI-algebras. In recent years, Shabir et al. [31] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set

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Xiangxiang Zeng is a master of School of Information and Mathematics, Yangtze University, Jingzhou, P.R. China e-mail: 202071164@yangtzeu.edu.cn

Kuanyun Zhu is a Lecturer of School of Information and Mathematics, Yangtze University, Jingzhou, P.R. China e-mail:kyzhu@yangtzeu.edu.cn

Jingru Wang is a Lecturer of School of Information and Mathematics, Yangtze University, Jingzhou, P.R. China e-mail:wangjingru@hnu.edu.cn

X. In order to solve these problems, Shabir modified the concept of soft rough set, which is called an MSR-set. The underlying concepts are very similar to Pawlak rough sets. As a result, the combination of the two aspects will be more effective when dealing with uncertain problems.

It is well known that, decision making in an imprecise environment has been showing more and more important role in real world applications. Researches on some concrete applications of above two types of uncertain theories as well as their hybrid models in decision making have attracted many researchers's widespread interest. In 2011, Feng [10] applied soft rough sets to multicriteria group decision making. Recently, Zhang et al. [46] applied soft rough sets to discuss a method for multi-attribute decision making. In 2019, Zhu [48] studied soft fuzzy rough rings (ideals) of rings and their application in decision making.

Based on the above ideas, it is an interesting work to discuss on this topic. This paper aims at providing a framework to combine soft sets, rough sets and hemirings all together, which propose the concept of soft rough hemirings (ideals) of hemirings with respect to MS-approximation spaces. The paper is organized as follows: In Section II, we recall some concepts and results on hemirings, soft sets and rough sets, which will be used throughout this paper. In Section III, we study some operations with respect to MSR-approximation spaces and explore some new soft rough operations over hemirings. In Section IV, lower and upper soft rough hemirings (ideals) are investigated. Finally, our researches are concluded in Section V.

II. Preliminaries

In this section, we recall some basic notions which shall be needed in the sequel. Firstly, we give a brief reminder of the definition of semirings as follows.

By a zero of a semiring $(H,+,\cdot)$, we mean an element $0\in H$ such that $0\cdot x=x\cdot 0=0$ and $0+x=x+0=x$ for all $x\in H$. A semiring with zero and a commutative semigroup $(H,+)$ is called a hemiring. For the sake of simplicity, we shall write ab for $a\cdot b$ $(a,b\in H)$. In this paper, H is always a hemiring.

A non-empty subset A of H is called a subhemiring if A is closed under addition and multiplication. A non-empty subset A of H is called a left (resp. right) ideal if A is closed under addition and $HA\subseteq A$ (resp. $AH\subseteq A$). Further, A is called an ideal of H if it is both a left ideal and a right ideal.

An ideal I of H is called a k-ideal of H if $x\in H$, $a,b\in I$ and $x+a=b$ implies $x\in I$. An ideal I of H is called an h-ideal if $x,z\in H$, $a,b\in I$ and $x+a+z=b+z$ implies $x\in I$. An ideal I of H is called a strong h-ideal if $x,y,z\in H$, $a,b\in I$ and $x+a+z=y+b+z$ implies $x\in y+I$.

Let U be an initial universe set, E be a set of parameters and $\mathscr{P}(U)$ be the power set over U.

Definition 2.1: [26] A pair $\mathfrak{S}=(F,A)$ is called a soft set over U, where $A\subseteq E$ and $F:A\to\mathscr{P}(U)$ is a set-valued mapping.

For a soft set $\mathfrak{S}=(F,A)$, the set Supp$(F,A)=\{x\in A|F(x)\neq\emptyset\}$ is called a soft support of (F,A).

Definition 2.2: [12] A soft set $\mathfrak{S}=(F,A)$ over U is called a full soft set if $\bigcup\limits_{a\in A}F(a)=U$.

Definition 2.3: [14] Let $\mathfrak{S}=(F,A)$ be a soft set over H. Then (F,A) is called a soft hemiring (ideal) over H if $F(x)$ is a subhemirings (ideal) of H for all $x\in$ Supp(F,A).

The soft bi-ideal (k-ideal, h-ideal, strong h-ideal) of H is defined similarly.

Next, we introduce the concept of rough sets as follows:

Definition 2.4: [28] Let R be an equivalence relation on the universe U and (U,R) be a Pawlak approximation space. A subset $X\subseteq U$ is called definable if $R_*X=R^*X$; in the opposite case, i.e., if $R_*X-R^*X\neq\emptyset$, X is said to be a rough set, where two operators are defined as:

$$R_*X=\{x\in U|[x]_R\subseteq X\},$$

$$R^*X=\{x\in U|[x]_R\cap X\neq\emptyset\}.$$

In what follows, we give the concept of soft rough sets as follows:

Definition 2.5: [13] Let $\mathfrak{S}=(F,A)$ be a soft set over U. Then the pair $P=(U,\mathfrak{S})$ is called a soft approximation space. Based on P, we define the following two operators:

$$\underline{apr}_P(X)=\{u\in U|\exists a\in A[u\in F(a)\subseteq X]\},$$

$$\overline{apr}_P(X)=\{u\in U|\exists a\in A[u\in F(a),F(a)\cap X\neq\emptyset]\},$$

assigning to every subset $X\subseteq U$.

Two sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the lower and upper soft rough approximations of X in P, respectively. If $\underline{apr}_P(X)=\overline{apr}_P(X)$, X is said to be soft definible; otherwise, X is called a soft rough set. In what follows, we call it Feng-soft rough set.

In order to resolve theoretical and practical aspects, we usually require the soft set to be full in the above definition. If not, it is often limits the research value by means of Feng-soft rough sets, which can be found in the following example.

Example 2.6: Let $\mathfrak{S}=(F,A)$ be a soft set over U which is given by Table 1.

Table 1 Soft set $\mathfrak{S}$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
e_1	1	0	1	0	1	0	0
e_2	0	1	0	0	0	1	0
e_3	1	0	1	0	1	0	1

We assume that $P=(U,\mathfrak{S})$ is a soft approximation space, then we can see that $\mathfrak{S}$ is not full.

Then for $X=\{u_1,u_2,u_4,u_6\}$. It follows from Definition 2.5 that $\underline{apr}_P(X)=\{u_2,u_6\}$ and $\overline{apr}_P(X)=\{u_1,u_2,u_3,u_5,u_6,u_7\}$. It's just a shame that $X\not\subseteq\overline{apr}_P(X)$. In order to avoid this situations, in 2013, Shabir discuss another approach to soft rough sets as follows.

Definition 2.7: [31] Let (F,A) be a soft set over U and $\xi:U\to\mathscr{P}(A)$ be a mapping defined as $\xi(x)=\{a|x\in F(a)\}$. Then the pair (U,ξ) is called MSR-approximation space and for any $X\subseteq U$, the lower MSR-approximation and upper MSR-approximation of X are denoted by $\underline{X}_\xi$ and $\overline{X}_\xi$, respectively, which two operators are defined as

$$\underline{X}_\xi=\{x\in X|\xi(x)\neq\xi(y)\text{ for all }y\in X^c\}$$

and

$$\overline{X}_\xi=\{x\in U|\xi(x)=\xi(y)\text{ for some }y\in X\}$$

If $\underline{X}_\xi=\overline{X}_\xi$, then the X is said to be MSR-definable, otherwise, X is said to be an MSR-set. In what follows, we call it Shabir-soft rough set.

III. Some operations of lower and upper soft rough approximations over hemrings

In this section, at first, we propose the concept of soft rough sets over hemirings, and then, we discuss some operations and fundamental properties of soft rough sets over hemirings.

Definition 3.1: Let $\mathfrak{S} = (F, A)$ be a soft set over H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a|x \in F(a)\}$. Let $\mathfrak{T} = (G, B)$ be another soft set defined over H. The lower and upper soft rough approximations of $\mathfrak{T}$ with respect to $\mathfrak{S}$ are denoted by $\underline{(G, B)}_\xi = (\underline{G}_\xi, B)$ and $\overline{(G, B)}_\xi = (\overline{G}_\xi, B)$, respectively, which are two operators defined as

$$\underline{G(e)}_\xi = \{x \in G(e)|\xi(x) \neq \xi(y) \text{ for all } y \in H - G(e)\}$$

and

$$\overline{G(e)}_\xi = \{x \in H|\xi(x) = \xi(y) \text{ for some } y \in G(e)\}$$

for all $e \in B, x \in X$.
If $\underline{(G, B)}_\xi = \overline{(G, B)}_\xi$, then $\mathfrak{T}$ is called soft definable, otherwise, $\mathfrak{T}$ is called a soft rough set over H.

Remark 3.2: It follows from Definition 3.1 that $\underline{(G, B)}_\xi \subseteq (G, B) \subseteq \overline{(G, B)}_\xi$ for any soft sets $\mathfrak{T} = (G, B)$.

In order to understand the above concept, we consider the following example.

Example 3.3: Let $S = \{0, a, b, c, d\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	c	d
b	b	b	b	d	c
c	c	c	d	c	b
d	d	d	c	b	d

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	a
c	0	a	a	a	a
d	0	a	a	a	a

Then H is a hemiring. $\mathfrak{S} = (F, A)$ is a soft set over H which is given by Table 2.

Table 2 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	1	1	1
e_2	1	1	1	1	1
e_3	0	0	0	1	1

Then the mapping $\xi : H \to \mathscr{P}(A)$ in MSR-approximation space (H, ξ) is given by $\xi(0) = \xi(a) = \xi(b) = \{e_1, e_2\}$, $\xi(c) = \{e_1, e_2, e_3\}$. Define another soft set $\mathfrak{T} = (G, B)$ over H which is given by Table 3.

Table 3 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	0	0	0
e_2	1	1	0	1	1
e_3	0	0	1	1	1
e_4	1	0	1	1	0
e_5	1	1	1	1	0

That is, $G(e_1) = \{0, a\}$, $G(e_2) = \{0, a, c, d\}$, $G(e_3) = \{b, c, d\}$, $G(e_4) = \{0, b, c\}$ and $G(e_5) = \{0, a, b, c\}$. By calculating, $\underline{G(e_1)}_\xi = \emptyset$, $\overline{G(e_1)}_\xi = \{0, a, b\}$, $\underline{G(e_2)}_\xi = \{c, d\}$, $\overline{G(e_2)}_\xi = \{0, a, b, c, d\}$, $\underline{G(e_3)}_\xi = \{c, d\}$, $\overline{G(e_3)}_\xi = \{0, a, b, c, d\}$, $\underline{G(e_4)}_\xi = \emptyset$, $\overline{G(e_4)}_\xi = \{0, a, b, c, d\}$, $\underline{G(e_5)}_\xi = \{0, a, b\}$, $\overline{G(e_5)}_\xi = \{0, a, b, c, d\}$. Then we can see that $\underline{(G, B)}_\xi \subseteq (G, B) \subseteq \overline{(G, B)}_\xi$ for any $e \in B$.

Now, we study some basic properties of lower and upper MSR-approximations of a soft set $\mathfrak{T} = (G, B)$ over a hemiring H. In order to illustrate the roughness in H w.r.t. MSR-approximation spaces over hemirings, we first recall two special kinds of soft sets over hemirings in [37].

Definition 3.4: [37] Let $\mathfrak{S} = (F, A)$ be a soft set over hemiring H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a|x \in F(a)\}$. Then $\mathfrak{S}$ is called a C-soft set over H if $\xi(a) = \xi(b)$ and $\xi(c) = \xi(d)$ imply $\xi(a + c) = \xi(b + d)$ and $\xi(ac) = \xi(bd)$ for all $a, b, c, d \in H$.

Definition 3.5: [37] Let $\mathfrak{S} = (F, A)$ be a C-soft set over H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a|x \in F(a)\}$. Then $\mathfrak{S}$ is called a CC-soft set over H if for all $c \in H$,

(i) $\xi(c) = \xi(x + y)$ for $x, y \in H$, there exist $a, b \in H$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $a + b = c$,

(ii) $\xi(c) = \xi(x + y)$ for $x, y \in H$, there exist $a, b \in H$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $ab = c$.

Remark 3.6: [37] Let $\mathfrak{S} = (F, A)$ be a CC-soft set over H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a|x \in F(a)\}$ if for all $c \in H$,

(i) $\xi(c) = \xi(x + y)$ if and only if for each $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$, we have $a + b = c$, $\forall a, b \in H$;

(ii) $\xi(c) = \xi(xy)$ if and only if for each $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$, we have $ab = c$, $\forall a, b \in H$.

Definition 3.7: Let $\mathfrak{T} = (G, B)$ and $\mathfrak{J} = (H, C)$ be two soft sets over H with $D = B \cap C \neq \emptyset$. The addition operation $+$ and a multiplication operation $\cdot$ of $\mathfrak{T} + \mathfrak{J}$ and $\mathfrak{T} \cdot \mathfrak{J}$ are defined as $\mathfrak{T} + \mathfrak{J} = (G, B) + (H, C) = (K, D)$ and $\mathfrak{T} + \mathfrak{J} = (G, B) + (H, C) = (L, D)$, where $K(a) = G(a) + H(a)$ and $L(a) = G(a) \cdot H(a)$ for all $a \in D$.

Proposition 3.8: Let $\mathfrak{S} = (F, A)$ be a C-soft set over H and (H, ξ) be an MSR-soft approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over H with $D = B \cap C \neq \emptyset$. Then

$$\overline{(G_1, B)}_\xi + \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 + G_2, D)}_\xi,$$

where $(G_1 + G_2)(e) = G_1(e) + G_2(e)$ for all $e \in D$.

Proof. For all $e \in D$, let $c \in \overline{G_1(e)}_\xi + \overline{G_2(e)}_\xi$. Then $c = a + b$, where $a \in \overline{G_1(e)}_\xi$ and $b \in \overline{G_2(e)}_\xi$, and so there exist $x \in \overline{G_1(e)}_\xi$, $y \in \overline{G_2(e)}_\xi$ such that $\xi(a) = \xi(x)$ and $\xi(b) = \xi(y)$. Since $\mathfrak{S}$ is a C-soft set, $\xi(a + b) = \xi(x + y)$ for $x + y \in \overline{G_1(e)}_\xi + \overline{G_2(e)}_\xi$. Hence $c = a + b \in \overline{G_1(e) + G_2(e)}_\xi$, i.e., $\overline{(G_1, B)}_\xi + \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 + G_2, D)}_\xi$. □

The following example shows that the containment in Proposition 3.8 is proper.

Example 3.9: Let $H = \{0, a, b, c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Then H is a hemiring. $\mathfrak{S} = (F, A)$ is a soft set over H which is given by Table 4.

Table 4 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	0	1

Then the mapping $\xi : H \to \mathscr{P}(A)$ in MSR-approximation space (H,ξ) is given by $\xi(0)=\xi(a)=\xi(b)=\{e_1\}$, $\xi(c)=\{e_1,e_2\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over H.

Define two soft sets $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ over H, where $B=\{e_1,e_2\}$ and $C=\{e_2,e_3\}$ with $B\cap C=\{e_2\}$ by $G_1(e_2)=\{a,c\}$ and $G_2(e_2)=\{c\}$. By calculating, $\overline{G_1(e_2)}_\xi=\{0,a,b,c\}$ and $\overline{G_2(e_2)}_\xi=\{c\}$, so $\overline{G_1(e_2)}_\xi+\overline{G_2(e_2)}_\xi=\{b,c\}$. Also we have $\overline{G_1(e_2)+G_1(e_2)}_\xi=\overline{\{b,c\}}_\xi=\{0,a,b,c\}$. Thus $\overline{G_1(e_2)}_\xi+\overline{G_1(e_2)}_\xi \subsetneqq \overline{G_1(e_2)+G_2(e_2)}_\xi$.

Proposition 3.10: Let $\mathfrak{S}=(F,A)$ be a C-soft set over X and (H,ξ) be an MSR-approximation space. Let $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ be two soft sets over H with $D=B\cap C\neq\emptyset$. Then

$$\overline{(G_1,B)}_\xi\cdot\overline{(G_2,C)}_\xi\subseteq\overline{(G_1\cdot G_2,D)}_\xi,$$

where $(G_1\cdot G_2)(e)=G_1(e)\cdot G_2(e)$ for all $e\in D$.

Proof. For all $e\in D$, let $c\in\overline{G_1(e)}_\xi\cdot\overline{G_2(e)}_\xi$. Then $c=\sum_{i=1}^{n}x_iy_i$, where $x_i\in\overline{G_1(e)}_\xi$ and $y_i\in\overline{G_2(e)}_\xi$, and so there exist $a\in\overline{G_1(e)}_\xi$, $b\in\overline{G_2(e)}_\xi$ such that $\xi(x_i)=\xi(a)$ and $\xi(y_i)=\xi(b)$. Since $\mathfrak{S}$ is a C-soft set, $\xi(\sum_{i=1}^{n}x_iy_i)=\xi(\sum_{i=1}^{n}ab)$ for $\sum_{i=1}^{n}ab\in\overline{G_1(e)}_\xi\cdot\overline{G_2(e)}_\xi$. Hence $c=\sum_{i=1}^{n}x_iy_i\in\overline{G_1(e)\cdot G_2(e)}_\xi$, i.e., $\overline{(G_1,B)}_\xi\cdot\overline{(G_2,C)}_\xi\subseteq\overline{(G_1\cdot G_2,D)}_\xi$. □

The following example shows that the containment in Proposition 3.10 is proper.

Example 3.11: Let $H=\{0,a,b,c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	c	a
c	c	c	a	b

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Then H is a hemiring. Now we define a soft set $\mathfrak{S}=(F,A)$ over H which is given by Table 5.

Table 5 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	1	1	1

Then the mapping $\xi : H\to\mathscr{P}(A)$ in MSR-approximation space (H,ξ) will be $\xi(0)=\xi(a)=\xi(b)=\xi(c)=\{e_1,e_2\}$. We can check that $\mathfrak{S}$ is a C-soft set over H.

Define two soft sets $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ over H, where $B=\{e_1,e_3\}$ and $C=\{e_2,e_3\}$ with $B\cap C=\{e_3\}$ by $G_1(e_3)=\{0,b\}$ and $G_2(e_3)=\{b,c\}$. By calculating, $\overline{G_1(e_2)}_\xi=\{0,a,b,c\}$ and $\overline{G_2(e_2)}_\xi=\{b,c\}$, so $\overline{G_1(e_2)}_\xi\cdot\overline{G_2(e_2)}_\xi=\{0,a\}$. Also we have $\overline{G_1(e_2)\cdot G_1(e_2)}_\xi=\overline{\{b,c\}}_\xi=\{0,a,b,c\}$. Thus $\overline{G_1(e_2)}_\xi\cdot\overline{G_1(e_2)}_\xi\subsetneqq\overline{G_1(e_2)\cdot G_2(e_2)}_\xi$.

If we strength the condition, we can obtain the following result.

Proposition 3.12: Let $\mathfrak{S}=(F,A)$ be a C-soft set over H and (H,ξ) be an MSR-soft approximation space. Let $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ be two soft sets over X with $D=B\cap C\neq\emptyset$. Then

$$\overline{(G_1,B)}_\xi+\overline{(G_2,C)}_\xi=\overline{(G_1+G_2,D)}_\xi.$$

Proof. It follows from Proposition 3.10 that $\overline{(G_1,B)}_\xi\cdot\overline{(G_2,C)}_\xi\subseteq\overline{(G_1\cdot G_2,D)}_\xi$. For all $e\in D$, let $c\in\overline{G_1(e)\cdot G_2(e)}_\xi$, so $\xi(c)=\xi(\sum_{i=1}^{n}x_iy_i)$ for some $x_i\in G_1(e)$ and $y_i\in G_2(e)$. Then there exist $a_i,b_i\in H$, such that $\xi(a_i)=\xi(x_i)$ and $\xi(b_i)=\xi(y_i)$ satisfying $c=\sum_{i=1}^{n}a_ib_i$ since $\mathfrak{S}$ is a CC-soft set over H. Thus $a_i\in\overline{G_1(e)}_\xi$ and $b_i\in\overline{G_2(e)}_\xi$. Hence $c\in\overline{G_1(e)}_\xi\cdot\overline{G_1(e)}_\xi$, i.e., $\overline{(G_1,B)}_\xi\cdot\overline{(G_2,C)}_\xi=\overline{(G_1\cdot G_2,D)}_\xi$. □

Proposition 3.13: Let $\mathfrak{S}=(F,A)$ be a C-soft set over X and (H,ξ) be an MSR-soft approximation space. Let $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ be two soft sets over X with $D=B\cap C\neq\emptyset$. Then

$$\overline{(G_1,B)}_\xi+\overline{(G_2,C)}_\xi=\overline{(G_1+G_2,D)}_\xi.$$

Proof. It follows from Proposition 3.10 that $\overline{(G_1,B)}_\xi+\overline{(G_2,C)}_\xi\subseteq\overline{(G_1+G_2,D)}_\xi$. For all $e\in D$, let $c\in\overline{G_1(e)+G_2(e)}_\xi$, so $\xi(c)=\xi(x+y)$ for some $x\in G_1(e)$ and $y\in G_2(e)$. Then there exist $a,b\in H$, such that $\xi(a)=\xi(x)$ and $\xi(b)=\xi(y)$ satisfying $c=a+b$ since $\mathfrak{S}$ is a CC-soft set over H. Thus $a\in\overline{G_1(e)}_\xi$ and $b\in\overline{G_2(e)}_\xi$. Hence $c\in\overline{G_1(e)}_\xi+\overline{G_1(e)}_\xi$, i.e., $\overline{(G_1,B)}_\xi+\overline{(G_2,C)}_\xi=\overline{(G_1+G_2,D)}_\xi$. □

Next, we consider lower soft rough approximations over hemirings.

Proposition 3.14: Let $\mathfrak{S}=(F,A)$ be a C-soft set over H and (H,ξ) be an MSR-soft approximation space. Let $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ be two soft sets over X with $D=B\cap C\neq\emptyset$. Then

$$\underline{(G_1,B)}_\xi\cdot\underline{(G_2,C)}_\xi\subseteq\underline{(G_1\cdot G_2,D)}_\xi.$$

Proof. Suppose that $\underline{(G_1,B)}_\xi\cdot\underline{(G_2,C)}_\xi=\underline{(G_1\cdot G_2,D)}_\xi$ does not hold. For all $e\in D$, there exists $c\in\underline{G_1(e)}_\xi\cdot\underline{G_2(e)}_\xi$ but $c\notin\underline{G_1(e)\cdot G_2(e)}_\xi$. Then $c=\sum_{i=1}^{n}a_ib_i$, where $a_i\in\underline{G_1(e)}_\xi$ and $b_i\in\underline{G_2(e)}_\xi$, and so, $\xi(a_i)\neq\xi(x_i)$ and $\xi(b_i)\neq\xi(y_i)$ for all $x_i\in G_1(e)^c$ and $y_i\in G_2(e)^c$.

On the other hand, $c\notin\underline{G_1(e)\cdot G_2(e)}_\xi$, then we may have the following two conditions:

(i) $c\notin G_1(e)\cdot G_2(e)$, which contradicts with $c\in\underline{G_1(e)}_\xi\cdot\underline{G_2(e)}_\xi\subseteq G_1(e)\cdot G_2(e)$;

(ii) $c\in G_1(e)\cdot G_2(e)$ and $\xi(c)=\xi(\sum_{i=1}^{n}x_i'y_i')$ for some $\sum_{i=1}^{n}x_i'y_i'\in(G_1(e)\cdot G_1(e))^c$. Thus $x_i'\in G_1(e)^c$ or $y_i'\in G_2(e)^c$. In fact, if $x_i'\notin G_1(e)^c$ and $y_i'\notin G_2(e)^c$, we have $\sum_{i=1}^{n}X_i'y_i'\in G_1(e)\cdot G_2(e)$, a contradiction. Since $\mathfrak{S}=(F,A)$ is a CC-soft set over H, then there exist $a_i',b_i'\in S$ such that $\xi(a_i')=\xi(x_i')$ and $\xi(b_i')=\xi(y_i')$ satisfying $\sum_{i=1}^{n}a_i'b_i'=c$, for some $x_i'\in G_1(e)^c$ or $y_i'\in G_2(e)^c$. This is contradiction. Hence $\underline{(G_1,B)}_\xi\cdot\underline{(G_2,C)}_\xi\subseteq\underline{(G_1\cdot G_2,D)}_\xi$. □

Proposition 3.15: Let $\mathfrak{S}=(F,A)$ be a C-soft set over H and (H,ξ) be an MSR-soft approximation space. Let $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ be two soft sets over X with $D=B\cap C\neq\emptyset$. Then

$$\underline{(G_1,B)}_\xi+\underline{(G_2,C)}_\xi\subseteq\underline{(G_1+G_2,D)}_\xi.$$

Proof. Suppose that $\underline{(G_1,B)}_\xi + \underline{(G_2,C)}_\xi = \underline{(G_1+G_2,D)}_\xi$ does not hold. For all $e \in D$, there exists $c \in \underline{G_1(e)}_\xi + \underline{G_2(e)}_\xi$ but $c \notin \underline{G_1(e)+G_2(e)}_\xi$. Then $c = ab$, where $a \in \underline{G_1(e)}_\xi$ and $b \in \underline{G_2(e)}_\xi$, and so, $\xi(a) \neq \xi(x)$ and $\xi(b) \neq \xi(y)$ for all $x \in G_1(e)^c$ and $y \in G_2(e)^c$.

On the other hand, $c \notin \underline{G_1(e)\cdot G_2(e)}_\xi$, then we may have the following two conditions:

(i) $c \notin G_1(e)\cdot G_2(e)$, which contradicts with $c \in \underline{G_1(e)}_\xi \cdot \underline{G_2(e)}_\xi \subseteq G_1(e)\cdot G_2(e)$;

(ii) $c \in G_1(e)\cdot G_2(e)$ and $\xi(c) = \xi(x_i'y_i')$ for some $x_i'y_i' \in (G_1(e)\cdot G_1(e))^c$. Thus $x_i' \in G_1(e)^c$ or $y_i' \in G_2(e)^c$. In fact, if $x_i' \notin G_1(e)^c$ and $y_i' \notin G_2(e)^c$, we have $X_i'y_i' \in G_1(e)\cdot G_2(e)$, a contradiction. Since $\mathfrak{S} = (F,A)$ is a CC-soft set over H, then there exist $a', b' \in S$ such that $\xi(a') = \xi(x')$ and $\xi(b') = \xi(y')$ satisfying $a'b' = c$, for some $x' \in G_1(e)^c$ or $y' \in G_2(e)^c$. This is contradiction. Hence $\underline{(G_1,B)}_\xi + \underline{(G_2,C)}_\xi \subseteq \underline{(G_1+G_2,D)}_\xi$. □

The following example shows that the containment in Propositions 3.14 and 3.15 are proper.

Example 3.16: Let $H = \{0,a,b,c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	a
c	c	c	c	b

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Then H is a hemiring. Now we define a soft set $\mathfrak{S} = (F,A)$ over H which is given by Table 6.

Table 6 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	1	1

Then the mapping $\xi : H \to \mathscr{P}(A)$ in MSR-approximation space (H,ξ) will be $\xi(0) = \xi(a) = \{e_1\}$. $\xi(b) = \xi(c) = \{e_1,e_2\}$. Then we can check that $\mathfrak{S}$ is a CC-soft set over H.

Define two soft sets $\mathfrak{T}_1 = (G_1,B)$ and $\mathfrak{T}_2 = (G_2,C)$ over H, where $B = \{e_1,e_3\}$ and $C = \{e_2,e_3\}$ with $B \cap C = \{e_3\}$ by $G_1(e_3) = \{a,b,c\}$ and $G_2(e_3) = \{0,b,c\}$. Then $\underline{G_1(e_3)}_\xi = \{b,c\}$ and $\underline{G_2(e_3)}_\xi = \{b,c\}$, so $\underline{G_1(e_3)}_\xi \cdot \underline{G_2(e_3)}_\xi = \{a\}$. Also we have $\underline{G_1(e_3)\cdot G_2(e_3)}_\xi = \underline{\{0,a\}}_\xi = \{0,a\}$, that is $\underline{G_1(e_3)}_\xi \cdot \underline{G_2(e_3)}_\xi \subsetneqq \underline{G_1(e_3)\cdot G_2(e_3)}_\xi$. On the other hand, if we take $G_1(e_3') = \{0,a,b\}$ and $G_2(e_3') = \{0,b,c\}$, we have $\underline{G_1(e_3')}_\xi + \underline{G_2(e_3')}_\xi \subsetneqq \underline{G_1(e_3')+G_2(e_3')}_\xi$.

The following example shows that Propositions 3.14 and 3.15 are not true if $\mathfrak{S}$ is not a CC-soft set over H.

Example 3.17: Consider the hemiring H in Example 3.16. Define a soft set $\mathfrak{S} = (F,A)$ over H which is given by Table 5.

Table 5 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	0	1

Then we know that $\mathfrak{S}$ is not a CC-soft set over H. Define two soft sets $\mathfrak{T}_1 = (G_1,B)$ and $\mathfrak{T}_2 = (G_2,C)$ over H, where $B = \{e_1,e_2\}$ and $C = \{e_2,e_3\}$ with $B \cap C = \{e_2\}$ by $G_1(e_2) = \{0,a,c\}$ and $G_2(e_2) = \{c\}$, then $\underline{G_1(e_2)}_\xi = \{0,a\}$ and $\underline{G_2(e_2)}_\xi = \{c\}$, so $\underline{G_1(e_2)}_\xi \cdot \underline{G_2(e_2)}_\xi = \{0,a\}$ and $\underline{G_1(e_2)}_\xi + \underline{G_1(e_2)}_\xi = \{c\}$. Also we have $\underline{G_1(e_2)\cdot G_2(e_2)}_\xi = \underline{\{0,a\}}_\xi = \emptyset$ and $\underline{G_1(e_2)+G_2(e_2)}_\xi = \underline{\{b,c\}}_\xi = \{c\}$, so $\underline{G_1(e_2)}_\xi \cdot \underline{G_2(e_2)}_\xi \nsubseteq \underline{G_1(e_2)\cdot G_2(e_2)}_\xi$ and $\underline{G_1(e_2)}_\xi + \underline{G_1(e_2)}_\xi \nsubseteq \underline{G_1(e_2)+G_2(e_2)}_\xi$.

IV. Some characterizations of soft rough hemirings (ideals) of hemirings

In this section, at first, we propose the concept of soft rough hemirings (ideals) of hemirings. Then, some characterizations of soft rough hemirings (ideals) of hemirings are also given.

Definition 4.1: Let $\mathfrak{S} = (F,A)$ be a soft set over H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Let $\mathfrak{T} = (G,B)$ be another soft set defined over H. The lower and upper soft rough approximations of $\mathfrak{T}$ with respect to $\mathfrak{S}$ are denoted by $\underline{(G,B)_\xi} = (\underline{G_\xi},B)$ and $\overline{(G,B)_\xi} = (\overline{G_\xi},B)$, respectively, which are two operators defined as

$$\underline{G(e)_\xi} = \{x \in G(e) | \xi(x) \neq \xi(y) \text{ for all } y \in H - G(e)\}$$

and

$$\overline{G(e)_\xi} = \{x \in H | \xi(x) = \xi(y) \text{ for some } y \in G(e)\}$$

for all $e \in B, x \in X$.

(i) If $\underline{(G,B)_\xi} = \overline{(G,B)_\xi}$, then $\mathfrak{T}$ is called definable.

(ii) If $\underline{(G,B)_\xi} \neq \overline{(G,B)_\xi}$ and $\overline{G(e)_\xi}$ ($\underline{G(e)_\xi}$) is a subhemiring (ideal) of H for all $e \in B$, then $\mathfrak{T}$ is called a lower (upper) soft rough hemiring (ideal) with respect to $\mathfrak{S}$ over H. Moreover, $\mathfrak{T}$ is called a lower (upper) soft rough hemiring (ideal) with respect to $\mathfrak{S}$ over H if $\overline{G(e)_\xi}$ and $\underline{G(e)_\xi}$ are subhemirings (ideals) with respect to $\mathfrak{S}$ over X for all $e \in B$.

Example 4.2: Let $H = \mathbb{Z}_6 = \{0,1,2,3,4,5\}$ be a hemiring of integers modulo 6 and $\mathfrak{S} = (F,A)$ be a soft set over H which is given by Table 6.

Table 6 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	0	1	1	1	0
e_2	0	1	0	0	0	1
e_3	1	1	0	1	0	1

Then the mapping $\xi : H \to \mathscr{P}(A)$ in MSR-approximation space (H,ξ) is given by $\xi(0) = \xi(3) = \{e_1,e_3\}$, $\xi(1) = \xi(5) = \{e_2,e_3\}$, $\xi(2) = \xi(4) = \{e_1\}$. Define a soft set $\mathfrak{T} = (G,B)$ as the following Table 7.

Table 7 Soft set $\mathfrak{S}$

	0	1	2	3	4	5
e_1	1	1	1	1	0	0
e_2	1	0	0	1	0	0
e_3	1	1	1	0	0	0

By calculating, $\underline{G(e_1)}_\xi = \{0,3\}$, $\overline{G(e_1)}_\xi = \{0,1,2,3,4,5\}$, $\underline{G(e_2)}_\xi = \{0,3\}$, $\overline{G(e_1)}_\xi = \{0,3\}$, $\underline{G(e_3)}_\xi = \emptyset$, $\overline{G(e_3)}_\xi = \{0,1,2,3,4,5\}$, We can obtain two soft sets $\underline{(G,B)}_\xi$ and $\overline{(G,B)}_\xi$, which are given by Table 8 and Table 9, respectively.

Table 8 Soft set $\underline{(G,B)}_\xi$

	0	1	2	3	4	5
e_1	1	1	1	1	1	1
e_2	1	0	0	1	0	0
e_3	1	1	1	1	1	0

Table 9 Soft set $\overline{(G,B)}_\xi$

	0	1	2	3	4	5
e_1	1	0	0	1	0	0
e_2	1	0	0	1	0	0
e_3	0	0	0	0	0	0

It is easy to check that $\underline{(G,B)}_\xi$ and $\overline{(G,B)}_\xi$ are subhemiring of H for all $e \in B$. In other words, $\mathfrak{T} = (G,B)$ is a soft rough hemiring with respect to $\mathfrak{S}$ of H.

Theorem 4.3: Suppose that (H,ξ) is an MSR-approximation space and (G_1,B) and (G_2,C) are lower MSR-hemirings of H with $D = B \cap C \neq \emptyset$. Then $(G_1,B) \cap (G_2,C) = (K,D)$ is a lower MSR-hemiring of H. where $K(e) = G_1(e) \cap G_2(e)$ for all $e \in D$.

Proof. It follows from Definition 4.1 that $\underline{G_1(e)}_\xi$ and $\underline{G_1(e)}_\xi$ are subhemirings of H for all $e \in D$. So $\underline{G_1(e)}_\xi \cap \underline{G_2(e)}_\xi$ is a subhemiring of H. It is clear that $\underline{G_1(e)}_\xi \cap \underline{G_2(e)}_\xi = \underline{G_1(e) \cap G_2(e)}_\xi = \underline{K(e)}_\xi$. Thus we have $\underline{K(e)}_\xi$ is a subhemiring of H for all $e \in D$. Hence, (K,D) is a lower MSR-hemiring of H. □

In general, $(G_1,B) \cap (G_2,C)$ is not an upper MSR-hemiring of H, if (G_1,B) and (G_2,B) are upper MSR-hemirings of H. Actually, we have the following example.

Example 4.4: Consider the hemiring H in Example 3.9 and the soft set $\mathfrak{S} = (F,A)$ in Example 3.16. Define $\mathfrak{T}_1 = (G_1,B)$ and $\mathfrak{T}_2 = (G_2,C)$ over H, where $B = \{e_1,e_2\}$ and $C = \{e_1,e_2\}$ with $B \cap C = \{e_2\}$ by $G_1(e_2) = \{0,c\}$ and $G_2(e_2) = \{a,c\}$, then $\overline{G_1(e_2)}_\xi = \{0,a,b,c\}$ and $\overline{G_1(e_2)}_\xi = \{0,a,b,c\}$ are subhemirings of H. That is $G_1(e_2)$ and $G_2(e_2)$ are upper MSR-hemirings of H. But $\overline{G_1(e_2) \cap G_2(e_2)}_\xi = \overline{\{c\}}_\xi = \{b,c\}$ is not an upper MSR-hemiring of H.

The following example shows that $(G_1,B) \cup (G_2,C)$ is also not a lower (an upper) MSR-hemiring of H, if (G_1,B) and (G_2,C) are MSR-hemirings of H.

Example 4.5: Let $H = \{0,a,b,c,d\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	c	d
b	b	b	b	d	c
c	c	c	d	c	b
d	d	d	c	b	d

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	a
c	0	a	a	a	a
d	0	a	a	a	a

Then H is a hemiring. $\mathfrak{S} = (F,A)$ is a soft set over H which is given by Table 10.

Table 10 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	0	0	1	1	1
e_2	1	1	0	1	0
e_3	0	0	0	1	1

Then the mapping $\xi : H \to \mathscr{P}(A)$ in MSR-approximation space (H,ξ) is given by $\xi(0) = \xi(a) = \{e_2\}$, $\xi(b) = \{e_1\}$, $\xi(c) = \{e_1,e_2,e_3\}$, $\xi(d) = \{e_1,e_3\}$.

Define $\mathfrak{T}_1 = (G_1,B)$ and $\mathfrak{T}_2 = (G_2,C)$ over H, where $B = \{e_1,e_2\}$ and $C = \{e_1,e_2\}$ with $B \cap C = \{e_2\}$ by $G_1(e_2) = \{0,a,b\}$ and $G_2(e_2) = \{0,a,c\}$. So $\mathfrak{T}_1 = (G_1,B)$ and $\mathfrak{T}_2 = (G_2,C)$ are MSR-hemirings of H. On the other hand, $\underline{G_1(e_2) \cup G_2(e_2)}_\xi = \underline{\{0,a,b,c\}}_\xi = \{0,a,b,c\}$ and $\overline{G_1(e_2) \cup G_2(e_2)}_\xi = \overline{\{0,a,b,c\}}_\xi = \{0,a,b,c\}$ is not a subhemirings of H, i.e., $(G_1,B) \cup (G_2,C)$ is not a lower (an upper) MSR-hemiring of H.

In the following, we study the upper and lower MSR-hemirings.

Theorem 4.6: Let $\mathfrak{S} = (F,A)$ be a C-soft set over H and $\mathfrak{T} = (G,B)$ be a soft hemiring of H. Then (G,B) is an upper soft rough hemiring of H.

Proof. Since $(G,B) \subseteq \overline{(G,B)}_\xi$, $0 \in \overline{G(e)}_\xi$ for all $e \in B$. For all $e \in B$, $m,n \in \overline{G(e)}_\xi$. It follows from Definition 3.1 that $\xi(m) = \xi(x)$ and $\xi(n) = \xi(y)$ for some $x,y \in G(e)$. Since $\mathfrak{S}$ is a C-soft set, $\xi(m+n) = \xi(x+y)$ for $x+y \in G(e) + G(e) \subseteq G(e)$, thus $m+n \in \overline{G(e)}_\xi$. Similarly, $mn \in \overline{G(e)}_\xi$. Hence $\overline{G(e)}_\xi$ is a subhemiring of H, i.e., (G,B) is a upper soft rough hemiring of H. □

Theorem 4.7: Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft hemiring of H. Then $\mathfrak{T} = (G,B)$ is a lower soft rough hemiring o H if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. Let $\underline{\mathfrak{T}}_\xi \neq \emptyset$, $m,n \in \underline{G(e)}_\xi$, for all $e \in B$. We suppose that $m+n \notin \underline{G(e)}_\xi$. Then we have $\xi(m) \neq \xi(x)$ for all $x \in G(e)^c$ and $\xi(n) \neq \xi(y)$ for all $y \in G(e)^c$.

On the other hand, $m+n \notin \underline{G(e)}_\xi$, then we may have the following two conditions:

(i) $m+n \notin G(e)$, which contradicts with $m+n \in \underline{G(e)}_\xi + \underline{G(e)}_\xi \subseteq G(e) + G(e) \subseteq G(e)$;

(ii) $m+n \in G(e)$ and $\xi(x') = \xi(m+n)$ for some $x' \in G(e)^c$. Since $\mathfrak{S} = (F,A)$ is a CC-soft set, there exist $x_1,y_1 \in H$ such that $\xi(m) = \xi(x_1)$ and $\xi(n) = \xi(y_1)$ satisfying $x_1 + y_1 = x' \in G(e)^c$. Thus, $x_1 \in G(e)^c$ or $y_1 \in G(e)^c$. In fact, if $x_1 \notin G(e)^c$ and $y_1 \notin G(e)^c$, we have $x_1+y_1 \in G(e)+G(e) \subseteq G(e)$, a contradiction. That is there exist $x_1 \in G(e)^c$ such that $\xi(m) = \xi(x_1)$ or $y_1 \in G(e)^c$ such that $\xi(n) = \xi(y_1)$. This is contradictory to $m+n \notin \underline{G(e)}_\xi$. Thus $m+n \in \underline{G(e)}_\xi$. Similarly, we have $m \cdot n \in \underline{G(e)}_\xi$. This implies $\underline{G(e)}_\xi$ is a subhemiring of H, i.e., (G,B) is a lower soft rough hemiring of H. □

Theorem 4.8: Let $\mathfrak{S} = (F,A)$ be a C-soft set over H and $\mathfrak{T} = (G,B)$ be a soft ideal of H. Then $\mathfrak{T} = (G,B)$ is a soft rough ideal of H.

Proof. Let $\mathfrak{T} = (G,B)$ be a soft ideal of H. It follows from Theorem 4.6 that $\overline{G(e)}_\xi$ is a subhemiring of H for all $e \in B$. If $r \in H$ and $s \in \overline{G(e)}_\xi$, then $\xi(s) = \xi(x)$ for some $x \in X$. Since $\xi(r) = \xi(r)$ and $\mathfrak{S}$ is a C-soft set, $\xi(rs) = \xi(rx)$ for some $rx \in S \cdot G(e) \subseteq G(e)$, thus $rs \in \overline{G(e)}_\xi$. Hence $\overline{G(e)}_\xi$ is a left ideal of H. Similarly, we have $\overline{G(e)}_\xi$ is a right ideal of H, i.e., $\overline{G(e)}_\xi$ is a soft rough ideal of H. □

Theorem 4.9: Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft ideal of H. Then $\mathfrak{T} = (G,B)$ is soft rough ideal of H if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. Let $\mathfrak{T} = (G,B)$ be a soft ideal of H. It follows from Theorem 4.7 that $\underline{G(e)}_\xi$ is a subhemiring of H for all $e \in B$. It follows from Proposition 3.16 that $H \cdot \underline{G(e)}_\xi = \underline{H}_\xi \cdot \underline{G(e)}_\xi \subseteq \underline{H \cdot G(e)}_\xi \subseteq \underline{G(e)}_\xi$. Similarly, $\underline{G(e)}_\xi \cdot H \subseteq \underline{G(e)}_\xi$. Therefore, $\underline{G(e)}_\xi$ is an ideal of H. Hence, $\mathfrak{T} = (G,B)$ is soft rough ideal of H. □

The Following example shows that the converse of Theorems 4.8 and 4.9 do not hold in general.

Example 4.10: We consider the hemiring H in Example 3.7. $\mathfrak{S} = (F,A)$ is a soft set over H which is given by Table 4 in Example 3.8. Then the mapping $\xi : H \to \mathscr{P}(A)$

in MSR-approximation space (H,ξ) is given by $\xi(0) = \xi(a) = \{e_1\}$, $\xi(b) = \xi(c) = \{e_1, e_2\}$. Then we can check that $\mathfrak{S}$ is a CC-soft set over H.

Now we define another soft set (G,B) over H by $G(e_1) = \{0,a,b\}$ and $G(e_2) = \{0,a,c\}$. So $\underline{G(e_1)}_\xi = \{0,a\}$, $\overline{G(e_1)}_\xi = \{0,a,b,c\}$ and $\underline{G(e_2)}_\xi = \{0,a\}$, $\overline{G(e_2)}_\xi = \{0,a,b,c\}$. Then we have $\overline{G(e)}_\xi$ and $\underline{G(e)}_\xi$ are ideals of H for all $e \in B$, but (G,B) is not a soft ideal H.

Theorem 4.11: Let $\mathfrak{S} = (F,A)$ be a C-soft set over H and (G,B) be a soft bi-ideal of H. Then (G,B) is a soft rough bi-ideal of H.

Proof. Let (G,B) be a soft bi-ideal of H. It follows from Theorem 4.6 that $\overline{G(e)}_\xi$ is a subhemiring of H for all $e \in B$. It follows from Proposition 3.10 that $\overline{G(e)}_\xi \cdot H \cdot \overline{G(e)}_\xi = \overline{G(e)}_\xi \cdot \overline{H}_\xi \cdot \overline{G(e)}_\xi \subseteq \overline{G(e) \cdot H \cdot G(e)}_\xi \subseteq \overline{G(e)}_\xi$. Hence $\overline{G(e)}_\xi$ is a bi-ideal of H. □

Theorem 4.12: Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft bi-ideal of H. Then $\mathfrak{T} = (G,B)$ is a soft rough bi-ideal of H if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. Let $\mathfrak{T} = (G,B)$ be a soft bi-ideal of H. It follows from Theorem 4.7 that $\underline{G(e)}_\xi$ is a subhemiring of H. It follows from Proposition 3.6 that $\underline{G(e)}_\xi \cdot H \cdot \underline{G(e)}_\xi = \underline{G(e)}_\xi \cdot \underline{H}_\xi \cdot \underline{G(e)}_\xi \subseteq \underline{G(e) \cdot H \cdot G(e)}_\xi \subseteq \underline{G(e)}_\xi$. Hence $\underline{G(e)}_\xi$ is a bi-ideal of H. □

Definition 4.13: [37] Let $\mathfrak{S} = (F,A)$ be a C-soft set over H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$. Let I be an ideal of H. Then $\mathfrak{S}$ is called a BC-soft set over H, $\forall a, b \in H$, $\xi(a) = \xi(b)$ if and only if there exist $i_1, i_2 \in I$ such that $a + i_1 = b + i_2$.

Theorem 4.14: Let $\mathfrak{S} = (F,A)$ be a BC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft k-ideal of H. Then $\mathfrak{T} = (G,B)$ is an upper soft rough k-ideal of H.

Proof. Let $\mathfrak{T} = (G,B)$ be a k-ideal of H. It follows from Theorem 4.8 that $\overline{G(e)}_\xi$ is an ideal of H for all $e \in B$. Let $c + a = b$ for $a, b \in \overline{G(e)}_\xi$ and $c \in H$. Now we prove $c \in \overline{G(e)}_\xi$. By the preceding description, we have $\xi(a) = \xi(x)$ and $\xi(b) = \xi(y)$ for some $x, y \in G(e)$. Since $\mathfrak{S} = (F,A)$ is a BC-soft set over H, there exist $i_1, i_2, i_3, i_4 \in X$ such that $a + i_1 = x + i_2$ and $b + i_3 = y + i_4$. And then $c + a = b \Rightarrow c + a + i_1 + i_3 = b + i_3 + i_1 \Rightarrow c + x + i_2 + i_3 = y + i_4 + i_1$, we can write $c + a' = b'$, where $a' = x + i_2 + i_3 \in G(e)$, $b' = y + i_4 + i_1 \in G(e)$. By the hypothesis, $\mathfrak{T} = (G,B)$ is a soft k-ideal of H, then $c \in G(e)$. Hence, $c \in G(e) \subseteq \overline{G(e)}_\xi$. Thus, $\mathfrak{T} = (G,B)$ is an upper soft rough k-ideal of H □

Theorem 4.15: Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft k-ideal of H. Then $\mathfrak{T} = (G,B)$ is a lower soft rough k-ideal of H if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. Let $\mathfrak{T} = (G,B)$ be a k-ideal of H. It follows from Theorem 4.9 that $\underline{G(e)}_\xi$ is an ideal of H for all $e \in B$. We suppose that $\underline{G(e)}_\xi$ is not a k-ideal of H, then there exist $a, b \in \underline{G(e)}_\xi \subseteq G(e)$ and $x' \in H$ from $x' + a = b$, we have $x' \in X/\underline{G(e)}_\xi$. Then $\xi(b) = \xi(x' + a)$, $\xi(a) \neq \xi(x)$ for all $x \in G(e)^c$ and $\xi(b) \neq \xi(y)$ for all $y \in G(e)^c$. Since $\mathfrak{S}$ is a CC-soft set over H and it follows from Remark 3.9 that for all $\xi(x') = \xi(e_i)$ and $\xi(a) = \xi(c)$, we have $e_i + c = b$, where $e_i, c \in H$. This means that there exists at least $e' \in G(e)^c$ with $\xi(x') = \xi(e')$ since $x' \in X/\underline{G(e)}_\xi$.

On the other hand, $\xi(c) = \xi(a) \neq \xi(x)$ for all $x \in G(e)^c$, that is, $c \in \underline{G(e)}_\xi \subseteq G(e)$. By the hypothesis, $\mathfrak{T} = (G,B)$ is a soft k-ideal of H, then $e' \in G(e)$, which contradicts with $e' \in G(e)^c$. Hence $\underline{G(e)}_\xi$ is a k-ideal of H. Thus $\mathfrak{T} = (G,B)$ is a lower soft rough k-ideal of H. □

Theorem 4.16: Let $\mathfrak{S} = (F,A)$ be a BC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft h-ideal of H. Then $\mathfrak{T} = (G,B)$ is an upper soft rough h-ideal of H.

Proof. Let $\mathfrak{T} = (G,B)$ be a soft h-ideal of H. It follows from Theorem 4.8 that $\overline{G(e)}_\xi$ is an ideal of H for all $e \in B$. Let $c + a + z = b + z$ for $a, b \in \overline{G(e)}_\xi$ and $c, z \in H$. Now we prove $c \in \overline{G(e)}_\xi$. By the preceding description, we have $\xi(a) = \xi(x)$ and $\xi(b) = \xi(y)$ for some $x \in G(e)$, $y \in G(e)$. Since $\mathfrak{S} = (F,A)$ is a BC-soft set over H, there exist $i_1, i_2, i_3, i_4 \in X$ such that $a + i_1 = x + i_2$ and $b + i_3 = y + i_4$. And then $c + a + z = b + z \Rightarrow c + a + i_1 + i_3 + z = b + i_3 + i_1 + z \Rightarrow c + x + i_2 + i_3 + z = y + i_4 + i_1 + z$, we can write $c + a' + z = b' + z$, where $a' = x + i_2 + i_3 \in X$, $b' = y + i_4 + i_1 \in G(e)$. By the hypothesis, $\mathfrak{T} = (G,B)$ is a soft h-ideal of H, then $c \in G(e)$. Hence, $c \in G(e) \subseteq \overline{G(e)}_\xi$ for all $e \in B$. Thus $\mathfrak{T} = (G,B)$ is an upper soft rough h-ideal of H. □

Lemma 4.17: [37] Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$, if $x, a, b, z \in H$ and $x + a + z = b + z$, then for each $\xi(x) = \xi(x_1)$, $\xi(a) = \xi(a_1)$, $\xi(b) = \xi(b_1)$ and $\xi(z) = \xi(z_1)$, we have $x_1 + a_1 + z_1 = b_1 + z_1$, $\forall x_1, a_1, b_1, z_1 \in H$.

Theorem 4.18: Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft h-ideal of H. Then $\mathfrak{T} = (G,B)$ is a lower soft rough h-ideal of H if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. Let $\mathfrak{T} = (G,B)$ be a soft h-ideal of H. By Theorem 4.9, $\underline{G(e)}_\xi$ is an ideal of H. We suppose that $\underline{G(e)}_\xi$ is not an h-ideal of H, then there exist $a, b \in \underline{G(e)}_\xi \subseteq X$ and $x', z \in S$ from $x' + a + z = b + z$, we have $x' \in X/\underline{G(e)}_\xi$. Then $\xi(x' + a + z) = \xi(b + z)$, $\xi(a) \neq \xi(x)$ for all $x \in G(e)^c$ and $\xi(b) \neq \xi(y)$ for all $y \in G(e)^c$. Since $\mathfrak{S}$ is a CC-soft set over H and by Lemma 4.17, then for all $\xi(x') = \xi(x_1)$, $\xi(a) = \xi(a_1)$, $\xi(b) = \xi(b_1)$ and $\xi(z) = \xi(z_1)$, we have $x_1 + a_1 + z_1 = b_1 + z_1$, where $x_1, a_1, b_1, z_1 \in S$. This means that there exists at least $x'_1 \in G(e)^c$ with $\xi(x') = \xi(x'_1)$ since $x' \in X/\underline{X}_\xi$. On the other hand, $\xi(a_1) = \xi(a) \neq \xi(x)$ for all $x \in G(e)^c$, that is, $a_1 \in \underline{G(e)}_\xi \subseteq G(e)$, similarly, $b_1 \in \underline{G(e)}_\xi \subseteq G(e)$. By the hypothesis, $\mathfrak{T} = (G,B)$ is a soft h-ideal of H, then $x'_1 \in G(e)$, which contradicts with $x'_1 \in G(e)^c$. Hence $\underline{G(e)}_\xi$ is an h-ideal of H. Thus, $\mathfrak{T} = (G,B)$ is a lower soft rough h-ideal of H. □

Theorem 4.19: Let $\mathfrak{S} = (F,A)$ be a BC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft h-ideal of H. Then $\mathfrak{T} = (G,B)$ is an upper soft rough strong h-ideal of H.

Proof. The proof is similar to that of Theorem 4.16. □

Lemma 4.20: [37] Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\xi : H \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a | x \in F(a)\}$, if $x, y, a, b, z \in S$ and $x + a + z = y + b + z$, then for each $\xi(x) = \xi(x')$, $\xi(a) = \xi(a')$, $\xi(b) = \xi(b')$, $\xi(y) = \xi(y')$, $\xi(z) = \xi(z')$, where $x', y', a', b', z' \in H$, we have $x' + a' + z' = y' + b' + z'$.

Theorem 4.21: Let $\mathfrak{S} = (F,A)$ be a CC-soft set over H and $\mathfrak{T} = (G,B)$ be a soft strong h-ideal of H. Then $\mathfrak{T} = (G,B)$ is a lower soft rough strong h-ideal of H if $\underline{\mathfrak{T}}_\xi \neq \emptyset$.

Proof. The proof is similar to that of Theorem 4.18. □

V. Conclusions

Recently, Shabir [31] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set X. In order to overcome the shortcomings of the Feng-soft rough sets, Shabir modified a new soft rough set (MSR)-set, which removes the limiting condition that full soft sets are required in Feng's soft rough set.

In this paper, we apply soft rough set theory to hemirings and propose soft rough hemirings. We discuss some operational properties and algebraic structures of lower and upper soft rough approximations over hemirings. Besides, several examples are presented in order to investigated their characterizations.

As future works, we will consider the following topics:

(1) Constructing soft rough sets to other algebras, such as hyperrings, BL-algebras, EQ-algebras and so on;

(2) Investigating decision making methods based on soft rough sets;

(3) Establishing soft rough sets to some applied some areas of applications, such as information sciences, intelligent systems and so on.

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