Permanence and Stability for a Competition and Cooperation Model of Two Enterprises with Feedback Controls on Time Scales

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Abstract—**By using some differential inequalities on time scales and constructing a suitable Lyapunov function, some new conditions are obtained for the permanence and uniformly asymptotical stability of a competition and cooperation model of two enterprises with feedback controls on time scales. Our results indicate that feedback controls are irrelevant to the permanence of this model which improve and complement some existing ones.**

Index Terms—**Permanence, Uniformly asymptotical stability, Competition and cooperation model, Feedback controls, Time scales.**

I. Introduction

As an effective tool to depict real ecological system, mathematical ecological model has become more and more important in the study of modern applied mathematics. Differential equations and difference equations are two main tools for the description of species relationship. However, due to the different concepts, theoretical knowledge and research methods, differential equations and difference equations always appear separately and people need to study twice for a complete and comprehensive understanding for systems. Furthermore, only using differential equations or difference equations is ineffective for describing the law of those species whose development process are both continuous and discrete in the real world [1, 2]. In order to unify both differential and difference analysis, Hilger [12] introduced the theory of time scales in his Ph.D. thesis. After then, many researchers pay attentions to the study of dynamic equations on time scales, such as permanence [5, 14, 15], global attractivity [11, 18], periodic solution and almost periodic solution [4, 7, 16, 19–21, 25] and so on. In particular, Zhi, Ding and Li [25] considered the following competitive and cooperation model of a satellite enterprises and a dominant enterprise with feedback controls on time scales $\mathbb{T}$

$$\begin{cases} x^{\Delta}(t) = a_1(t) - b_1(t)\exp\{x(t)\} \\ \qquad\qquad - c_1(t)[\exp\{y(t)\} - d_2(t)]^2 - f_1(t)u(t), \\ y^{\Delta}(t) = a_2(t) - b_2(t)\exp\{y(t)\} \\ \qquad\qquad + c_2(t)[\exp\{x(t)\} - d_1(t)]^2 - f_2(t)v(t), \\ u^{\Delta}(t) = -\gamma_1(t)u(t) + \eta_1(t)\exp\{x(t)\}, \\ v^{\Delta}(t) = -\gamma_2(t)v(t) + \eta_2(t)\exp\{y(t)\}, \end{cases} \tag{1}$$

in which $\{a_i(t)\}$, $\{b_i(t)\}$, $\{c_i(t)\}$, $\{d_i(t)\}$, $\{f_i(t)\}$, $\{\gamma_i(t)\}$ and $\{\eta_i(t)\}$ are all bounded nonnegative functions on $\mathbb{T}$ satisfying

$$\begin{aligned} &0 < a_i^l \le a_i(t) \le a^u, \quad 0 < b_i^l \le b_i(t) \le b_i^u, \\ &0 < c_i^l \le c(t) \le c^u, \quad 0 < d_i^l \le d_i(t) \le d_i^u, \\ &0 < f_i^l \le f_i(t) \le f_i^u, \quad 0 < \gamma_i^l \le \gamma_i(t) \le \beta_i^u, \\ &0 < \eta_i^l \le \eta_i(t) \le \eta_i^u, \ i = 1, 2, \end{aligned}$$

where we using the following notations:

$$h^l = \inf_{t\in\mathbb{T}} h(t), \quad h^u = \sup_{t\in\mathbb{T}} h(t),$$

for any $h(t)$ which is a continuous bounded function defined on $\mathbb{T}$. We also suppose that $1-\mu(t)\gamma_i(t) > 0$ ($\mu(t)$ is defined in Section II) and there exists a positive constant L such that $\mu(t) \le L$. Consider system (1) together with the following initial conditions:

$$x(0) > 0, y(0) > 0, u(0) > 0, v(0) > 0, \tag{2}$$

by using the comparison theorem of dynamic equations on time scales, Zhi, Ding and Li [25] got the following permanent result for system (1):

Theorem A ([25]). *Assume*

$$-b_i^l, -\gamma_i^l \in \mathcal{R}^+ \tag{Q_1}$$

and

$$a_1^l - c_1^u(e^{y^*} + d_2^u)^2 - f_1^u u^* > 0, \ a_2^l - f_2^u v^* > 0 \tag{Q_2}$$

hold, where $y^* = \frac{a_2^u - b_2^l + c_2^u e^{2x^*}}{b_2^l}$, $x^* = \frac{a_1^u - b_1^l}{b_1^l}$, $u^* = \frac{\eta_1^u e^{x^*}}{\gamma_1^l}$ *and* $v^* = \frac{\eta_2^u e^{y^*}}{\gamma_2^l}$, *then system (1) is permanent.*

Remark 1.1. *There is a mistake in* (H_2) *of Proposition 12 in Zhi, Ding and Li [25]. Although* $\exp\{y(t)\} - d_2(t) \le e^{y^*} - d_2^l$, *however we can not obtain* $[\exp\{y(t)\} - d_2(t)]^2 \le (e^{y^*} - d_2^l)^2$ *but* $[\exp\{y(t)\} - d_2(t)]^2 \le (e^{y^*} + d_2^u)^2$. *So inequation (35) in [25] is invalid and* (H_2) *i.e.* $a_1^l - c_1^u(e^{y^*} - d_2^l)^2 - f_1^u u^* > 0, \ a_2^l - f_2^u v^* > 0$ *should be changed to* (Q_2).

According to Theorem A, feedback controls can affect the permanence of system (1) which is also supported by Lu, Lian and Li [26] who investigated the discrete version of system (1) with time delays. However, some results (see such as [8–10, 14, 15] and so on) have shown that feedback controls have no impact on the permanence of ecological system. In particular, by using some differential inequalities on time scales, Wang and Fan [15] showed that feedback term is irrelevant to the permanence of a Nicholson's blowflies model with feedback control on time scales. Their results motivated us to consider the permanence of system (1) again.

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In fact, in this paper, by utilizing the analytical skills of Wang and Fan [15] , we ultimately get the following result:
Theorem B. *Assume*

$$a_1^l - c_1^u(e^{W_2} + d_2^u)^2 > 0 \qquad (A_1)$$

holds, where $W_1 = a_1^u L + \ln\frac{a_1^u}{b_1^l}$ *and* $W_2 = [a_2^u + c_2^u(e^{W_1} + d_1^u)^2]L + \ln\frac{a_2^u + c_2^u(e^{W_1}+d_1^u)^2}{b_2^l}$, *then system* (1) *is permanent.*

One can easily find that (A_1) in Theorem B is weaker than (Q_1) and (Q_2) in Theorem A and feedback terms are harmless to the permanence of system (1), hence our results improve those in [25, 26]. For more similar problems, one could refer to [3, 6, 17, 22–24] and references therein.

The organization of this paper is as follows. In Section II, we give some foundational definitions and results on time scales. The permanence and uniform asymptotical stability of the model are discussed in Section III and IV. Then, in Section V, our results are verified by one example with numerical simulations. Finally, we conclude in Section VI.

II. Preliminaries

In this section, we shall present some foundational definitions and results on time scales and one can refer to [13] for more detail.

Definition 2.1. ([13]) *A time scale is an arbitrary nonempty closed subset* $\mathbb{T}$ *of the real numbers* $\mathbb{R}$. *The set* $\mathbb{T}$ *inherits the standard topology of* $\mathbb{R}$.

Definition 2.2. ([13]) *For* $t \in \mathbb{T}$, *the forward jump operator, the backward jump operator* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, *and the graininess* $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, +\infty)$ *are defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

$$\mu(t) = \sigma(t) - t,$$

respectively. If $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$, *then* t *is called right-dense, and if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$, *then* t *is called left-dense.*

Definition 2.3. ([13]) *A function* $f : \mathbb{T} \to \mathbb{R}$ *is said to be rd-continuous if it is continuous at right-dense points in* $\mathbb{T}$ *and its left-sided limits exist (finite) at left-dense points in* $\mathbb{T}$. *The set of rd-continuous functions is denoted by* $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.4. ([13]) *Suppose* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}$. *Then we define* $f^\Delta(t)$, *the delta-derivative of* f *at* t, *to be the number (provided it exists) with the property that, given any* $\varepsilon > 0$, *there is a neighborhood* U *of* t *(i.e.,* $U = (t - \delta, t + \delta) \cap \mathbb{T}$*) for some* $\delta > 0$ *such that*

$$|[f(\sigma(t))-f(s)]-f^\Delta(t)[\sigma(t)-s]| \le \varepsilon|\sigma(t)-s|, \ \text{for all } s \in U.$$

Thus, f *is said to be delta-differentiable if its delta-derivative exists. The set of functions* $f : \mathbb{T} \to \mathbb{R}$ *that are delta-differentiable and whose delta-derivative are rd-continuous functions is denoted by* $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.5. ([13]) *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called a delta-antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *provided* $F^\triangle(t) = f(t)$, *for all* $t \in \mathbb{T}$. *Then, we write*

$$\int_r^s f(t)\Delta t = F(s) - F(r), \ \text{for all } s, r \in \mathbb{T}.$$

Definition 2.6. ([13]) *A function* $f : \mathbb{T} \to \mathbb{R}$ *is regressive if* $1 + \mu(t)f(t) \neq 0$ *for all* $t \in \mathbb{T}$ *and is positively regressive if* $1 + \mu(t)f(t) > 0$ *for all* $t \in \mathbb{T}$. *Denote by* $\mathcal{R}$ *and* $\mathcal{R}^+$ *the set of regressive and positively regressive functions from* $\mathbb{T}$ *to* $\mathbb{R}$, *respectively. If* $p \in \mathcal{R}$, *we define the exponential function by*

$$e_p(a, b) = \exp\Big\{ \int_b^a \xi_{\mu(t)}(p(t))\Delta t\Big\}, \ a, b \in \mathbb{T},$$

where the cylinder transformation $\xi_\mu(z) = (1/\mu)\log(1+z\mu)$, *for* $\mu > 0$ *and* $\xi_0(z) = z$, *for* $\mu = 0$.

Lemma 2.1. ([13]) *Suppose that* $p, q \in \mathcal{R}^+$*; then for all* $a, b \in \mathbb{T}$,

(i) $e_p(a, b) > 0$*;*

(ii) *if* $p(a) \le q(a)$ *for all* $a \in \mathbb{T}$, *then* $e_p(a, b) \le e_q(a, b)$ *for all* $a \ge b$.

Lemma 2.2. ([13])

(i) $(\nu_1 f + \nu_2 g)^\Delta = \nu_1 f^\Delta + \nu_2 g^\Delta$, *for any constants* ν_1, ν_2*;*

(ii) *if* $f^\Delta \ge 0$, *then* f *is nondecreasing.*

Lemma 2.3. ([14]) *Suppose* $A, B > 0$ *and* $x(0) > 0$, *further assume that*

(i)

$$x^\Delta(t) \le B - A\exp\{x(t)\}, \ \forall\, t \ge 0,$$

then

$$\limsup_{t\to+\infty} x(t) \le BL + \ln\frac{B}{A}.$$

(ii)

$$x^\Delta(t) \ge B - A\exp\{x(t)\}, \ \forall\, t \ge 0,$$

and there exists a constant $M > 0$, *such that* $\limsup_{t\to+\infty} x(t) < M$, *then*

$$\liminf_{t\to+\infty} x(t) \ge (B - A\exp\{M\})L + \ln\frac{B}{A}.$$

Lemma 2.4. ([14]) *Assume that* $C(t)$, $D(t) > 0$ *are bounded and rd-continuous functions,* $-C \in \mathcal{R}^+$ *and* $C^l > 0$. *Further suppose that*

(i)

$$x^\Delta(t) \le -C(t)x(t) + D(t), \ \forall\, t \ge T_0,$$

then there exists a constant $T_1 > T_0$, *such that for* $t > T_1$,

$$x(t) \le x(T_1)e_{(-C)}(t, T_1) + \frac{D(t)}{C^l}.$$

Especially, if $D(t)$ *is bounded above with respect to* H_1, *then*

$$\limsup_{t\to+\infty} x(t) \le \frac{H_1}{C^l}.$$

(ii)

$$x^\Delta(t) \ge -C(t)x(t) + D(t), \ \forall\, t \ge T_0,$$

then there exists a constant $T_2 > T_0$, such that for $t > T_2$,

$$x(t) \geq \Big(x(T_2) - \frac{D(T_2)}{C^u}\Big) e_{(-C)}(t, T_2) + \frac{D(T_2)}{C^u}.$$

Especially, if $D(t)$ is bounded below with respect to h_1, then

$$\liminf_{t\to+\infty} x(t) \geq \frac{h_1}{C^u}.$$

Definition 2.7. *System (1) is said to be permanent if for any solution $(x(t), y(t), u(t), v(t))^T$ of system (1), there exist four constants w_i, k_i, W_i and K_i $(i = 1, 2)$ such that*

$$w_1 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq W_1,$$

$$w_2 \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq W_2,$$

$$k_1 \leq \liminf_{t\to+\infty} u(t) \leq \limsup_{t\to+\infty} u(t) \leq K_1,$$

$$k_2 \leq \liminf_{t\to+\infty} v(t) \leq \limsup_{t\to+\infty} v(t) \leq K_2.$$

III. Permanence

We shall investigate the permanence of system (1) in this part. Similarly to the proof of [15, Lemma 18], we can obtain

Lemma 3.1. *For any solution $(x(t), y(t), u(t), v(t))^T$ of system (1) with initial condition (2), we have*

$$\exp\{x(t)\} > 0,\ \exp\{y(t)\} > 0,\ u(t) > 0,\ v(t) > 0,\ \forall\, t \in \mathbb{T}.$$

Lemma 3.2. *For any solution $(x(t), y(t), u(t), v(t))^T$ of system (1) with initial condition (2), we have*

$$\limsup_{t\to+\infty} x(t) \leq W_1,\ \limsup_{t\to+\infty} y(t) \leq W_2$$
$$\limsup_{t\to+\infty} u(t) \leq K_1,\ \limsup_{t\to+\infty} v(t) \leq K_2,$$

where

$$W_1 = a_1^u L + \ln\frac{a_1^u}{b_1^l},\ K_1 = \frac{\eta_1^u e^{W_1}}{\gamma_1^l},\ K_2 = \frac{\eta_2^u e^{W_2}}{\gamma_2^l}$$

$$W_2 = [a_2^u + c_2^u(e^{W_1} + d_1^u)^2]L + \ln\frac{a_2^u + c_2^u(e^{W_1} + d_1^u)^2}{b_2^l}.$$

Proof. From the positivity of $u(t)$ and the first equation of system (1), we get

$$\begin{aligned} x^{\triangle}(t) \leq& a_1(t) - b_1(t)\exp\{x(t)\} \\ \leq& a_1^u - b_1^l \exp\{x(t)\}. \end{aligned}$$

According to Lemma 2.3 (i), we obtain

$$\limsup_{t\to+\infty} x(t) \leq a_1^u L + \ln\frac{a_1^u}{b_1^l} \triangleq W_1. \tag{3}$$

Thus, for any $\varepsilon_0 > 0$, there exists a large enough $t_0 \in \mathbb{T}^+$, such that for all $t > t_0$, we have

$$x(t) \leq W_1 + \varepsilon_0.$$

Then, for $t > t_0$, we can get from the second equation and the third equation of system (1) that

$$\begin{aligned} y^{\triangle}(t) \leq& a_2^u - b_2^l \exp\{y(t)\} + c_2^u(e^{W_1+\varepsilon_0} + d_1^u)^2, \\ u^{\triangle}(t) \leq& -\gamma_1(t)u(t) + \eta_1^u e^{W_1+\varepsilon_0}. \end{aligned}$$

Using Lemma 2.3 (i) and Lemma 2.4 (i), we further obtain

$$\begin{aligned} \limsup_{t\to+\infty} y(t) \leq& [a_2^u + c_2^u(e^{W_1+\varepsilon_0} + d_1^u)^2]L \\ &+ \ln\frac{a_2^u + c_2^u(e^{W_1+\varepsilon_0} + d_1^u)^2}{b_2^l} \\ \limsup_{t\to+\infty} u(t) \leq& \frac{\eta_1^u e^{W_1+\varepsilon_0}}{\gamma_1^l}. \end{aligned} \tag{4}$$

Setting $\varepsilon_0 \to 0$, it follows from (4) that

$$\begin{aligned} \limsup_{t\to+\infty} y(t) \leq& [a_2^u + c_2^u(e^{W_1} + d_1^u)^2]L \\ &+ \ln\frac{a_2^u + c_2^u(e^{W_1} + d_1^u)^2}{b_2^l} \triangleq W_2 \\ \limsup_{t\to+\infty} u(t) \leq& \frac{\eta_1^u e^{W_1}}{\gamma_1^l} \triangleq K_1. \end{aligned} \tag{5}$$

Similarly, we have

$$\limsup_{t\to+\infty} v(t) \leq \frac{\eta_2^u e^{W_2}}{\gamma_2^l} \triangleq K_2.$$

The proof is completed. □

Lemma 3.3 *Assume*

$$a_1^l - c_1^u(e^{W_2} + d_2^u)^2 > 0, \tag{A_1}$$

then there exists two constants w and k such that

$$\liminf_{t\to+\infty} x(t) \geq w_1,\ \liminf_{t\to+\infty} u(t) \geq k_1,$$

where w_1 and k_1 can be found in the proof.

Proof. It follows from the third equation of system (1) that

$$u^{\triangle}(t) \leq -\gamma_1(t)u(t) + \eta_1^u \exp\{x(t)\}.$$

By Lemma 2.4 (i), there exists a constant $t_1 > t_0$, such that for $t > t_1$,

$$u(t) \leq u(t_1)e_{(-\gamma_1)}(t, t_1) + \frac{\eta_1^u \exp\{x(t)\}}{\gamma_1^l}.$$

Since $u(t_1)e_{(-\gamma_1)}(t, t_1) \to 0$ as $t \to +\infty$, then there exists a positive integer $t_2 > t_1$ such that

$$f_1^u u(t_1)e_{(-\gamma_1)}(t_2, t_1) \leq \frac{1}{2}[a_1^l - c_1^u(e^{W_2} + d_2^u)^2]. \tag{6}$$

Fix t_2, for $t > t_2$, we have

$$u(t) \leq u(t_1)e_{(-\gamma_1)}(t_2, t_1) + \frac{\eta_1^u \exp\{x(t)\}}{\gamma_1^l}. \tag{7}$$

One can get from (6), (7) and the first equation of system (1) that

$$\begin{aligned} x^{\triangle}(t) \geq& a_1^l - b_1^u \exp\{x(t)\} - c_1^u(e^{W_2} + d_2^u)^2 - f_1^u u(t) \\ \geq& a_1^l - b_1^u \exp\{x(t)\} - c_1^u(e^{W_2} + d_2^u)^2 \\ &- f_1^u\Big[u(t_1)e_{(-\gamma_1)}(t_2, t_1) + \frac{\eta_1^u \exp\{x(t)\}}{\gamma_1^l}\Big] \\ =& a_1^l - c_1^u(e^{W_2} + d_2^u)^2 - f_1^u u(t_1)e_{(-\gamma_1)}(t_2, t_1) \\ &- (b_1^u + \frac{f_1^u \eta_1^u}{\gamma_1^l})\exp\{x(t)\} \\ \geq& \frac{1}{2}[a_1^l - c_1^u(e^{W_2} + d_2^u)^2] - (b_1^u + \frac{f_1^u \eta_1^u}{\gamma_1^l})\exp\{x(t)\}, \end{aligned} \tag{8}$$

for $t > t_2$. Using this and Lemma 2.3 (ii), we get

$$\liminf_{t\to+\infty} x(t) \geq \Big(\frac{1}{2}[a_1^l - c_1^u(e^{W_2} + d_2^u)^2] - (b_1^u + \frac{f_1^u\eta_1^u}{\gamma_1^l})e^{W_1}\Big)L + \ln\frac{\gamma_1^l[a_1^l - c_1^u(e^{W_2} + d_2^u)^2]}{2(\gamma_1^l b_1^u + f_1^u\eta_1^u)} \triangleq w_1. \tag{9}$$

So for any $\varepsilon_1 > 0$, there exists enough large $t_3 > t_2$, such that for $t > t_3$,

$$x(t) \geq w_1 - \varepsilon_1.$$

This together with the third equation of system (1) results in

$$u^{\triangle}(t) \geq -\gamma_1(t)u(t) + \eta_1^l e^{w_1-\varepsilon_1}, \text{ for } t > t_3. \tag{10}$$

It follows from (10) and Lemma 2.4 (ii) that

$$\liminf_{t\to+\infty} u(t) \geq \frac{\eta_1^l e^{w_1-\varepsilon_1}}{\gamma_1^u}. \tag{11}$$

Setting $\varepsilon_1 \to 0$, we get from (11) that

$$\liminf_{t\to+\infty} u(t) \geq \frac{\eta_1^l e^{w_1}}{\gamma_1^u} \triangleq k_1. \tag{12}$$

The proof is completed. □

Lemma 3.4 *Assume (A_1) holds, then there exists two constants w_2 and k_2 such that*

$$\liminf_{t\to+\infty} y(t) \geq w_2,\ \liminf_{t\to+\infty} v(t) \geq k_2,$$

where w_2 and k_2 can be found in the proof.

Proof. For any $\varepsilon_2 > 0$ small enough, by Lemma 3.2 and Lemma 3.3, there exists a point $t_4 > 0$, such that for $t \geq t_4$,

$$y(t) \leq W_1 + \varepsilon_2,\ x(t) \geq w_1 - \varepsilon_2,\ v(t) \leq K_2 + \varepsilon_2. \tag{13}$$

One can get from the fourth equation of system (1) that

$$v^{\triangle}(t) \leq -\gamma_2(t)v(t) + \eta_2^u \exp\{y(t)\}.$$

By Lemma 2.4 (i), there exists a constant $t_5 > t_4$, such that for $t > t_5$,

$$v(t) \leq v(t_5)e_{(-\gamma_2)}(t, t_5) + \frac{\eta_2^u \exp\{y(t)\}}{\gamma_2^l}.$$

Since $v(t_5)e_{(-\gamma_2)}(t, t_5) \to 0$ as $t \to +\infty$, then there exists a positive integer $t_6 > t_5$ such that

$$f_2^u v(t_5)e_{(-\gamma_2)}(t_6, t_5) \leq \frac{a_2^l}{2}. \tag{14}$$

Fix t_6, for $t > t_6$, we have

$$v(t) \leq v(t_5)e_{(-\gamma_2)}(t_6, t_5) + \frac{\eta_2^u \exp\{y(t)\}}{\gamma_2^l}. \tag{15}$$

One can get from (14), (15) and the second equation of system (1) that

$$\begin{aligned} y^{\triangle}(t) \geq& a_2^l - b_2^u \exp\{y(t)\} - f_2^u v(t) \\ \geq& a_2^l - b_2^u \exp\{y(t)\} \\ &- f_2^u\Big[v(t_5)e_{(-\gamma_2)}(t_6, t_5) + \frac{\eta_2^u \exp\{y(t)\}}{\gamma_2^l}\Big] \\ =& a_2^l - f_2^u v(t_5)e_{(-\gamma_2)}(t_6, t_5) - (b_2^u + \frac{f_2^u\eta_2^u}{\gamma_2^l})\exp\{y(t)\} \\ \geq& \frac{a_2^l}{2} - (b_2^u + \frac{f_2^u\eta_2^u}{\gamma_2^l})\exp\{y(t)\}, \end{aligned} \tag{16}$$

for $t > t_6$. Using this and Lemma 2.3 (ii), we get

$$\liminf_{t\to+\infty} y(t) \geq \Big(\frac{a_2^l}{2} - (b_2^u + \frac{f_2^u\eta_2^u}{\gamma_2^l})e^{W_2}\Big)L + \ln\frac{\gamma_2^l a_2^l}{2(\gamma_1^2 b_2^u + f_2^u\eta_2^u)} \triangleq w_2. \tag{17}$$

So for the above $\varepsilon_2 > 0$, there exists enough large $t_7 > t_6$, such that for $t > t_6$,

$$y(t) \geq w_2 - \varepsilon_2.$$

This together with the fourth equation of system (1) results in

$$v^{\triangle}(t) \geq -\gamma_2(t)v(t) + \eta_2^l e^{w_2-\varepsilon_2}, \text{ for } t > t_7. \tag{18}$$

It follows from (18) and Lemma 2.4 (ii) that

$$\liminf_{t\to+\infty} v(t) \geq \frac{\eta_2^l e^{w_2-\varepsilon_2}}{\gamma_2^u}. \tag{19}$$

Setting $\varepsilon_2 \to 0$, we get from (19) that

$$\liminf_{t\to+\infty} v(t) \geq \frac{\eta_2^l e^{w_2}}{\gamma_2^u} \triangleq k_2. \tag{20}$$

The proof is completed. □

Theorem B can be obtained directly from Lemma 3.2-Lemma 3.4.

IV. UNIFORM ASYMPTOTICAL STABILITY

In this part, we will investigate the uniform asymptotical stability of system (1) by the method of Lyapunov function.

Theorem 4.1. *Assume (A_1), further suppose that*

$$\gamma_1^l > f_1^u,\ b_1^l - c_2^u(2e^{W_1} + 2d_1^u) > \eta_1^u, \tag{A_2}$$

$$\gamma_2^l > f_2^u,\ b_2^l - c_1^u(2e^{W_2} + 2d_2^u) > \eta_2^u, \tag{A_3}$$

where W_i $(i = 1, 2)$ is defined in Lemma 3.2, then system (1) with initial conditions (2) *is uniformly asymptotically stable.*

Proof. It follows from (A_2) and (A_3) that there exists a small enough $\varepsilon > 0$ such that

$$\begin{aligned} &\gamma_1^l - f_1^u > \varepsilon,\ e^{w_1-\varepsilon}\Big[b_1^l - c_2^u(2e^{W_1+\varepsilon} + 2d_1^u) - \eta_1^u\Big] > \varepsilon, \\ &\gamma_2^l - f_2^u > \varepsilon,\ e^{w_2-\varepsilon}\Big[b_2^l - c_1^u(2e^{W_2+\varepsilon} + 2d_2^u) - \eta_2^u\Big] > \varepsilon. \end{aligned} \tag{21}$$

Suppose $Z_1(t) = (x_1(t), y_1(t), u_1(t), v_1(t))^T$, $Z_2(t) = (x_2(t), y_2(t), u_2(t), v_2(t))^T$ are two solutions of system (1) with initial conditions (2). For above ε, according to Lemma 3.2-Lemma 3.4, there exist a $t_8 > 0$, when $t > t_8$, for $i = 1, 2$, we have

$$\begin{aligned} w_1 - \varepsilon \leq x_i(t) \leq W_1 + \varepsilon,\ w_2 - \varepsilon \leq y_i(t) \leq W_2 + \varepsilon \\ k_1 - \varepsilon \leq u_i(t) \leq K_1 + \varepsilon,\ k_2 - \varepsilon \leq v_i(t) \leq K_2 + \varepsilon. \end{aligned} \tag{22}$$

Consider the following Lyapunov function

$$\begin{aligned} V(t, Z_1, Z_2) =& |x_1(t) - x_2(t)| + |y_1(t) - y_2(t)| \\ &+ |u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|. \end{aligned}$$

Calculating $D^+V^\Delta(t, Z_1, Z_2)$ of $V(t, Z_1, Z_2)$ along system (1) leads to

$$
\begin{aligned}
&D^+V^\Delta(t, Z_1, Z_2)\\
=&\text{sgn}(x_1(t) - x_2(t))\Big[- b_1(t)[\exp\{x_1(t)\} - \exp\{x_2(t)\}]\\
&- c_1(t)\Big([\exp\{y_1(t)\} - d_2(t)]^2 - [\exp\{y_2(t)\} - d_2(t)]^2\Big)\\
&- f_1(t)(u_1(t) - u_2(t))\Big]\\
&+ \text{sgn}(y_1(t) - y_2(t))\Big[- b_2(t)\big[\exp\{y_1(t)\} - \exp\{y_2(t)\}\big]\\
&+ c_2(t)\Big([\exp\{x_1(t)\} - d_1(t)]^2 - [\exp\{x_2(t)\} - d_1(t)]^2\Big)\\
&- f_2(t)(v_1(t) - v_2(t))\Big]\\
&+ \text{sgn}(u_1(t) - u_2(t))\Big[- \gamma_1(t)(u_1(t) - u_2(t))\\
&+ \eta_1(t)\big[\exp\{x_1(t)\} - \exp\{x_2(t)\}\big]\Big]\\
&+ \text{sgn}(v_1(t) - v_2(t))\Big[- \gamma_2(t)(v_1(t) - v_2(t))\\
&+ \eta_2(t)\big[\exp\{y_1(t)\} - \exp\{y_2(t)\}\big]\Big]\\
=&\text{sgn}(x_1(t) - x_2(t))\Big[- b_1(t)\big[\exp\{x_1(t)\} - \exp\{x_2(t)\}\big]\\
&- c_1(t)\big[\exp\{y_1(t)\} - \exp\{y_2(t)\}\big]\big[\exp\{y_1(t)\}\\
&+ \exp\{y_2(t)\} - 2d_2(t)\big] - f_1(t)(u_1(t) - u_2(t))\Big]\\
&+ \text{sgn}(y_1(t) - y_2(t))\Big[- b_2(t)\big[\exp\{y_1(t)\} - \exp\{y_2(t)\}\big]\\
&+ c_2(t)\big[\exp\{x_1(t)\} - \exp\{x_2(t)\}\big]\big[\exp\{x_1(t)\}\\
&+ \exp\{x_2(t)\} - 2d_1(t)\big] - f_2(t)(v_1(t) - v_2(t))\Big]\\
&- \gamma_1(t)|u_1(t) - u_2(t)| - \gamma_2(t)|v_1(t) - v_2(t)|\\
&+ \text{sgn}(u_1(t) - u_2(t))\eta_1(t)\big[\exp\{x_1(t)\} - \exp\{x_2(t)\}\big]\\
&+ \text{sgn}(v_1(t) - v_2(t))\eta_2(t)\big[\exp\{y_1(t)\} - \exp\{y_2(t)\}\big].
\end{aligned}
\tag{23}
$$

Using the mean value theorem, we get

$$
\begin{aligned}
\exp\{x_1(t)\} - \exp\{x_2(t)\} = \xi_1(t)(x_1(t) - x_2(t)),\\
\exp\{y_1(t)\} - \exp\{y_2(t)\} = \xi_2(t)(y_1(t) - y_2(t)),
\end{aligned}
\tag{24}
$$

where $\xi_1(t)$ lies between $\exp\{x_1(t)\}$ and $\exp\{x_2(t)\}$ and $\xi_2(t)$ lies between $\exp\{y_1(t)\}$ and $\exp\{y_2(t)\}$. We can obtain from (21)-(24) that

$$
\begin{aligned}
&D^+V^\Delta(t, Z_1, Z_2)\\
\leq&\xi_1(t)|x_1(t) - x_2(t)|\Big[- b_1(t) + c_2(t)(2e^{W_1+\varepsilon} + 2d_1(t))\\
&+ \eta_1(t)\Big] + (f_1(t) - \gamma_1(t))|u_1(t) - u_2(t)|\\
&+ \xi_2(t)|y_1(t) - y_2(t)|\Big[- b_2(t) + c_1(t)(2e^{W_2+\varepsilon} + 2d_2(t))\\
&+ \eta_2(t)\Big] + (f_2(t) - \gamma_2(t))|v_1(t) - v_2(t)|\\
\leq& - e^{w_1-\varepsilon}|x_1(t) - x_2(t)|\Big[b_1^l - c_2^u(2e^{W_1+\varepsilon} + 2d_1^u) - \eta_1^u\Big]\\
&- (\gamma_1^l - f_1^u)|u_1(t) - u_2(t)|\\
&- e^{w_2-\varepsilon}|y_1(t) - y_2(t)|\Big[b_2^l - c_1^u(2e^{W_2+\varepsilon} + 2d_2^u) - \eta_2^u\Big]\\
&- (\gamma_2^l - f_2^u)|v_1(t) - v_2(t)|\\
\leq& - \varepsilon V(t, Z_1, Z_2), \text{ for } t > t_8.
\end{aligned}
\tag{25}
$$

Therefore, $V(t, Z_1, Z_2)$ is non-increasing. Integrating (25) from t_8 to t ($t > t_8$) leads to

$$V(t, Z_1, Z_2)+\varepsilon \int_{t_8}^{t} V(s, Z_1, Z_2)\Delta s \leq V(t_8, Z_1, Z_2) < +\infty.$$

Hence,

$$\int_{t_8}^{+\infty} V(s, Z_1, Z_2)\Delta s < +\infty,$$

which means that

$$\lim_{t\to+\infty} |x_1(t) - x_2(t)| = \lim_{t\to+\infty} |y_1(t) - y_2(t)| = 0,$$

$$\lim_{t\to+\infty} |u_1(t) - u_2(t)| = \lim_{t\to+\infty} |v_1(t) - v_2(t)| = 0.$$

Therefore, system (1) is uniformly asymptotically stable. □

Remark 4.1. *By constructing a different Lyapunov function with ours, Zhi, Ding and Li [25] established sufficient conditions on the uniformly asymptotical stability of the system (1) (see Theorem 15 in [25]) which are more complex than conditions (A_2) and (A_3) in Theorem 4.1.*

V. EXAMPLE AND NUMERIC SIMULATION

In this part, we will give some numerical simulations to support our results.

Example 5.1. Consider the following system:

$$
\begin{cases}
x^\Delta(t) =0.35 + 0.02\sin(2t) - 0.33\exp\{x(t)\}\\
\qquad - 0.02[\exp\{y(t)\} - 0.06]^2 - 0.2u(t),\\
y^\Delta(t) =0.38 + 0.01\cos(3t) - 3\exp\{y(t)\}\\
\qquad + 0.03[\exp\{x(t)\} - 0.05]^2 - 0.55v(t),\\
u^\Delta(t) = - (0.6 + 0.03\sin(\sqrt{7}t))u(t) + 0.2\exp\{x(t)\},\\
v^\Delta(t) = - (0.57 + 0.01\cos(\sqrt{5}t))v(t) + 1.5\exp\{y(t)\}.
\end{cases}
\tag{26}
$$

By simple calculation, we get

$$a_2^l - f_2^u v^* \approx -0.231 < 0,$$

which implies that we can't judge the permanence by Theorem A since (Q_2) does not hold.

On the other hand, $z_1(t) = \exp\{x(t)\}$ and $z_2(t) = \exp\{y(t)\}$, then system (26) reduces to the following continuous system:

$$
\begin{cases}
\dot{z_1}(t) =z_1(t)\Big(0.35 + 0.02\sin(2t) - 0.33z_1(t)\\
\qquad - 0.02(z_2(t) - 0.06)^2 - 0.2u(t)\Big),\\
\dot{z_2}(t) =z_2(t)\Big(0.38 + 0.01\cos(3t) - 3z_2(t)\\
\qquad + 0.03(z_1(t) - 0.05)^2 - 0.55v(t)\Big),\\
\dot{u}(t) = - (0.6 + 0.03\sin(\sqrt{7}t))u(t) + 0.2z_1(t),\\
\dot{v}(t) = - (0.57 + 0.01\cos(\sqrt{5}t))v(t) + 1.5z_2(t).
\end{cases}
\tag{27}
$$

Since $\mu(t) \equiv 0$, we can choose $L = 0$ for convenience. Thus, for system (27), we have

$$a_1^l - c_1^u(e^{W_2} + d_2^u)^2 \approx 0.3292 > 0,$$

so (A_1) holds and system (26) is permanent according to Theorem B.

Moreover, since

$$
\begin{aligned}
\gamma_1^l - f_1^u = 0.37 > 0,\ \gamma_2^l - f_2^u = 0.01 > 0,\\
b_1^l - c_2^u(2e^{W_1} + 2d_1^u) - \eta_1^u \approx 0.0597 > 0,\\
b_2^l - c_1^u(2e^{W_2} + 2d_2^u) > \eta_2^u \approx 1.4921 > 0,
\end{aligned}
$$

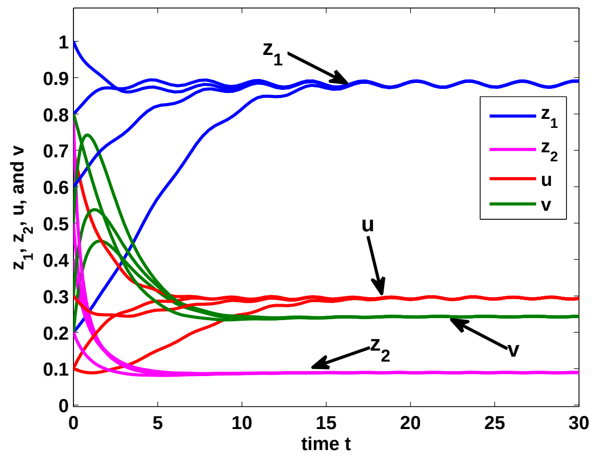

Fig. 1. Numeric simulations of system (27) with the initial condition $(z_1(0), z_2(0), u(0), v(0))^T = (0.2, 0.4, 0.1, 0.2)^T$, $(1, 0.5, 0.7, 0.3)^T$, $(0.8, 0.8, 0.1, 0.5)^T$ and $(0.6, 0.2, 0.3, 0.8)^T$, respectively.

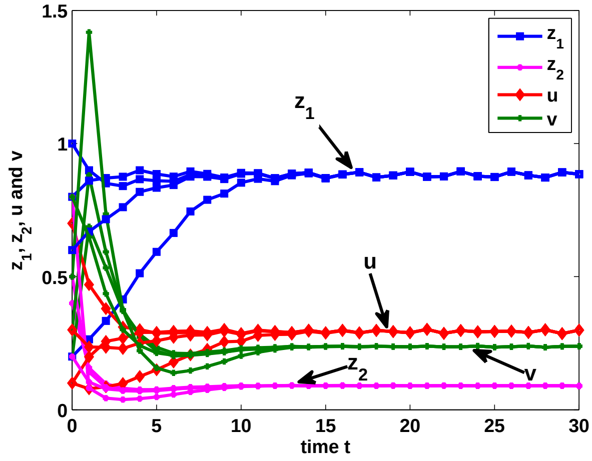

Fig. 2. Numeric simulations of system (28) with the initial condition $(z_1(0), z_2(0), u(0), v(0))^T = (0.2, 0.4, 0.1, 0.2)^T$, $(1, 0.5, 0.7, 0.3)^T$, $(0.8, 0.8, 0.1, 0.5)^T$ and $(0.6, 0.2, 0.3, 0.8)^T$, respectively.

so all conditions in Theorem 4.1 are satisfied and system (27) is permanent and uniformly asymptotically stable which is supported by Fig. 1.

When $\mathbb{T} = \mathbb{Z}$, if we also set $z_1(t) = \exp\{x(t)\}$ and $z_2(t) = \exp\{y(t)\}$, then system (26) reduces to the following discrete system:

$$\begin{cases} z_1(t+1) = z_1(t)\exp\Big[0.35 + 0.02\sin(2t) - 0.33z_1(t) \\ \qquad\qquad - 0.02(z_2(t) - 0.06)^2 - 0.2u(t)\Big], \\ z_2(t+1) = z_2(t)\exp\Big[0.38 + 0.01\cos(3t) - 3z_2(t) \\ \qquad\qquad + 0.03(z_1(t) - 0.05)^2 - 0.55v(t)\Big], \\ \Delta u(t) = -(0.6 + 0.03\sin(\sqrt{7}t))u(t) + 0.2z_1(t), \\ \Delta u(t) = -(0.57 + 0.01\cos(\sqrt{5}t))v(t) + 1.5z_2(t), \end{cases} \tag{28}$$

Since $\mu(t) \equiv 1$, we choose $L = 1$ for convenience. Thus, we have

$$a_1^l - c_1^u(e^{W_2} + d_2^u)^2 \approx 0.3292 > 0,$$
$$\gamma_1^l - f_1^u = 0.37 > 0, \ \gamma_2^l - f_2^u = 0.01 > 0,$$
$$b_1^l - c_2^u(2e^{W_1} + 2d_1^u) - \eta_1^u \approx 0.0296 > 0,$$
$$b_2^l - c_1^u(2e^{W_2} + 2d_2^u) > \eta_2^u \approx 1.4881 > 0,$$

so all conditions in Theorem B and Theorem 4.1 are satisfied, system (28) is permanent and uniformly asymptotically stable. Our numerical simulation also supports this result (see Fig. 2).

VI. CONCLUSION

In this paper, we consider a competition and cooperation model of two enterprises with feedback controls on time scales which was investigated by Zhi, Ding and Li [25]. By using some differential inequalities on time scales, we obtain a new condition on the permanence of system (1) which is weaker than those in [25] and [26]. This result shows that feedback terms are irrelevant to the permanence of this model. By constructing a different Lyapunov function with Zhi, Ding and Li [25], we established some new sufficient conditions on the uniformly asymptotical stability of the considered system which are more simpler and easier to verify then those in [25]. Therefore, our results improve and complement those in [25, 26].

REFERENCES

[1] H. Li and L. Dai, "Positive solutions for nonlinear differential equations with sign changing nonlinearity on a measure chain," *Journal of Mathematics*, vol. 32, no. 1, pp. 9-16, 2012.

[2] R. McKellar and K. Knight, "A combined discrete-continuous model describing the lag phase of Listeria monocytogenes," *International Journal of Food Microbiology*, vol. 54, no. 3, pp. 171-180, 2000.

[3] G. Liu, Y. Luo and L. Shu, "Asymptotic synchronization of complex dynamical networks with time-varying delays on time scales," *Engineering Letters*, vol. 26, no. 2, pp. 210-215, 2018.

[4] M. Hu and L. Wang, "Almost periodic solution for a Nabla BAM neural networks on time scales,"*Engineering Letters*, vol. 25, no.3, pp. 290-295, 2017.

[5] L. Wang and P. Xie, "Permanence and extinction of delayed stage-structured predator-prey system on time scales,"*Engineering Letters*, vol. 25, no. 2, pp. 147-151, 2017.

[6] H. Alfifi and T. Marchant, "Feedback control for a diffusive delay Logistic equation: semi-analytical solutions," *IAENG International Journal of Applied Mathematics*, vol. 48, no. 3, pp. 317-323, 2018.

[7] L. Yang, Y. Yang, Y. Li and T. Zhang, "Almost periodic solution for a Lotka-Volterra recurrent neural networks with harvesting terms on time scales,"*Engineering Letters*, vol. 24, no. 4, pp. 455-460, 2016.

[8] W. Yang and X. Li, "Feedback control has no influence on the persistent property of a single species discrete model with time delays," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2007-2013, 2011.

[9] J. Chen and S. Yu, "Permanence for a discrete ratio-dependent predator-prey system with Holling type III functional response and feedback controls," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 326848, 6 pages, 2013.

[10] F. Chen, " Permanence of a discrete N-species cooperation system with time delays and feedback controls," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 23-29, 2007.

[11] Y. Yang and T. Zhang, "Permanence and uniform asymptotical stability in consumer-resource system with mutual interference on time scales," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 3, pp. 334-338, 2019.

[12] S. Hilger, "Analysis on measure chains-a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18-56, 1990.

[13] M. Bohner, A. Peterson, "Dynamic Equations on Time Scales: An Introduction with Applications," *Birkhäuser, Boston*, 2001.

[14] Y. Fan, Y. Yu and L. Wang, "Some differential inequalities on time scales and their applications to feedback control systems," *Discrete Dynamics in Nature and Society*, vol. 2017, Article ID 9195613, 11 pages, 2017.

[15] L. Wang and Y. Fan, "Permanence and existence of periodic solutions for Nicholson's blowflies model with feedback control and delay on time scales," *Discrete Dynamics in Nature and Society*, vol. 2018, Article ID 3403127, 7 pages, 2018.

[16] Y. Li, L. Yang and H. Zhang, "Permanence and uniformly asymptotical stability of almost periodic solutions for a single-species model with feedback control on time scales," *Asian-European Journal of Mathematics*, vol. 7, no. 1, Article ID 1450004, 15 pages, 2014.

[17] Q. Yue, "The influence of positive feedback control to a single species stage structure system," *Engineering Letters*, vol. 28, no. 2, pp. 322-330, 2020.

[18] Z. Li, J. Zhou and T. Zhang, "Permanence and uniform asymptotical stability of a ratio-dependent Leslie system with feedback controls on time scales," *ScienceAsia*, vol. 46, no. 4, pp. 494-502, 2020.

[19] S. Yu, H. Wu and J. Chen, "Multiple periodic solutions of delayed predator-prey systems with type IV functional responses on time scales," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 271672, 12 pages, 2012.

[20] L. Wang, "Dynamics of an almost periodic single-species system with harvesting rate and feedback control on time scales," *IAENG International Journal of Computer Science*, vol. 46, no. 2, pp. 237-242, 2019.

[21] X. Chen, C. Shi and D. Wang, "Dynamic behaviors for a delay Lasota-Wazewska model with feedback control on time scales," *Advances in Difference Equations*, vol. 2020, Article ID 17, pp. 1-13, 2020.

[22] S. Yu, "Effect of predator mutual interference on an autonomous Leslie-Gower predator-prey model," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 2, pp. 229-233, 2019.

[23] J. Zhang, S. Yu, Q. Wang, "Extinction and stability of a discrete competitive system with Beddington-DeAngelis functional," *Engineering Letters*, vol. 28, no. 2, pp. 406-411, 2020.

[24] Y. Liu, Y. Liao, Y. Yang, S. Han and T. Zhang, "Multiplicity of positive almost periodic solutions and local asymptotical stability for a kind of time-varying fishing model with harvesting term," *Engineering Letters*, vol. 28, no. 4, pp. 1232-1237, 2020.

[25] Y. Zhi, Z. Ding and Y. Li, " Permanence and almost periodic solution for an enterprise cluster model based on ecology theory with feedback controls on time scales," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 639138, 14 pages, 2013.

[26] L. Lu, Y. Lian and C. Li, "Dynamics for a discrete competition and cooperation model of two enterprises with multiple delays and feedback controls," *Open Mathematics*, vol. 15, no. 1, pp. 218-232, 2017.