Analysis of the Single-Vendor–Multi-Buyer Inventory Model for Imperfect Quality with Controllable Lead Time

Rubono Setiawan, Salmah*, Widodo, Irwan Endrayanto, and Indarsih

Abstract— **The two-echelon single vendor–multi-buyer inventory model is an interesting topic and is suitable to many real conditions in the supply chain system. The existence of imperfect quality items and controllable lead time is one of the important assumptions in modern inventory analysis. This study analyzes the two-echelon single-vendor–multi-buyer inventory model with imperfect quality items presented as random variables, which follow the Binomial distribution. We use boundedness service level constraint to replace shortage cost term in the objective function. We also use the integrated scheme to formulate optimization problems. We prove a theorem about the nonconvexity properties of the objective function and then apply this result with the Karush–Kuhn–Tucker (KKT) conditions and the Lagrange multiplier method to obtain an optimum solution. Using numerical examples, we show that our model can help a vendor minimize the total cost of the inventory system and the number of lots in which a product is delivered from the vendor to all buyers under an uncertain lead time by reducing the number of imperfect items.**

Index Terms—**Multi-buyer, Controllable lead time, Imperfect quality, Service level.**

I. Introduction

IMPERFECT quality items are products in many shipments with defective quality, that is, products in several shipments with imperfect quality. According to Lin [1], imperfect quality items affect the inventory level, service level constraint (SLC) for a customer, and order quantity in the supply chain. The traditional assumption, which argues that products are 100% perfect, is no longer suitable to the real condition. The first analysis of the inventory model for imperfect quality has been proposed by Porteus [2] and Rosenblatt and Lee [3]. Another research is encouraged to study these topics further. Paknejad et al. [4] proposed an inventory model for imperfect quality under some conditions, such as constant lead time demand, stochastic demand, and allowed shortage. Wu and Ouyang [5] simplified Paknejad's model by adding an analysis on the opportunity of the back-ordering process, which can be delivered by mixture backorder (full and partial back ordering), and a lost sale analysis. Lin [1] analyzed the inventory model for imperfect quality in which the random variable of a defective item follows a Binomial distribution under some conditions, such as allowed shortage and partial back-ordering process. Ko et al. [6] discussed about preventive maintenance programs in the imperfect production process. Other results of imperfect quality item analysis in inventory can be found in several studies, such as [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], and [19].

A study in a two-echelon inventory model has been extended to the multi-player approach. The coordination policy and synchronization in production flow are essential in the analysis of inventory with multiple buyers. Several papers have discussed coordination and synchronization assumptions in the inventory model for a single vendor and multiple buyers. For example, Banerjee and Banerjee [21] worked with a single-vendor–multi-buyer model by using electronic data interchange under the assumption of the coordination policy between a vendor and multiple buyers. A similar result is proposed by Chu and Leon [26] with their analysis of the inventory model for a single vendor and multiple buyers under private information. Banerjee and Burton [22] focused on the replenishment policy under the coordination policy. This coordination can reduce inventory cost, rather than the traditional approach of each vendor and buyer implementing their own optimal policy. This coordination policy can be implemented into synchronization assumption in the production flow between a vendor and all buyers. Hoque [23] provided the application of this type of synchronization on the inventory model with a single vendor and multiple buyers under the coordination policy. Jha and Shanker [20] also shared an analysis of the inventory with a single vendor and multiple buyers under coordination and synchronization, but in uncertain lead time conditions. This uncertainty lead time is handled by a common method to control lead time cost, that is, controllable lead time. The vendor and all buyers still agree to take the coordination policy and synchronization in the production flow. In this model, they included SLC but did not consider stockout cost in the objective function. Moreover, this model excluded imperfect quality items. Another result of inventory analysis involving a single vendor and multiple buyers for imperfect quality is given by Mandal and Giri [27]. In their analysis,

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they assumed that all buyers follow a partial backorder and the defective item is regarded as a percentage from all products.

In this research, we are interested in analyzing a single-vendor–multi-buyer inventory model, which extends and modifies the model proposed by Jha and Shanker [20] into an inventory model for imperfect quality items. We still adopt the coordination policy and synchronization in the production flow, including controllable lead time. We add assumptions about the existence of imperfect quality, transportation cost, and boundedness SLC into the model. Because the product quality from the production process is not always 100% perfect, then the assumption of imperfect quality items is essential to obtain a realistic inventory model. Considering the existence of imperfect quality, all buyers must take different holding costs for non-defective and defective items. So, there is modification in holding cost term which from single holding cost into two different holding cost. Then, we also give modification with add fixed transportation cost to the objective function of the model. We use the boundedness SLC to replace the shortage cost term in the objective function and then take this service level as a constraint of the optimization problem. Therefore, we work with a constrained optimization problem in which total cost is the objective function and service level is a constraint. In this research, we assume that all buyers follow full-ordering. We assume that imperfect quality items are random variables and follow the Binomial distribution. We use the expectation value concept of Binomial distribution to find the joint total cost. Those assumptions is different with other work like by Mandal and Giri [27] which assume buyers follow a partial backorder and the defective item is regarded as a percentage from all products. Furthermore, we consider the synchronization of the production flow between a vendor and all buyers. This type of synchronization can minimize the total average cost. The synchronization of production flow is yet to be discussed in a single-vendor–multi-buyer setting under the combination of assumption imperfect quality items and service level as a constraint. We pay further attention to the mathematical analysis of the non-convexity of the objective function of the optimization problem. Non-convexity properties are a common problem in the objective function of an inventory model with a complex form. However, according to the best of author information on inventory research, the analysis of non-convexity properties has never been discussed before. Therefore, we prove a theorem about such properties. Subsequently, we apply this result with Karush–Kuhn–Tucker (KKT) conditions and the Lagrange multiplier method to obtain an optimum solution of constrained optimization problems. We also provide an algorithm procedure for the optimal solution and numerical examples by using appropriate simulation data. From these numerical results, controlling the number of imperfect quality items can reduce the total cost for the system.

II. Model Formulation

We determine the analytical and numerical results of the inventory model, which involves a single vendor and multiple buyers, for an imperfect quality product. Assumptions about the controllable lead time and the synchronization of the production flow between all buyers and the vendor are referred to the assumptions by Jha and Shanker [20]. We consider imperfect quality items as a random variables, which follow the Binomial distribution. An integrated scheme and some mathematical optimization techniques are applied to determine the optimal value. We describe the difference between our result and that obtained by Jha and Shanker [20]. Jha and Shanker [20] excluded the existence of imperfect quality in their model. By contrast, we define the existence of imperfect quality items in the products. Such an existence implies two different types of holding cost: holding cost for defective items and holding cost for non-defective items. We add fixed transportation costs to the objective function of the model. For the optimization process, we work with constrained optimization problems where expected cost is an objective function and boundedness service level is a constraint. We also provide a theorem with its proof to explain the non-convexity condition of the objective function. Optimization methods using KKT conditions and the Lagrange multiplier are applied for non-convexity conditions. Before going into any details about the mathematical model, some notations, including decision variables and parameters, are provided in Table I.

TABLE I
NOTATIONS

Notation	Explanation
q_i	Order quantity for each buyer, a decision variable.
k_i	Safety factor, a decision variable.
L_i	Lead time, a decision variable
m	Number of lots in which a product is delivered from from the vendor to each buyer-i.
D_i	Average demand per unit time.
O_i	Ordering cost.
B_i	Unit purchase cost.
r_i	Reorder point.
σ_i	Standard deviation of demand per unit time.
X_i	Lead time demand which has a finite mean $D_i L_i$ and standard deviation $\sigma_i.\sqrt{L_i}$.
K_i	Fixed freight cost.
γ_i	Defective rate in the order lot.
Y_i	Random variable for a defective items.
h_{gi}	Holding cost for a non-defective items per unit time.
h_{bi}	Holding cost for a defective items per unit time.
x	Buyers' inspection time.
$E[.]$	Expectation.
q	Order quantity of a buyer, including defective items, a decision variable.
P	Production rate.
T	Length of the cycle.
S	Fixed setup cost.
h_v	Vendor's holding cost per unit time.
ω	Treatment cost for imperfect quality.
ETC^U	Expected average total cost per unit time.
W	Set of buyers with an active SLC.
V	Set of buyers with an inactive SLC.

A. Assumptions

In this section, we provide the details and explanations about the assumptions used to construct the proposed model.

1) The supply chain system consists of one vendor producing a single item and delivering it to several numbers of i buyers in m shipments.
2) Average demand from each buyer-i is D_i and with a standard deviation σ_i.
3) The inventory is continuously reviewed. Each buyer-i places an order immediately when the inventory level falls into a reorder point.
4) The lead time related to each buyer-i has several n_i mutually independent components. The rth component, which is $r \leq n_i$, has a minimum duration $a_{i,r}$ and normal duration $b_{i,r}$ and a crushing cost per unit time $c_{i,r}$. Index i and r refer to each buyer-i and each component, respectively. Without loss of generality, we assume that $c_{i,1} \leq c_{i,2} \leq ... \leq c_{i,n_i}, \forall i$. The components of lead time can be crashed one at a time starting with the component of least $c_{i,r}, \forall i$, and so on. Crashing costs are fully transferred to each buyer-i if any. Let, $L_{i,r} = \sum_{r+1}^{n_i} b_{i,r} + \sum_{j=1}^{r} a_{i,j}$, $r = 1, 2, 3, \ldots, n_i, \forall i$, denote the length of lead time for buyer-i whose components $1, 2, 3, ..., r$ are crashed to their minimum duration. The maximum duration of lead time for the buyers is denoted by $L_{i,0}$, with $L_{i,0} = \sum_{r=1}^{n_i} b_{i,r}, \forall i$. Then, the lead time crashing cost $C_i(L_i)$ per-cycle of the ith buyer is given by
$$C_i(L_i) = c_{i,r}\left(L_{i,r-1} - L_{i,r}\right) + \textstyle\sum_{j=1}^{r-1} c_{i,j}\left(b_{i,j} - a_{i,j}\right), \quad (1)$$
where $L_{i,r} \in \left[L_{i,r}, L_{i,r-1}\right], \forall i \in \mathbf{N}$.
5) A fixed cost is applied for transporting products from the vendor to each buyer-i per cycle, and the cost is assumed to be the same for all buyer-i.
6) Some defective items with probability $\gamma_i\,(0 \leq \gamma_i < 1)$ exist in a lot that arrives on each buyer-i's side. For simplification, this probability is assumed to have the same value for each buyer-i, that is, $\gamma_i = \gamma, \forall i$. These defective items are discovered after the inspection process with screening rate x and are returned to the vendor in the next lot of shipments. The number of defective items denoted by Y_i has a binomial distribution with parameters q and γ.

B. Buyers' Expected Average Total Cost

Given that $\gamma_i = \gamma$, then $Y_i = Y$. We denote Y as a random variable, which represents the number of the defective items with a defective rate of $\gamma\ (0 \leq \gamma < 1)$ within a lot of q_i and follow the Binomial distribution
$$P_r(Y) = C_Y^{q_i}\gamma^Y(1-\gamma)^{q_i - Y)}, Y = 0, 1, 2, \ldots, q_i, \quad (2)$$
where $E[Y] = q_i\gamma$, $E\left[Y^2\right] = q_i^2\gamma^2 + q_i\gamma(1-\gamma)$. The expected length of the cycle time and expected total cost under a lot of q_i are formulated by
$$E[T] = E\left[\frac{q_i - Y}{D}\right] = \frac{q_i(1-\gamma)}{D}. \quad (3)$$
$$EJTC = \frac{E[JTC]}{E[T]}. \quad (4)$$
The total cost for each buyer is the sum of several components of cost, such as fixed ordering cost, fixed transportation cost, holding cost for both non-defective items and defective items, and crashing lead time cost. SLC replaces stockout cost because shortage cost is a constraint in the inventory model. All of the buyers run the inspection process after a lot of q_i arrives at their's side. The inspection process is delivered with the rate of inspection x and inspection period $\frac{q_i}{x}$. The main purpose of this process is to detect non-defective items in a lot of q_i. These defective items are discovered after the inspection process with screening rate x and are returned to the vendor in the next lot of shipments. To illustrate the dynamic in the buyers' side and the vendor's side, we give the inventory level for the buyers and the vendor in Figure 1. The inventory level per-cycle for each buyer-i for non-defective items is evaluated on the basis of the number of defective items in order lot, which arrives in each buyer-i and is reduced by the number of defective items that are not detected when inspection period $\frac{q_i}{x}$ is launched by each buyer.

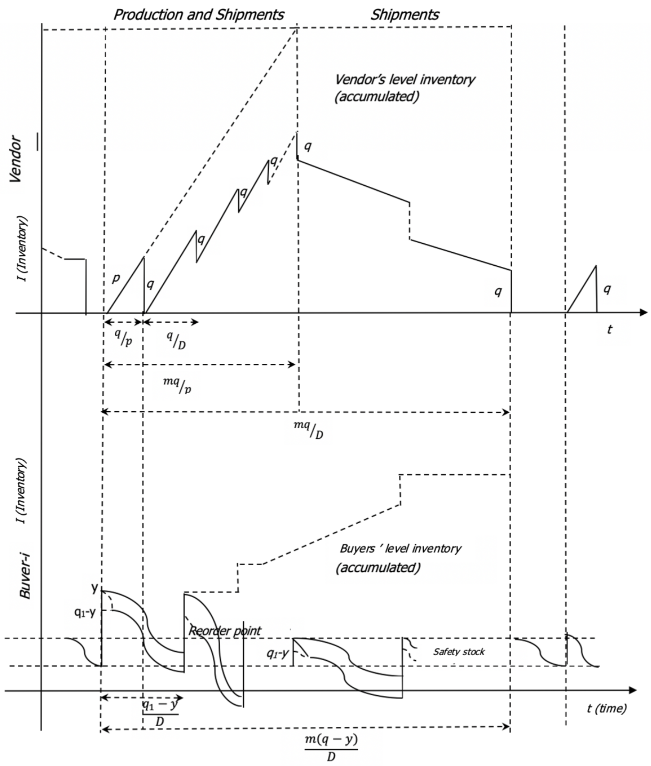

Fig. 1. Inventory Level for the Vendor and Buyers

We note that all the arrival items are accounted for the non-defective items until they are gradually found in an inspection process. Therefore, we can calculate the inventory level for defective items ($\widehat{I_{bi}}$) by subtracting the accumulated defective items found during inspection time from defective items throughout the cycle time. Therefore, we have
$$\widehat{I_{bi}} = \left(\frac{(q_i - y)y}{D_i} - \frac{q_i y}{2x}\right). \quad (5)$$
Thus, the holding cost term for defective items is obtained when Equation (5) is multiplied by holding cost for defective items and unit purchase cost, that satisfy
$$h_{bi}B_i\widehat{I_{bi}} = h_{bi}B_i\left(\frac{(q_i - y)y}{D_i} - \frac{q_i y}{2x}\right). \quad (6)$$
The average inventory level per cycle for non-defective items is obtained from the sum of average inventory level for non-defective items, which come to buyers' side and average

inventory level for buffer stock for each buyer-i. This formula is given by

$$\widehat{I_{gi}} = \frac{q_i - y}{D}\left(\frac{q_i y D_i}{2x(q_i - y)} + \frac{q_i - y}{2} + k_i \sigma_i \sqrt{L_i}\right). \quad (7)$$

Holding cost term for non-defective items per cycle is obtained when Equation (7) is multiplied by holding cost for defective items and unit purchase cost, that satisfy

$$h_{gi} B_i \widehat{I_{gi}} = h_{gi} B_i \frac{q_i - y}{D}\left(\frac{q_i y D_i}{2x(q_i - y)} + \frac{q_i - y}{2} + k_i \sigma_i \sqrt{L_i}\right). \quad (8)$$

Furthermore, we arrange a total cost per cycle for the buyers as a sum of a fixed ordering cost, a fixed transportation cost, holding cost term for defective items (Equation (6)), and holding cost term for non-defective items (Equation (8)) as

$$\begin{aligned} TC_{bi}(q_i, k_i, L_i; y) &= O_i + K_i \\ &+ h_{gi} B_i \frac{(q_i - y)}{D}\left(\frac{q_i y D_i}{2x(q_i - y)} + \frac{q_i - y}{2} + k_i \sigma_i \sqrt{L_i}\right) \\ &+ h_{bi} B_i \left(\frac{(q_i - y) y}{D_i} - \frac{q_i y}{2x}\right) + C_i(L_i). \quad (9) \end{aligned}$$

We consider the synchronization assumption, that is, $q_i = D_i \frac{q}{D}$, then Equation (9) can be written in a new form. In this form, q_i is replaced by q, therefore

$$\begin{aligned} TC_{bi}(q, k_i, L_i; y) &= O_i + K_i \\ &+ h_{gi} B_i \frac{\frac{D_i q}{D} - y}{D_i}\left(\frac{\frac{D_i q y D_i}{D}}{2x(\frac{D_i q}{D} - y)} + \frac{\frac{D_i q}{D} - y}{2} + k_i \sigma_i \sqrt{L_i}\right) \\ &+ h_{bi} B_i \left(\frac{(\frac{q D_i}{D} - y) y}{D_i} - \frac{\frac{q D_i y}{D}}{2x}\right) + C_i(L_i). \quad (10) \end{aligned}$$

From Equation (10) the expected total cost for buyers per cycle is given by

$$\begin{aligned} ETC_{bi}(q, k_i, L_i) &\equiv E\left[TC_i\left(q, k_i, L_i; Y\right)\right] = O_i + K_i \\ &+ h_{gi} B_i \frac{q(1-\gamma)}{D}\left(\frac{q D_i \gamma}{2x(1-\gamma)} + \frac{q D_i (1-\gamma)}{2D} + k_i \sigma_i \sqrt{L_i}\right) \\ &+ h_{bi} B_i \frac{q^2 \gamma}{D}\left((1-\gamma) - \frac{D_i}{2x}\right) + C\left(L_i\right). \quad (11) \end{aligned}$$

Based on Equation (4), we obtain the formulation of the expected average total cost per unit time for each buyer as

$$ETC_{bi}^{U}(q, k_i, L_i) = \frac{ETC_i\left(q, k_i, L_i\right)}{E\left[T\right]} = \frac{ETC_i\left(q, k_i, L_i\right) D}{q(1-\gamma)},$$

Hence,

$$\begin{aligned} ETC_{bi}^{U}(q, k_i, L_i) &= \frac{D}{q(1-\gamma)}\left(O_i + K_i + C_i\left(L_i\right)\right) \\ &+ h_{gi} B_i \left(\frac{q D_i \gamma}{2x(1-\gamma)} + \frac{q D_i (1-\gamma)}{2D} + k_i \sigma_i \sqrt{L_i}\right) \\ &+ h_{bi} B_i q \gamma \left(1 - \frac{D_i}{2x(1-\gamma)}\right). \quad (12) \end{aligned}$$

C. Vendor's Expected Average Total Cost

Cost for the vendor has three components, such as setup cost, cost of treatment for defective items, and holding costs. Considering that the rate production of the vendor is higher than the demand rate from all of the buyers, the vendor inventory level increases gradually. After the first lot of production finishes in the vendor, it is delivered to each buyer-i. The production process continues until the quantity of mq is reached in one production cycle, and the process is stopped immediately. Each buyer-i receives each of size q_i's in m lots of shipments until the level inventory in vendor reaches to zero. As illustrated in Figure 1, the level inventory of the vendor $(\widehat{I_v})$ can be obtained by subtracting the accumulated all buyer-i inventory level from the accumulated vendor inventory level as

$$\begin{aligned} \widehat{I_v} &= m\left(q.\frac{q}{P} + q\frac{q(m-1)q}{D}\right) - \int_0^{\frac{mq}{P}} Pt dt \\ &- \sum_{i=1}^{N}\left(1 + 2 + ... + (m-1) q_i \frac{q_i}{D_i}\right). \end{aligned}$$

$\Leftrightarrow$

$$\widehat{I_v} = m\frac{q^2}{P} + m(m-1)\frac{q^2}{D} - \frac{1}{2}\frac{Pm^2q^2}{P^2} - \frac{q^2(m-1)m}{2D^2}\sum_{i=1}^{N} D_i.$$

$$\widehat{I_v} = \left(mq\left(\frac{q}{P} + (m-1)\frac{q}{D}\right) - \frac{m^2q^2}{2P}\right) - \frac{q^2(m-1)mD}{2D^2}.$$

$\Leftrightarrow$

$$\widehat{I_v} = mq\left(\frac{q}{P} + (m-1)\frac{q}{D} - \frac{mq}{2P}\right) - \frac{(m-1)mq^2}{2D}.$$

$\Leftrightarrow$

$$\widehat{I_v} = \frac{mq^2}{2}\left(\frac{m-1}{D} - \frac{m-2}{P}\right). \quad (13)$$

Based on Equation (13), we obtain the holding cost term for the vendor is

$$h_v \frac{mq^2}{2}\left(\frac{m-1}{D} - \frac{m-2}{P}\right). \quad (14)$$

The cost of treatment for defective items is $mY\omega$. Therefore, the random variable of the vendor's total cost is sum of the fixed setup cost, cost of treatment for defective items, and holding cost term. This give

$$C_v(q, m) = S + mY\omega + h_v \frac{mq^2}{2}\left((\frac{m-1}{D} - \frac{m-2}{P}\right). \quad (15)$$

We apply the expectation formula in (4) to Equation (15), for obtaining the expected average total cost per unit time as follows:

$$ETC_v^U(q, m) = \frac{D\left(S + mq\gamma\omega + h_v \frac{mq^2}{2}\left(\frac{m-1}{D} - \frac{m-2}{P}\right)\right).}{mq(1-\gamma)} \quad (16)$$

Given that integrated scheme exists between the vendor and all buyers, then the expected average total cost per unit time is the sum of the vendor's expected average total cost per unit time and buyers' expected average total cost per unit time, as given by

$$\begin{aligned} &JETC^U(q, k_1, k_2, ..., k_N, L_1, L_2, ..., L_N, m) = \\ &\sum_{i=1}^{N} ETC_{bi}^U(q, k_i, L_i) + ETC_v^U(q, m). \quad (17) \end{aligned}$$

D. SLC and Optimization Problem

Obtaining quantification about penalty costs associated with shortage conditions is difficult. SLC can be used to replace the stock-out cost term. SLC formula for each buyer-i is given by

$$\frac{E\left[(X_i-r_i)^+\right]}{q_i}\le\eta_i, \tag{18}$$

where η_i is a real constant value. We consider that the lead time demand analysis is following distribution-free method and satisfies the suitable theorem proposed by Mandal and Giri [27].

$$E\left[(X_i-r_i)^+\right]\le\frac{\sigma_i\sqrt{L_i}}{2}\left(\sqrt{1+k_i^2}-k_i\right), \tag{19}$$

Inequality (19) is equivalent with

$$\frac{\frac{\sigma_i\sqrt{L_i}}{2}\left(\sqrt{1+k_i^2}-k_i\right)}{q_i}\le\eta_i,\forall i. \tag{20}$$

Given that all buyers and the vendor agree to the synchronization process, then Equation (20) can be transformed into

$$\frac{D\sigma_i\sqrt{L_i}\left(\sqrt{1+k_i^2}-k_i\right)}{2D_iq}\le\eta_i,\forall i. \tag{21}$$

Inequality (21) is set to be a constraint in our optimization problem. Therefore, constrained optimization problem by using (17) and (21) is

$$\min JETC^U, \tag{22}$$

s.t.

$$\frac{D\sigma_i\sqrt{L_i}\left(\sqrt{1+k_i^2}-k_i\right)}{2D_iq}\le\eta_i,\forall i. \tag{23}$$

III. Solution Properties

Before going into any details about optimal analysis, the convexity analysis of the objective function of the optimization problem (22)−(23) is provided through the following theorem.

Theorem 3.1: The joint expected average total cost function $JETC^U = F(q,k_i,L_i,m)$ of (22)−(23) is not convex in $(q,k_1,k_2,\ldots,k_N,L_1,L_2,\ldots,L_N,m)$.

Proof: The Hessian matrix of the objective function $JETC^U$ is given by

$$H=\begin{pmatrix} c_1 & 0 & 0 & c_2\\ 0 & 0 & c_3 & 0\\ 0 & c_3 & -c_4 & 0\\ c_5 & 0 & 0 & c_6\end{pmatrix},$$

where $c_1=\frac{D\left(\frac{S}{m}+(O_i+K_i+C_i(L_i))\right)}{q^3(1-\gamma)}$, $c_2=\frac{2SD}{m^2q(1-\gamma)}+\frac{Dh_v\left(\frac{1}{D}-\frac{1}{P}\right)}{2(1-\gamma)}$,

$c_3=\frac{h_{gi}B_i\sigma_i}{2\sqrt{L_i}}$, $c_4=\frac{h_{gi}B_ik_i\sigma_i}{4\sqrt{L_i^3}}$,

$c_5=\frac{D\left(\frac{S}{q^2m^2}+\frac{h_v}{2}\left(\frac{1}{D}-\frac{1}{P}\right)\right)}{1-\gamma}$, and $c_6=\frac{2SD}{m^3q(1-\gamma)}$.

Using the counter-example process, for $i=1$, F is proven not convex in (q,k_1,L_1,m). For $i=1$, the eigenvalue of H is given by $|\lambda I-H|=0$. Therefore, we obtain

$$\begin{vmatrix} \lambda-c_{11} & 0 & 0 & -c_2\\ 0 & \lambda & -c_{31} & 0\\ 0 & -c_{31} & \lambda+c_{41} & 0\\ c_5 & 0 & 0 & \lambda-c_6\end{vmatrix}=0,$$

where $c_{11}=\frac{D\left(\frac{S}{m}+(O_i+K_i+C_i(L_i))\right)}{q^3(1-\gamma)}$, $c_{31}=\frac{h_{g1}B_1\sigma_1}{2\sqrt{L_1}}$, and $c_{41}=\frac{h_{g1}B_1k_1\sigma_1}{4\sqrt{L_1^3}}$. If we compute those determinant equation, then we obtain the following polynomial.

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0, \tag{24}$$

where

$$a_0=1,$$
$$a_1=-\left(\frac{D\left(\frac{2S}{m^2}+\frac{(S+(O_1+K_1+C_1(L_1))m)}{q^2}\right)}{q(1-\gamma)m}-c_{41}\right),$$
$$a_2=\left(c_{41}c_6+(c_{31})^2+c_{11}(c_{41}-c_6)\right),$$
$$a_3=c_{11}+c_{41}\left(c_6+c_{41}c_6{L_1}^2+c_{41}{L_1}^2\right),$$
$$a_4=-c_{11}{c_{41}}^2c_6{L_1}^2.$$

If we apply Routh−Hurwitz's criteria, then we know that not all of roots of (24) are negative. If we apply Descartes's rule of signs and Vieta's formula, then we conclude that obtaining all positive roots is impossible. Hence, for $i=1$, H is an indefinite matrix, and F is not a convex function in (q,k_1,L_1,m). Function F is also not a convex function in $(q,k_i,L_i,m), i\in N$. ■

Because Function F is also not a convex function in $(q,k_i,L_i,m), i\in N$ and also the complexity of the form of function F, then it is very difficult to find the optimal solution for all variables simultaneously. However, for practical purposes and decision-making in the inventory problem, we can obtain the optimum value when at least another decision variable is given in a specific value. The analytical result of the optimum value is delivered in two different approaches. First, we temporarily ignore the SLC, resulting in an unconstrained optimization. Second, we consider buyers with active (at least one) SLC. We begin with the first case. For the fixed valued of m and L_i, we take the first partial derivative of the joint total for q and L_i and obtain

$$\begin{aligned}\frac{\partial JETC^U}{\partial q}&=-\frac{D}{q^2(1-\gamma)}\left(\frac{S}{m}+\textstyle\sum_{1=1}^N\left(O_i+K_i+C_i(L_i)\right)\right)\\&+\textstyle\sum_{1=1}^N\frac{D_iB_i}{(1-\gamma)}\left(\frac{h_{gi}\gamma}{2x}+\frac{h_{gi}(1-\gamma)^2}{2D}+\frac{h_{bi}\gamma\left(1-\gamma-\frac{D_i}{2x}\right)}{D_i}\right)\\&+\frac{Dh_v}{2(1-\gamma)}\left(\frac{m-1}{D}-\frac{m-2}{P}\right),\end{aligned} \tag{25}$$

and

$$\frac{\partial JETC^U}{\partial L_i}=\frac{h_{gi}B_ik_i\sigma_i}{2\sqrt{L_i}}. \tag{26}$$

Considering that the value of the second partial derivative of $JETC^U$ with respect to q is always positive

$\left(\frac{\partial^2 JETC^U}{\partial q^2} > 0\right)$, then $JETC^U$ is a convex function in q for any given fixed m and L_i. For the specific value of (q, m), the following equation is satisfied.

$$\frac{\partial^2 JETC^U}{\partial L_i^2} = -\frac{h_{bi} B_i k_i \sigma_i}{4\sqrt{L^3}} < 0, \tag{27}$$

Hence, $JETC^U$ is a concave function in L_i, implying that the minimum values of $JETC^U$ occur at the endpoints of the interval $L_i \in [L_{i,r}, L_{i,r-1}], \forall i$, for a given fixed value of (q, m). If we take (25) equal to zero, then we obtain:

$$q^* = \sqrt{\frac{D\left(\frac{S}{m} + \sum_{i=1}^{N}\left(O_i + K_i + C_i(L_i)\right)\right)}{c_7 + \frac{Dh_v}{2}\left(\frac{m-1}{D} + \frac{m-2}{P}\right)}}, \tag{28}$$

where $c_7 = \sum_{i=1}^{N} D_i B_i \left(\frac{h_{gi}\gamma}{2x} + \frac{h_{gi}(1-\gamma)^2}{2D} + \frac{h_{bi}\gamma\left(1-\gamma-\frac{D_i}{2x}\right)}{D_i}\right)$.

If the joint total cost is obtained based on the value of q in (28) and the safety factor for all buyers is set equal to zero, then SLC is not linearly related to the joint total cost and it can be ignored. If value of q in (28) does not meet at least one buyer from all of the buyers which have zero safety factor, then value of q in (28) is not the optimum solution. It implies that one or more than one buyer has an active SLC. In this condition, q also depends on the safety factor from buyer-i who takes an active SLC. To determine the optimum solution whether some buyers with an active SLC exist, we use the Lagrange function method. The Lagrange function of (22)−(23) is given by

$$F(q, k_i, L_i, m, \lambda_w) = \sum_{i=1}^{N} ETC_{bi}^U(q, k_i, L_i) + ETC_v^U(q, m)$$
$$+\sum_{w\in W} \lambda_w \left(D\sigma_w\sqrt{L_W}\left(\sqrt{1+k_w^2} - k_w\right) - \eta_w 2D_w q\right). \tag{29}$$

If we take $\eta_w = \frac{D\sigma_w\sqrt{L_w}D_w}{w}$, and then substitutes this value into (29), then we have

$$F(q, k_i, L_i, m, \lambda_w) = \sum_{i=1}^{N} ETC_{bi}^U(q, k_i, L_i) + ETC_v^U(q, m)$$
$$+\sum_{w\in W} \lambda_w D\sigma_w\sqrt{L_w}\left(\left(\sqrt{1+k_w^2} - k_w\right) - 2q\right). \tag{30}$$

where λ_w is the Lagrange multiplier related to the buyer which sets an active SLC. The set of buyers can be divided into two groups. Let $w \in W$ and $v \in V$ denote the set of buyers with an active SLC and an inactive SLC, respectively, which can be identified by checking the SLC of each buyer with their safety factor and whether q satisfies (28). If we set $i \in \{1, 2, \ldots, N\}$ and

$$\frac{D\sigma_i\sqrt{L_i}\left(\sqrt{1+k_i^2} - k_i\right)}{2D_i q} > \eta_i, k_i = 0,$$

then $i \in W$; otherwise $i \in V$. For a fixed given value of m and $L \in [L_{i,r}, L_{i,r-1}], \forall i$, the optimum solution can be found from solving set of partial differential equations

$$\frac{\partial F}{\partial q} = 0,\ \frac{\partial F}{\partial k_w} = 0, \frac{\partial F}{\partial \lambda_w} = 0,$$

where

$$\frac{\partial F}{\partial q} = -\frac{D}{q^2(1-\gamma)}\left(\frac{S}{m} + \sum_{i=1}^{N}\left(O_i + K_i + C_i(L_i)\right)\right) + c_7$$
$$+\ \frac{Dh_v}{2}\left(\frac{m-1}{D} + \frac{m-2}{P}\right) - \sum_{w\in W} 2\lambda_w D\sigma_w\sqrt{L_w} = 0, \tag{31}$$

$$\frac{\partial F}{\partial k_w} = \sqrt{L_w}\sigma_w\left(h_{bw}B_w + \lambda_w D\left(\frac{k_w}{\sqrt{1+k_w^2}} - 1\right)\right) = 0, \tag{32}$$

and

$$\frac{\partial F}{\partial \lambda_w} = D\sigma_w\sqrt{L_w}\left(\left(\sqrt{1+k_w^2} - k_w\right) - 2q\right) = 0. \tag{33}$$

By solving (31), (32), and (33) simultaneously, we have the optimum value when the SLC is active as

$$q^* = \sqrt{\frac{D\left(\frac{S}{m} + \sum_{i=1}^{N}\left(O_i + K_i + C_i(L_i)\right)\right)}{c_8 + 2(1-\gamma)\sum_{w\in W}\frac{h_{bw}c_w}{D\left(1-\frac{k_w{}^*}{\sqrt{1+(k_w^*)^2}}\right)}}}, \tag{34}$$

where $c_8 = c_7 + \frac{Dh_v}{2}\left(\frac{m-1}{D} + \frac{m-2}{P}\right)$ and

$$\lambda_w^* = \frac{h_{bw}c_w}{D\left(1 - \frac{k_w{}^*}{\sqrt{1+(k_w^*)^2}}\right)}, \tag{35}$$

$$\sqrt{1 + (k_w^*)^2} - k_w{}^* = 2q^*. \tag{36}$$

Equations (34), (35), and (36) are found in implicit mathematical form. Information about the values of the others are needed to process such equations. However, we can present it in parametric form. Suppose for some value of $k_w = \mu, \mu \in N$, which satisfies (36), then we can find the values of q^* in (34). In this research, we use a numerical process to get a representation about the optimum value of our problem. Procedure algorithm is proposed to find an optimum solution based. By taking several values of k_w and q through an iterative numerical process, we obtain the approximation value of the optimal solution of this process. We provide this result through numerical examples in the next section.

IV. Numerical Process and Example

We provide some numerical examples and sensitivity analysis parameters of our analytical result. For the numerical process of q^* and k^*, we consider the algorithms presented in Table II. We can process these algorithms with a common spreadsheet computation software. Then, we give a numerical example for the case when SCL is inactive. We consider the system consisting of one vendor and three different buyers with the following data: P = 3,000 units per year, S = IDR 5600000, h_v = IDR 2400 per unit per year, and ω =IDR 1400 per unit per year. Parameters O_i and K_i are calculated per order; h_{gi}, h_{bi}, x, and σ are calculated per unit per year; B is calculated per unit.

TABLE II
ALGORITHM TO FIND AN OPTIMUM SOLUTION

Algorithm

Begin
For each $i \in N$ do
Begin
Step 1. Set m=1.
Step 2. For each $L_{i,j}$, $j = 1, 2, 3, ..., r$ do Steps (i)−Step (iv), do
Begin
set $k_{ij1} = 0$, do
(i). Subtitute $k_{ij1} = 0$ into (36) to obtain q_{j1}.
(ii). Use q_{j1} from Step (i) to find k_{1j2} by solving (35). Any simple root finding methods can be applied.
(iii). Set $k_{ij1} = k_{1j2}$.
Repeat Steps (i) and (ii).
Until
$|q_{jn} - q_{jn-1}| \leq \varepsilon_q$ and $|k_{jn} - k_{jn-1}| \leq \varepsilon_k, \ \varepsilon_q, \varepsilon_k \in \mathbf{R}$.
(iv). Compute $JETC^U(q_j, k_{ij}, L_{ij}, m)$.
End
Step 3. Determine $\min_{(j=1,2,3,...,r)} JETC^U(q_j, k_{ij}, L_{ij}, m)$.
If $JETC^U(q^*_m, k^*_{im}, L^*_{im}, m)$
$= \min_{(j=1,2,3,...,r)} JETC^U(q_j, k_{ij}, L_{ij}, m)$,
Then
$(q^*_m, k^*_{im}, L^*_{im}, m)$ is optimum value for $m = 1$.
Else go to Step.4.
Step 4. For $m = m + 1$, do
Repeat Steps 2 and 3.
Until optimum solution $JETC^U(q^*_m, k^*_{im}, L^*_{im}, m)$ is obtained.
End
Step 5. **If** $JETC^U(q^*_m, k^*_{im}, L^*_{im}, m) \leq$
$JETC^U(q^*_{m-1}, k^*_{im-1}, L^*_{im-1}, m-1)$.
Then go back to Step 4.
Else go to Step 6.
Step 6. Choose $JETC^U(q^*_m, k^*_{im}, L^*_{im}, m) =$
$JETC^U(q^*_{m-1}, k^*_{im-1}, L^*_{im-1}, m-1)$,
and $(q^*_m, k^*_{im}, L^*_{im}, m)$ as an optimum solution.
End

The complete data for each buyer-i are given in Table III. Such data includes demand from each buyer, setup cost, transportation cost, screening cost, buyers' holding cost for non-defective items, buyers' holding cost for defective items, unit purchasing cost, and standard deviation of demand per unit time. Lead time for every buyer has three components, such as lead ordering time and order transit time, set-up time, and delivery time. Buyers' component lead-time data are given in Table IV. In this simulation, we only consider the case in which all SLCs on the buyer side are inactive. That is, $k_i = 0, i = 1, 2, 3$ and q follows Equation (28).

TABLE III
PARAMETER VALUE FOR EACH BUYER-i (IN IDR1000)

i	D_i	O_i	K_i	x	h_{gi}	h_{bi}	B_i	σ_i
1	720	98	50	1000	3.1	1.6	90	360
2	800	140	50	1000	3.1	1.6	90	400
3	900	195	50	1000	3.2	1.7	90	450

TABLE IV
BUYERS' COMPONENT LEAD-TIME DATA

Buyer-i	**r**	$\mathbf{b_{i,r}}$ **(days)**	$\mathbf{a_{i,r}}$ **(days)**	**Unit crashing cost (in IDR1000/days)**
1	1	20	6	1.4
	2	20	6	16.8
	3	16	9	70
2	1	20	6	7
	2	16	9	18.2
	3	13	6	71.4
3	1	25	11	5.6
	2	20	6	35
	3	18	11	70

For the numerical simulation, we attempt several values of defective rate γ. We explore the effect of the variation of the defective rate on the expected joint total cost of the inventory system. We apply $\gamma = 0.01; 0.02; 0.03; 0.04; 0.05; 0.06; 0.07; 0.08; 0.09; 0.10; 0.11; 0.12; 0.13; 0.14; 0.15; 0.16; 0.17; 0.18; 0.19; 0.20; 0.21; 0.22; 0.23; 0.24; 0.25; 0.26; 0.27; 0.28; 0.29; 0.30; 0.31$. If we process these data into the procedure algorithm, then we obtain the following result in Table V.

TABLE V
OPTIMUM VALUE (SLC IS INACTIVE) IN DAYS AND IN IDR

γ	L_1^*	L_2^*	L_3^*	q^*	m^*	ETC_b^*	ETC_v^*	$JETC^*$
0.01	56	49	63	109	25	26861	5777	32638
0.02	56	49	63	107	26	27415	5774	33189
0.03	56	49	63	106	26	27997	5912	33909
0.04	56	49	63	105	26	28579	6052	34631
0.05	56	49	63	104	26	29162	6193	35355
0.06	56	49	63	103	26	29744	6335	36079
0.07	56	49	63	102	26	30328	6479	36807
0.08	56	49	63	101	26	30193	6624	36817
0.09	56	49	63	100	26	31500	6771	38271
0.10	56	49	63	100	26	32088	6920	39008
0.11	56	49	63	99	26	32679	7070	39749
0.12	56	49	63	98	26	33273	7222	40495
0.13	56	49	63	98	26	33869	7377	41246
0.14	56	49	63	97	26	34469	7533	42002
0.15	56	49	63	97	26	35072	7691	42763
0.16	56	49	63	96	26	35678	7852	43530
0.17	56	49	63	96	26	36289	8015	44304
0.18	56	49	63	95	26	36904	8180	45084
0.19	56	49	63	95	26	37524	8348	45872
0.20	56	49	63	94	27	38113	8327	46440
0.21	56	49	63	94	27	38742	8497	47239
0.22	56	49	63	94	27	39378	8670	48048
0.23	56	49	63	93	27	40472	8971	49443
0.24	56	49	63	93	27	40667	9025	49692
0.25	56	49	63	93	27	41321	9027	50348
0.26	56	49	63	93	28	41946	9195	51141
0.27	56	49	63	92	28	42614	9381	51995
0.28	56	49	63	92	28	43291	9570	52861
0.29	56	49	63	92	28	43975	9763	53738
0.30	56	49	63	91	28	44668	9961	54629
0.31	56	49	63	91	28	45370	10162	55532

From Table V, we resume the relation between variation of the joint expected total cost with the defective rate and different value of a number of lots m^* (from $m^* = 25$ to $m^* = 28$) in the following figure.

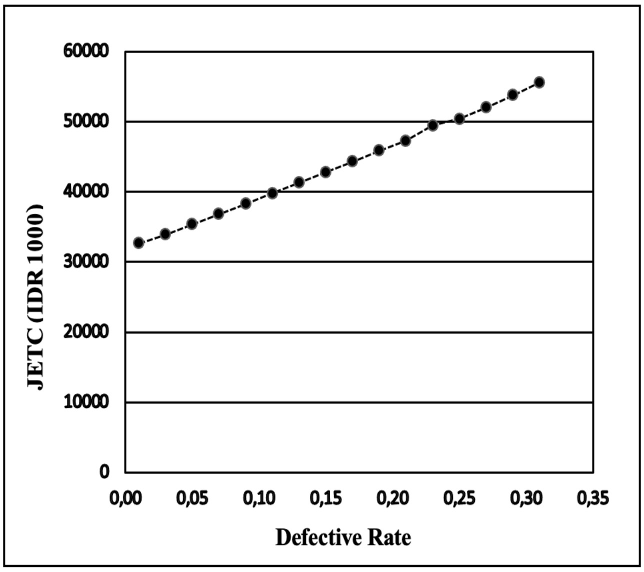

Fig. 2. Variation of the joint expected total cost with the defective rate

The optimum lead-time length option for buyer-1 is $L_1 = 56$ days, optimum lead time for buyer-2 is $L_2 = 49$ days, and for buyer-3 is $L_3 = 63$ days. According to Table V and Figure 2, if the value of the defective rate is increasing, then all buyers' expected average total cost and the vendor's average total cost gradually rise. The number of the lots also rises steadily when the defective rate increases. Although in our numerical example, we only present this pattern from $\gamma = 0.01$ to $\gamma = 0.31$, this increasing pattern also continues for a high defective rate. However, economic order quantity slightly decreases. On the range of the variation of the defective rate (from $\gamma = 0.01$ to $\gamma = 0.01$), we observe the number of lot changes three times in four different values of the number of the lots, namely, $m^* = 25, m^* = 26, m^* = 27$, and $m^* = 28$. According to these simulations, if we can reach a low but reasonable defective rate value, for example, $\gamma = 0.01$ or less, then we can obtain the maximum number of the lots at $m^* = 25$ with $q^* = 109$. To reach a low joint expected cost, the vendor can handle some parameters properly, for example, by reducing the number of defective rates. Such reduction must be considered with another parameter. Therefore, no new parameter appears when we reduce the number of defective rates.

Next, we observe the effect of another parameter on (joint) expected total cost for our model. Let $\gamma = 0.02$, then we attempt to take a different value of P, that is, $P = 3000, 3500, 4000, 4500, 5000, 5500, 6000, 6500, 7000$, and 7500. If the vendor raises P (per year), then it can benefit all buyers because the expected total cost for them decreases. However, it still increasing for the vendor (and the joint total cost). The number of lots m^* also increases (in our simulation from $m^* = 27$ to $m^* = 30$). We illustrate these results in the following Table VI and Figure 3 as follows.

TABLE VI
CHANGE IN THE OPTIMUM RESULT DUE TO THE CHANGE IN P FOR $\gamma = 0.02$

P	m^*	q^*	ETC_b	ETC_v	$JETC$
3000	26	107	27415	5774	33189
3500	26	105	27371	6179	33550
4000	26	105	27341	6478	33819
4500	27	103	27293	6602	33895
5000	27	103	27277	6788	34065
5500	28	102	27242	6853	34095
6000	29	101	27213	6907	34120
6500	29	101	27205	7019	34224
7000	30	100	27182	7051	34233
7500	30	100	27194	7195	34389

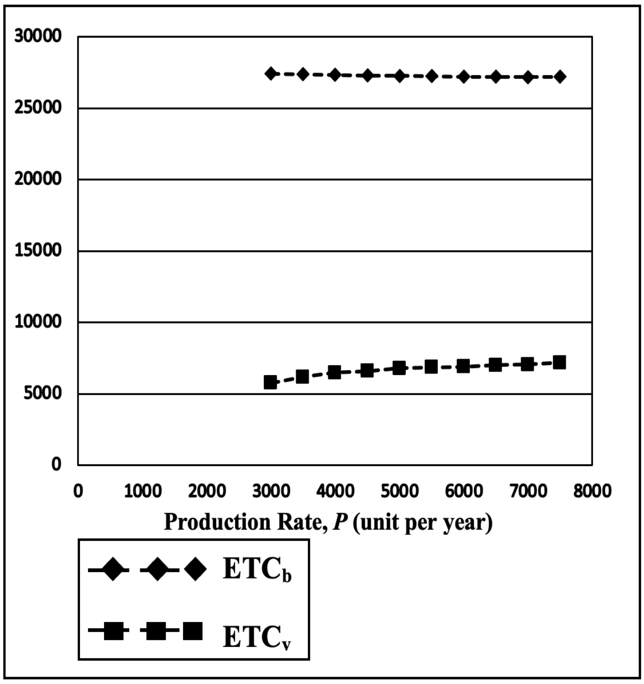

Fig. 3. Change in the Expected Total cost with the change in P

Because of the existence of imperfect quality, then the vendor have to prepare treatment cost for the existence of imperfect quality item per unit time to handle those kinds of product. Those become as a warranty cost for the buyers. If the imperfect quality product can be detected earlier in a vendor's side, then it can be reference as a cost of repairs. The numerical result which presented on the Table VI is computed using the same value of $\omega = 1400$ per unit per year. We will observe for two different value of ω, that is, $\omega = 700$ per unit per year and $\omega = 2800$ per unit per year. However, we do not do this for all value of P in Table VI. We attempt to take a simulation for $P = 3000, 3500, 6500$, and 7000. The higher value of a treatment cost per unit time will increase the expected total cost for the vendor for each value of P. However, it does not that big a difference with previous one. Based on our simulation, the changes of ω does not effect to the optimum quantity q^* for each different value of P. We present those numerical result in the Table VII.

TABLE VII
CHANGE IN THE OPTIMUM RESULT DUE TO THE CHANGE IN P AND ω FOR $\gamma = 0.02$

P	ω	m^*	q^*	ETC_b	ETC_v	$JETC$
3000	700	26	107	27415	5739	33154
3000	1400	26	107	27415	5774	33189
3000	2800	26	107	27415	5843	33258
3500	700	26	105	27371	6144	33515
3500	1400	26	105	27371	6179	33550
3500	2800	26	105	27371	6248	33619
6500	700	29	101	27205	6984	34189
6500	1400	29	101	27205	7019	34224
6500	2800	29	101	27205	7088	34293
7000	700	30	100	27182	7017	34199
7000	1400	30	100	27182	7051	34233
7000	2800	30	100	27182	7120	34302

According to the numerical result, the vendor must be careful about the defective rate, ω and P. If defective items and ω can be reduced, then it can be a long-term benefit for all the buyers and the vendor. They can reach a low joint expected total cost by reducing the defective rate.

V. CONCLUSION

By extending the results in a two-echelon single-vendor$-$multi-buyer model, we have obtained an integrated scheme to find an optimum solution to the system with some exact boundaries. Given the non-convexity properties of the objective function in all decision variables, we obtained the optimum value by using the Lagrange function methods for order quantity of a buyer and safety factor, if number of lots and lead time are known. Our algorithm can be applied to find an optimal solution based on a numerical process. Based on numerical examples, the expected total cost and the number of lots are affected by the defective item rate and production rate. Our model can help a vendor minimize the total cost of the inventory system and the number of lots under an uncertain lead time by reducing the defective rate. There are some open problems for future research related to this topic. For example, by adding stochastic demand for this model and applying a decentralized approach to find the optimum solution for this kind of inventory model.

REFERENCES

[1] H. J. Lin, "An integrated supply chain inventory model with imperfect-quality items, controllable lead time and distribution-free demand," *Yugoslav Journal of Operations Research*, vol. 23, no.1, pp87-109, 2013.

[2] E. L. Porteus, "Optimal lot sizing, process quality improvement, and setup cost reduction," *Operations Research*, vol. 34, no.1, pp137-144, 1986.

[3] M. J. Rosenblatt and H. L. Lee, "Economic production cycles with imperfect production processes," *IIE Transactions*, vol. 18, no.1, pp48-55, 1986.

[4] M. J. Paknejad, F. Nasri, and J.F. Afflsco, "Defective units in a continuous review (s, Q) system," *International Journal of Production Research*, vol.33, pp2767-2777, 1995.

[5] K. S. Wu and L. Y. Ouyang, "(Q,r,L) inventory model with defective items," *Computers and Industrial Engineering*, vol. 39, pp173-185, 2001.

[6] M.-D. Ko, C.-C. Wang, and M.-F. Yang, "Optimal Integrated Inventory Model with Preventive Maintenance Program based on Rapid Inspection," *IAENG International Journal of Applied Mathematics*, vol. 49, no.4, pp401-407, 2019.

[7] K. Annadurai and R. Uthayakumar, "Controlling setup cost in (Q, r, L) inventory model with defective items," *Applied Mathematical Modelling*, vol.34, pp1418-1427, 2010.

[8] A. Eroglu and G. Ozdemir, "An economic order quantity model with defective items and shortages," *International Journal of Production Economics*, vol.106, pp544-549, 2007.

[9] M. Y. Jaber, S. K. Goyal, and M. Imran, "Economic production quantity model for items with imperfect quality subject to learning effects," *International Journal of Production Economics*, vol. 115, no.1, pp143-150, 2008.

[10] S. Papachristos and I. Konstantaras, "Economic ordering quantity models for items with imperfect quality," *International Journal of Production Economics*, vol. 100, no.1, pp148-154, 2006.

[11] M. K. Salameh and M. Y. Jaber, "Economic production quantity model for items with imperfect quality," *International Journal of Production Economics*, vol. 64, no. 1, pp59-64, 2000.

[12] S. S. Sana, "A production-inventory model of imperfect quality products in a three-layer supply chain," *Decision Support Systems*, vol. 50, no. 2, pp158-170, 2011.

[13] H. M. Wee, J. Yu, and M. C. Chen, "Optimal inventory model for items with imperfect quality and shortage backordering," *Omega*, vol. 35, no.1, pp7-11, 2007.

[14] S. H. Yoo, D. S. Kim, and M. S. Park, "Economic production quantity model with imperfect-quality items, two-way imperfect inspection and sales return." *International Journal of Production Economics*, vol. 121, no.1, pp255- 265, 2009.

[15] J. T. Hsu and L. F. Hsu, "An Integrated Vendor-Buyer Cooperative Inventory Model for Items with Imperfect Quality and Shortage Backordering," *Advances in Decision Sciences*, vol. 2012, 2012.

[16] J. Bhowmick and G. P. Samanta, "Optimal inventory policies for imperfect inventory with price dependent stochastic demand and partially backlogged shortages," *Yugoslav Journal of Operations Research*, vol. 22, no.2, pp199 – 223, 2012.

[17] B. Pal, S. S. Sana, and K. Chaudhuri, "Maximising profits for an EPQ model with unreliable machine and rework of random defective items," *International Journal of Systems Science*, vol. 44, no.3, pp582-594, 2013.

[18] I. Konstantaras, K. Skouri, and M. Y. Jaber, "Inventory models for imperfect quality items with shortages and learning in inspection," *Applied Mathematical Modelling*, vol. 36, no.11, pp5334-5343, 2012.

[19] A. A. Alamri, I. Harris, and A. A. Syntetos, "Efficient inventory control for imperfect quality items," *European Journal of Operational Research*, vol. 254, no.1, pp92-104, 2016.

[20] J. K. Jha and K. Shanker, "Single-vendor multi-buyer integrated production-inventory model with controllable lead time and service level constraints," *Applied Mathematical Modelling*, vol. 37, no.4, pp1753-1767, 2013.

[21] A. Banerjee and S. Banerjee, "A coordinated order-up-to inventory control policy for a single supplier and multiple buyers using electronic data interchange," *International Journal of Production Economics*, vol.35, pp85-91, 1994.

[22] A. Banerjee and J. S. Burton, "Coordinated vs. independent inventory replenishment policies for a vendor and multiple buyers," *International Journal of Production Economics*, vol. 35, pp215 – 222, 1994.

[23] M. A. Hoque, "Synchronization in the single-manufacturer multi-buyer integrated inventory supply chain," *European Journal of Operational Research*, vol.188, no.3, pp811 – 825, 2008.

[24] M. A. Hoque, "Generalized single-vendor multi-buyer integrated inventory supply chain models with a better synchronization," *International Journal of Production Economics*, vol.131, no.2, pp463 – 472, 2011.

[25] M. A. Hoque, "An optimal solution technique to the single-vendor multi-buyer integrated inventory supply chain by incorporating some realistic factors," *European Journal of Operational Research*, vol. 215, no.1, pp80-88, 2011.

[26] C. L. Chu and V. J. Leon, "Single-vendor multi-buyer inventory coordination under private information," *European Journal of Operational Research*, vol.191, pp484-502, 2008.

[27] P. Mandal and B.C. Giri, "A single-vendor multi-buyer integrated model with controllable lead time and quality improvement through reduction in defective items," *International Journal of Systems Science: Operations and Logistics*, vol.2, no.1, pp1-14, 2015.

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