# Comparison and Analysis of Novel Score-Variance Portfolio Models based on Methods for Ranking Fuzzy Numbers 

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#### Abstract

Ranking the fuzzy numbers plays an important role in fuzzy decision-making, but it is challenging to rank multiple fuzzy numbers comprehensively and effectively. In previous studies, various properties and factors of fuzzy numbers are considered to construct the ranking scoring formula. Therefore, in theory, the uncertainty of fuzzy numbers can be better defuzzified, and the formula can measure the characteristics of fuzzy numbers. In application, it is meaningful and valuable to investigate the use of fuzzy number's ranking score in fuzzy portfolio to measure characteristics of the asset. Based on the classical mean-variance ( $\mathrm{M}-\mathrm{V}$ ) model and four kinds of ranking score formulas, this paper constructs the score-variance ( $\mathrm{S}-\mathrm{V}$ ) models. Finally, a numerical experiment is presented to illustrate the feasibility and validity of our proposed S-V model. Compared with the M-V model, on one hand the results of our proposed models are generally consistent with the classic M-V model; on the other hand, the return of our proposed model is slightly smaller in some part, but it performs better in skewness and Sharpe Ratio.


Index Terms-portfolio, ranking fuzzy numbers, ranking score, skewness

## I. Introduction

T'HE modern portfolio theory constructed by Markowitz [1] in 1950s is regarded as the beginning of modern investment quantification. The proposed $\mathrm{M}-\mathrm{V}$ model is based on probability theory. It describes the return rate of securities as a random variable and obtains the distribution of the random variable based on historical data. However, as a complex system, financial market exists a lot of non-random factors. Fuzzy set theory [2] can not only describe non-probabilistic factors of the financial market, but also express the vagueness and uncertainties. Therefore, the portfolio selection models in a fuzzy environment have good theoretical and practical value. In addition to classical objectives of mean and variance, the application of

[^0]higher-order moments was increasingly taken into account. Markowitz's M-V efficient frontier was extended to mean-variance-skewness (M-V-S) efficient hyper-surface by Adcock [3]. Deng and Pan [4] presented a mean-variance -skewness-entropy (M-S-E) portfolio model and solved the intuitionistic fuzzy (IF) multi-objective model into a single-objective model by three methods. Considered that the calculation method and reliability of risk measures and return are questionable, Alali and Tolga [5] analyzed the correlation between them based on historical data. Nazir [6] first proposed the multi-core and GP-GPU accelerated implementations of Anticor algorithm, which can accelerate the highly intensive computations. Complex practical factors and subjective attitude of investors are also worth considering in the portfolio selection. By combining Data Envelopment Analysis (DEA) prospect cross-efficiency approach and the maverick index, a novel mean-variance -maverick was designed by Deng and Fang [7] for fuzzy portfolio selection. Li and Yi [8] proposed a new trapezoidal fuzzy number with an adaptive index to measure the coherence of investor's expectation, and constructed fuzzy $\mathrm{M}-\mathrm{V}$ model and M-V-S model to analyze its feasibility and effectiveness. To explore the impact of background risks and mental accounts on investment decisions, Deng and Liu [9] constructed portfolio models with and without background risk for comparative study. The Ensemble Empirical Mode Decomposition (EEMD) was applied by Zheng and Yao [10] to de-noise the data, and the impact on the portfolio optimization was investigated. Based on the improved entropy-weighted method, Deng and Chen [11] proposed a new portfolio model with prospect value constraint and risk preference to adjust investment plan. In order to deal with fuzzy multi-objective and multi-period portfolio models, a hybrid genetic algorithm with wavelet neural network was applied [12].
Ranking the fuzzy numbers plays an important role in fuzzy decision-making. Since fuzzy numbers contain a lot of uncertain information, it is necessary to comprehensively consider various factors about fuzzy numbers when ranking fuzzy numbers. Modarres and Sadi-Nezhad [13] defined the preference ratio of fuzzy numbers to rank them. Chen et al. [14] presented a method for ranking generalized fuzzy numbers with different left heights and right heights, and developed a new method for fuzzy risk analysis. In order to overcome the shortcomings of Ezzati et al.'s [15] model, a new method for ranking fuzzy numbers based on the magnitude concepts was developed by Yu et al. [16]. Consider the optimistic and pessimistic attitudes of decision makers, Thanh-Lam [17] proposed a unified index with
subjective attitude parameters. By weighting defuzzified value, height and spread according to their importance, the ordered weighted averaging (OWA) operator was designed by Wu et al. [18] to rank generalized fuzzy numbers.

In this paper, a novel portfolio model based on methods for ranking fuzzy numbers is proposed. The ranking score can comprehensively measure the characteristics of fuzzy numbers, furthermore it can also be used as an important factor to measure the characteristics of stocks. Therefore, the objective of maximizing the mean is replaced by the objective of maximizing the ranking score to construct the $\mathrm{S}-\mathrm{V}$ models. By comparing with the classical M-V model, we testify the feasibility and effectiveness of our developed S-V models.
The rest of this paper is unfolded as follows. In Section II, we review the preliminaries about some definitions and properties of generalized fuzzy numbers. In Section III, we review several existing methods for ranking fuzzy numbers. Based on the four fuzzy number ranking formulas, RS-V and M-V models are constructed in Section IV. In Section V, a numerical experiment is given to demonstrate the feasibility and validity of the proposed S-V models. Conclusions are summarized in Section VI.

## II. PRELIMINARIES

In this section, we first review some definitions and properties of generalized fuzzy numbers.

## A. Definitions of Generalized Fuzzy Numbers

Definition $1^{[2]}$ : If $A$ has a membership function $\mu_{A}(x): R \rightarrow[0,1]$, called $A$ is a fuzzy number on the universe $U$, then $A$ must satisfy the following conditions:
(1) $A$ is normal, that is, there exists $x_{0}$ belonging to $U$ such that $\mu_{A}\left(x_{0}\right)=1$;
(2) $A$ is convex, that is, if all $x_{1}$ and $x_{2}$ belong to $U$, then:
$\mu_{A}\left(\lambda x_{1}+(1-\lambda) x_{2}\right) \geq \min \left(\mu_{A}\left(x_{1}\right), \mu_{A}\left(x_{2}\right)\right) ;$
(3) $\mu_{A}(x)$ is bounded and upper semicontinuous, and $\left\{x \in R \mid \mu_{A}(x) \leq \varepsilon\right\}$ is a closed set;
(4) $\left\{x \in R \mid \mu_{A}(x) \geq 0\right\}$ is a compact set.

Chen [19] developed the concept of generalized fuzzy numbers. And the membership function [20] $\mu_{A}(x)$ of a generalized trapezoidal fuzzy number $A=\left(a, b, c, d ; \omega_{A}\right)$ is defined as follows:
$\mu_{A}(x)=\left\{\begin{array}{cc}\mu_{A}^{L}(x), & a \leq x<b ; \\ \omega_{A}, & b \leq x<c ; \\ \mu_{A}^{R}(x), & c \leq x<d ; \\ 0, & \text { otherwise. }\end{array}\right.$
Where $\mu_{A}^{L}(x)$ and $\mu_{A}^{R}(x)$ are continuous mapping functions and $\omega_{A} \in[0,1]$.

According to Chen et al. [14], the membership function $\mu_{A}(x)$ of a generalized trapezoidal fuzzy number $A=\left(a, b, c, d ; \mu_{L}, \mu_{R}\right)$ with the different left height $\mu_{L}$ and right height $\mu_{R}$ is defined as follows:
$\mu_{A}(x)=\left\{\begin{array}{cc}\mu_{A}^{L}(x), & a \leq x<b ; \\ \mu_{A}^{T}(x), & b \leq x<c ; \\ \mu_{A}^{R}(x), & c \leq x<d ; \\ 0, & \text { otherwise. }\end{array}\right.$
Where $\mu_{A}^{L}(x)$ and $\mu_{A}^{R}(x)$ are continuous mapping functions. Similarly, Chen et al. [14] extended the definition of arithmetic operations between the generalized fuzzy numbers [21] to the generalized fuzzy numbers with different left heights and right heights.

Definition 2: Let $A=\left(a_{1}, b_{1}, c_{1}, d_{1} ; \mu_{L A}, \mu_{R A}\right)$ and $B=$ $\left(a_{2}, b_{2}, c_{2}, d_{2} ; \mu_{L B}, \mu_{R B}\right)$ be two generalized fuzzy numbers with different left heights and right heights, then the arithmetic operations between the generalized fuzzy numbers $A$ and $B$ are defined as follows:
(1) Generalized fuzzy numbers addition $\oplus$ :

$$
\begin{align*}
A \oplus B= & \left(a_{1}, b_{1}, c_{1}, d_{1} ; \mu_{L A}, \mu_{R A}\right) \oplus\left(a_{2}, b_{2}, c_{2}, d_{2} ; \mu_{L B}, \mu_{R B}\right) \\
= & \left(a_{1}+a_{2}, b_{1}+b_{2}, c_{1}+c_{2}, d_{1}+d_{2} ;\right.  \tag{4}\\
& \left.\min \left(\mu_{L A}, \mu_{L B}\right), \min \left(\mu_{R A}, \mu_{R B}\right)\right),
\end{align*}
$$

where $0 \leq a_{1} \leq b_{1} \leq c_{1} \leq d_{1} \leq 1$ and $0 \leq a_{2} \leq b_{2} \leq c_{2} \leq d_{2} \leq 1$.
(2) Generalized fuzzy numbers addition $\otimes$ :

$$
\begin{align*}
A \otimes B= & \left(a_{1}, b_{1}, c_{1}, d_{1} ; \mu_{L A}, \mu_{R A}\right) \otimes\left(a_{2}, b_{2}, c_{2}, d_{2} ; \mu_{L B}, \mu_{R B}\right) \\
= & \left(a_{1} \times a_{2}, b_{1} \times b_{2}, c_{1} \times c_{2}, d_{1} \times d_{2} ;\right.  \tag{5}\\
& \left.\min \left(\mu_{L A}, \mu_{L B}\right), \min \left(\mu_{R A}, \mu_{R B}\right)\right),
\end{align*}
$$

where $0 \leq a_{1} \leq b_{1} \leq c_{1} \leq d_{1} \leq 1$ and $0 \leq a_{2} \leq b_{2} \leq c_{2} \leq d_{2} \leq 1$.
When $\mu_{L}=\mu_{R}=1$ or $\omega_{A}=1, A$ turns into a normal trapezoidal fuzzy number, denoted as $A=(a, b, c, d)$, and its membership function is as follows:
$\mu_{A}(x)=\left\{\begin{array}{cc}\frac{x-a}{b-a}, & a \leq x \leq b ; \\ 1, & b \leq x \leq c ; \\ \frac{d-x}{d-c}, & c \leq x \leq d ; \\ 0, & \text { otherwise. }\end{array}\right.$

## B. Numerical Characteristics of Fuzzy Numbers

In previous studies, there are many methods to define the numerical characteristics of fuzzy numbers. Li et al. [22] defined the possibility density function formula of fuzzy number $A$ according to its membership function.

Definition $3^{[22]}$ : Let $A$ be a fuzzy number with piecewise differentiable membership function $\mu_{A}(x)$, then the possibility density function of fuzzy number $A$ can be defined as $f(x)=\mu(x)\left|\mu^{\prime}(x)\right|$ and has the properties: $f(x) \geq 0$ and $\int_{-\infty}^{+\infty} f(x) d x=1$.
By defining the possibility density function of fuzzy numbers, the numerical characteristics such as possibilistic mean, variance and skewness can be calculated in the same way as the numerical characteristic formula of random numbers.

Definition $4^{[22]}$ : Let $A$ be a fuzzy number with differentiable membership function $\mu_{A}(x)$, then the possibilistic mean, variance and skewness can be defined as:
$E(A)=\int_{-\infty}^{+\infty} x f(x) d x$,
$V(A)=\int_{-\infty}^{+\infty}[x-E(A)]^{2} f(x) d x$.
$S(A)=\int_{-\infty}^{+\infty}[x-E(A)]^{3} f(x) d x$.
Then, we can obtain the possibility density function of trapezoidal fuzzy number $A=(a, b, c, d)$ by Definition 3 as follows:

$$
f(x)=\mu_{A}(x)\left|\mu_{A}^{\prime}(x)\right|=\left\{\begin{array}{cl}
\frac{x-a}{(b-a)^{2}}, & a \leq x \leq b ;  \tag{10}\\
\frac{d-x}{(d-c)^{2}}, & c \leq x \leq d ; \\
0, & \text { otherwise. }
\end{array}\right.
$$

Furthermore, according to (7)-(10), the possibilistic mean, variance and skewness of trapezoidal fuzzy number $A=(a, b, c, d)$ can be calculated easily, shown as follows:

$$
\begin{align*}
E(A)= & \int_{a}^{b} x \cdot \frac{x-a}{(b-a)^{2}} d x+\int_{c}^{d} x \cdot \frac{d-x}{(d-c)^{2}} d x  \tag{11}\\
= & \frac{1}{6} a+\frac{1}{3} b+\frac{1}{3} c+\frac{1}{6} d, \\
V(A)= & \int_{a}^{b}[x-E(A)]^{2} \cdot \frac{x-a}{(b-a)^{2}} d x \\
& +\int_{a}^{b}[x-E(A)]^{2} \cdot \frac{d-x}{(d-c)^{2}} d x \\
= & \frac{1}{18}(b-a)^{2}+\frac{1}{4}(c-b)^{2}  \tag{12}\\
& +\frac{1}{18}(d-c)^{2}+\frac{1}{6}(b-a)(c-b) \\
& +\frac{1}{18}(b-a)(d-c)+\frac{1}{6}(c-b)(d-c), \\
S(A)= & \int_{a}^{b}[x-E(A)]^{3} \cdot \frac{x-a}{(b-a)^{2}} d x \\
+ & \int_{a}^{b}[x-E(A)]^{3} \cdot \frac{d-x}{(d-c)^{2}} d x \\
= & \frac{1}{1080}\left[-19(b-a)^{3}-45(b-a)^{2}(c-b)\right.  \tag{13}\\
& -15(b-a)^{2}(d-c)+15(b-a)(d-c)^{2} \\
& \left.+45(c-b)(d-c)^{2}+19(d-c)^{3}\right] .
\end{align*}
$$

## III. SEVERAL EXisting MEthods For Ranking Fuzzy NUMBERS

In this section, we review several methods for ranking fuzzy numbers in previous studies [14, 16-18].

## A. Ranking Generalized Fuzzy Numbers by Ranking Score

Chen et al. [14] presented a method for ranking generalized fuzzy numbers with different left heights and right heights. In this method, both the areas of the positive side and negative side were taken into consideration as factors for calculating the ranking score, and the centroid values of generalized fuzzy numbers were also applied.

Suppose there are $N$ generalized fuzzy numbers $\left\{A_{i} \mid i=1, \ldots, N\right\}$ to be ranked, where $A_{i}=$ $\left(a_{i}, b_{i}, c_{i}, d_{i} ; \mu_{i L}, \mu_{i R}\right), \mu_{i L}$ and $\mu_{i \mathbb{R}}$ denote the left and right height of the fuzzy number $A_{i}$. The specific steps of this method are as follows:
Step 1: Transform each generalized fuzzy number $A_{i}$ into a standardized generalized fuzzy number $A_{i}^{*}$, that is,

$$
\begin{equation*}
A_{i}^{* *}=\left(\frac{a_{i}}{k}, \frac{b_{i}}{k}, \frac{c_{i}}{k}, \frac{d_{i}}{k} ; \mu_{i L}, \mu_{i R}\right), \tag{14}
\end{equation*}
$$

where $k=\max _{i}\left(\left|a_{i}\right|,\left|b_{i}\right|,\left|c_{i}\right|,\left|d_{i}\right|, 1\right), i=1, \ldots, N$.
Step 2: Divide the area of standardized generalized fuzzy number $A_{i}^{*}$ into the left negative area $L N_{i}$, the right negative area $R N_{i}$, the left positive area $L P_{i}$, and the right positive area $R P_{i}$, where $i=1, \ldots, N$, as shown in (15)-(18).
$L N_{i}=\left\{\begin{array}{lc}\int_{-1}^{x_{2}} f(x) d x-\int_{x_{i}}^{x_{2}} \mu_{A^{*}}(x) d x, & \mu_{L}>\mu_{R} \text { or } \mu_{L}>\mu_{R} ; \\ \int_{-1}^{x_{3}} f(x) d x-\int_{x_{1}}^{x_{1}} \mu_{A_{i}}(x) d x, & \mu_{L}<\mu_{R} .\end{array}\right.$
$R N_{i}=\left\{\begin{array}{lc}\int_{-1}^{x_{3}} f(x) d x+\int_{x_{3}}^{x_{4}} \mu_{A^{\prime}}(x) d x, & \mu_{L}<\mu_{R} \text { or } \mu_{L}>\mu_{R} ; \\ \int_{-1}^{x_{2}} f(x) d x+\int_{x_{2}}^{x_{4}} \mu_{A_{i}}(x) d x, & \mu_{L}>\mu_{R} .\end{array}\right.$
$L P_{i}=\left\{\begin{array}{lc}\int_{x_{2}}^{1} f(x) d x+\int_{x_{1}}^{x_{2}} \mu_{A_{i}}(x) d x, & \mu_{L}>\mu_{R} \text { or } \mu_{L}>\mu_{R} ; \\ \int_{x_{2}}^{1} f(x) d x+\int_{x_{1}}^{x_{5}} \mu_{A_{i}}(x) d x, & \mu_{L}<\mu_{R} .\end{array}\right.$
$R P_{i}= \begin{cases}\int_{x_{3}}^{1} f(x) d x-\int_{x_{3}}^{x_{4}} \mu_{A_{i}}(x) d x, & \mu_{L}<\mu_{R} \text { or } \mu_{L}>\mu_{R} ; \\ \int_{x_{3}}^{1} f(x) d x-\int_{x_{2}}^{x_{4}} \mu_{A_{i}}{ }^{*}(x) d x, & \mu_{L}>\mu_{R} .\end{cases}$
Where $f(x)$ is the membership function of $(-1,-1,1,1 ; 1,1)$ and $\mu_{A^{\prime}}(x)$ is the membership function of standardized generalized fuzzy number $A_{i}^{*}$.
Step 3: Calculate the sum $M_{i}$ and $N_{i}$ of the negative areas and positive areas of each standardized generalized fuzzy number $A_{i}^{*}$, respectively.

$$
\begin{align*}
M_{i} & =L N_{i}+R N_{i},  \tag{19}\\
N_{i} & =L P_{i}+R P_{i} . \tag{20}
\end{align*}
$$

Step 4: Calculate the centroid $c\left(A_{i}^{*}\right)$ of each generalized fuzzy number $A_{i}^{*}$ according to (21).

Step 5: Calculate the ranking score $S\left(A_{i}^{*}\right)$ of each generalized fuzzy number $A_{i}^{*}$ according to (22).
$S\left(A_{i}^{*}\right)=\frac{M_{i}-N_{i}}{M_{i}+N_{i}+\left(1-c\left(A_{i}^{*}\right)\right)}$.

## B. Ranking Fuzzy Numbers Based on The Magnitude Concepts

Yu et al. [16] developed a method for ranking fuzzy numbers based on the magnitude concepts. In this method,
the attitude of decision makers towards fuzzy numbers was also taking into account.

Suppose there are $N$ arbitrary fuzzy numbers $\left\{A_{i} \mid i=1, \ldots, N\right\}$ to be ranked, where $A_{i}=\left(a_{i}, b_{i}, c_{i}, d_{i} ; \omega_{i}\right)$ and the membership function $\mu_{A_{i}}(x)$ is shown as follows:
$\mu_{A_{i}}(x)=\left\{\begin{array}{cc}\mu_{A_{i}}^{L}(x), & a_{i} \leq x<b_{i} ; \\ \omega_{i}, & b_{i} \leq x<c_{i} ; \\ \mu_{A_{i}}^{R}(x), & c_{i} \leq x<d_{i} \\ 0, & \text { otherwise. }\end{array}\right.$
Then the magnitude of arbitrary fuzzy number $A_{i}$ with the index of optimism $\alpha \in\left[0, h_{i}\right]$ was defined as:
$\operatorname{Mag}\left(A_{i}^{\alpha}\right)=\frac{1}{2}\left\{\begin{array}{l}\alpha \int_{0}^{\omega_{i}}\left[\bar{\mu}\left(r_{i}\right)-\left(a_{i}\right)_{\min }\right] d r_{i} \\ +(1-\alpha) \int_{0}^{\omega_{i}}\left[\underline{\mu}\left(r_{i}\right)-\left(a_{i}\right)_{\text {min }}\right] d r_{i}\end{array}\right\}$,
where $\left(a_{i}\right)_{\text {min }}=\inf S, S=\bigcup_{i=1}^{N} S_{i}, S_{i}=\left\{a_{i} \mid \mu_{A_{i}}(x)>0\right\}$.
The degree of optimism of a decision maker is measured by the index $\alpha$. A larger $\alpha$ represents a higher degree of optimism. When $A_{i} \mid i=1, \ldots, N$ are trapezoidal fuzzy numbers, i.e., $\omega_{i}=1$, (24) degradation to:
$\operatorname{Mag}\left(A_{i}^{\alpha}\right)=\frac{1}{2}\left\{\begin{array}{l}\alpha\left[\frac{c_{i}+d_{i}}{2}-\left(a_{i}\right)_{\text {min }}\right] \\ +(1-\alpha)\left[\frac{a_{i}+b_{i}}{2}-\left(a_{i}\right)_{\text {min }}\right]\end{array}\right\}$.

## C. Ranking Fuzzy Numbers by A Unified Index

Thanh-Lam [17] proposed a method for ranking generalized fuzzy numbers based on a unified index. In this method, centroid value (weighted mean) and attitude-incorporated left-and-right area (weighted area) was combined to construct the index.
Suppose there are $N$ fuzzy numbers $\left\{A_{i} \mid i=1, \ldots, N\right\}$ to be ranked, where $A_{i}=\left(a_{i}, b_{i}, c_{i}, d_{i} ; \omega_{i}\right)$ and the membership function $\mu_{A_{i}}(x)$ is shown as follows:
$\mu_{A_{i}}(x)=\left\{\begin{array}{cc}\mu_{A_{i}}^{L}(x), & a_{i} \leq x<b_{i} ; \\ \omega_{i}, & b_{i} \leq x<c_{i} ; \\ \mu_{A_{i}}^{R}(x), & c_{i} \leq x<d_{i} ; \\ 0, & \text { otherwise } .\end{array}\right.$
The specific steps:
Step 1: Calculate the centroid value $C V_{i}$ of each fuzzy number $A_{i}$ according to (27).
$C V_{i}=\frac{\int_{a_{i}}^{d_{i}} x \mu_{A_{i}}(x) d x}{\int_{a_{i}}^{d_{i}} \mu_{A_{i}}(x) d x}$
Step 2: The left area $S_{i}^{L}$ and right area $S_{i}^{R}$ of each fuzzy number $A_{i}$ are given by
$S_{i}^{L}=\left|\int_{0}^{\omega_{i}} g_{A_{i}}^{L}(y) d y\right|$,
$S_{i}^{R}=\left|\int_{0}^{\omega_{i}} g_{A_{i}}^{R}(y) d y\right|$.

Where $g_{A_{i}}^{L}(y)$ and $g_{A_{i}}^{R}(y)$ are inverse functions of $\mu_{A_{i}}^{L}(x)$ and $\mu_{A_{i}}^{R}(x)$, respectively. Then calculate the attitude-incorporated left area and right area $A A_{i}$ of each fuzzy number $A_{i}$ according to (30).
$A A_{i}=\lambda S_{i}^{R}+(1-\lambda) S_{i}^{L}$,
where $\lambda \in[0,1]$ is a level of optimism reflecting a data revelation optimism degree of the decision maker. The larger the $\lambda$ set by the decision maker is, the more optimistic attitude the decision maker has on the data revelation.
Step 3: Calculate the unified index $U I_{i}$ of each fuzzy number $A_{i}$ according to (31).
$U I_{i}=\left(C V_{i}+\varepsilon_{i}\right)\left[\lambda S_{i}^{R}+(1-\lambda) S_{i}^{L}\right]$,
where $\varepsilon_{i}$ is a very small real number which is quantifiable and rational for comparing the targeted fuzzy numbers whose centroid values take a value of 0 .

## D. Ranking Fuzzy Numbers by Improved Ranking Score

Wu et al. [18] presented a method for ranking generalized fuzzy numbers based on OWA operator. In this method, the priority of three ranking factors defuzzified value, height and spread were taking into consideration.

Suppose there are $N$ generalized fuzzy numbers $\left\{A_{i} \mid i=1, \ldots, N\right\}$ to be ranked, where $A_{i}=$ $\left(a_{i}, b_{i}, c_{i}, d_{i} ; \mu_{i L}, \mu_{i R}\right)$ and the membership function $\mu_{A_{i}}(x)$ is defined as Eq. (32).
$\mu_{A_{i}}(x)=\left\{\begin{array}{cc}\mu_{A_{i}}^{L}(x), & a_{i} \leq x<b_{i} ; \\ \mu_{A_{i}}^{T}(x), & b_{i} \leq x<c_{i} ; \\ \mu_{A_{i}}^{R}(x), & c_{i} \leq x<d_{i} ; \\ 0, & \text { otherwise } .\end{array}\right.$
$\mu_{i L}$ and $\mu_{i R}$ denote the left and right height of the fuzzy number $A_{i}$. The specific steps of this method are as follows:
Step 1: Calculate the defuzzified value $x_{A_{i}}$ and the height
$h_{A_{i}}$ of each generalized fuzzy number $A_{i}$ according to (33).
$x_{A_{i}}=\frac{\int_{a_{i}}^{d_{i}} x \mu_{A_{i}}(x) d x}{\int_{a_{i}}^{d_{i}} \mu_{A_{i}}(x) d x}$,
$h_{A_{i}}=\left\{\begin{array}{cl}\mu_{i L}, & \mu_{i L}=\mu_{i R} ; \\ \frac{\int_{\mu_{L L}}^{\mu_{i R}} y g_{A_{i}}^{R}(y) d y-\int_{\mu_{L}}^{\mu_{i_{R}}} y g_{A_{i}}^{T}(y) d y}{\int_{\mu_{L L}}^{\mu_{i,}} g_{A_{i}}^{R}(y) d y-\int_{\mu_{L L}}^{\mu_{L_{R}}} g_{A_{i}}^{T}(y) d y}, & \mu_{i L} \neq \mu_{i R} .\end{array}\right.$
Where $g_{A_{i}}^{R}(y)$ and $g_{A_{i}}^{T}(y)$ are inverse functions of $\mu_{A_{i}}^{L}(x)$ and $\mu_{A_{i}}^{R}(x)$, respectively. $x_{A_{i}}$ is the value of the centroid point on the horizontal axis of generalized fuzzy number $A_{i}$ and $h_{A_{i}}$ is the value of the centroid point on the vertical axis.
Step 2: Calculate the spread $S T D_{A_{i}}$ of each generalized fuzzy number $A_{i}$ according to Eq. (35).

$$
\begin{equation*}
S T D_{A_{i}}=\sqrt{\frac{\left(a_{i}-\bar{x}\right)^{2}+\left(b_{i}-\bar{x}\right)^{2}+\left(c_{i}-\bar{x}\right)^{2}+\left(d_{i}-\bar{x}\right)^{2}}{4-1}} \tag{35}
\end{equation*}
$$

where $\bar{x}=\frac{a_{i}+b_{i}+c_{i}+d_{i}}{4}$.
Step 3: Construct the vector $V$ with regard to the ordered parameters. Since the value of $x_{A_{i}}$ is the most important factor that influences the ranking priority of a generalized fuzzy number, the value of $h_{A_{i}}$ is the second important and $S T D_{A_{i}}$ is the least important. Then, the ranking order of importance is: $x_{A_{i}}>h_{A_{i}}>S T D_{A_{i}}, x_{A_{i}}>\mu h_{A_{i}}>\frac{\mu}{1+S T D_{A_{i}}}$, which is consistent with the analytical geometry. Therefore, the vector $V$ is defined as follows:
$V^{T}=\left(x_{A_{i}}, \mu h_{A_{i}}, \frac{\mu}{1+S T D_{A_{i}}}\right)$,
$\mu= \begin{cases}1, & x_{A_{i}} \in[0,+\infty) ; \\ -1 & x_{A_{i}} \in(-\infty, 0) .\end{cases}$
Step 4: Calculate the transpose of weighting vector $\omega^{T}=\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ of the three elements $x_{A_{i}}, \mu h_{A_{i}}$ and $\frac{\mu}{1+S T D_{A_{i}}}$ of vector $V$ based on (38).
$\left\{\max \quad \operatorname{Disp}(\omega)=-\sum_{i=1}^{3} \omega_{i} \ln \omega_{i}\right.$
$\operatorname{arness}(\omega)=\alpha=\frac{1}{n-1} \sum_{i=1}^{n}(n-i) \omega_{i}$,
$\sum_{i=1}^{3} \omega_{i}=1,0 \leq \omega_{i} \leq 1,0 \leq \alpha \leq 1$.
Cause the priority of these three factors is: $x_{A_{i}}>\mu h_{A_{i}}>\frac{\mu}{1+S T D_{A_{i}}}$. Therefore, we need to give $x_{A_{i}}$ the maximal weight $\omega_{1}, \mu h_{A_{i}}$ the slightly larger weight $\omega_{2}$ and $\frac{\mu}{1+S T D_{A_{i}}}$ the minimum weight $\omega_{3}$. In this situation, $\omega_{1}>\omega_{2}>\omega_{3}$, then $0.5<\alpha<1$. Generally, the value of $\alpha$ takes the mean value of the interval $(0.5,0.9]$, that is $\alpha=0.7$.
Step 5: Calculate the ranking score $S\left(A_{i}\right)$ of each generalized fuzzy number $A_{i}$ according to (39).
$S\left(A_{i}\right)=\omega^{T} \cdot V=\omega_{1} x_{A_{i}}+\mu \omega_{2} h_{A_{i}}+\omega_{3} \frac{\mu}{1+S T D_{A_{i}}}$.

## E. Ranking Trapezoidal Fuzzy Numbers by Above Four Methods

Based on the above four methods of ranking fuzzy numbers, we can effectively obtain four kinds of ranking scores of each fuzzy number $A_{i}$ in a set of fuzzy numbers, and rank them according to their scoring size. Suppose there are $N$ trapezoidal fuzzy numbers $\left\{A_{i} \mid i=1, \ldots, N\right\}$ to be ranked, where $A_{i}=\left(a_{i}, b_{i}, c_{i}, d_{i}\right)$
(1) Ranking trapezoidal fuzzy numbers by ranking score

According to Subsection 3.1, we can obtain the ranking score $R S\left(A_{i}\right)$ of trapezoidal fuzzy numbers $A_{i}$ as follows:
$R S\left(A_{i}\right)=\frac{2\left(\frac{a_{i}}{k}+\frac{b_{i}}{k}+\frac{c_{i}}{k}+\frac{d_{i}}{k}\right)}{10-\left|\frac{b_{i}}{k}+\frac{c_{i}}{k}\right|}$,
where $k=\max _{i}\left(\left|a_{i}\right|,\left|b_{i}\right|,\left|c_{i}\right|,\left|d_{i}\right|, 1\right), i=1, \ldots, N$.
(2) Ranking trapezoidal fuzzy numbers based on the magnitude concepts
According to Subsection 3.2, we can obtain the magnitude $\operatorname{Mag}\left(A_{i}\right)$ of trapezoidal fuzzy numbers $A_{i}$ as follows:
$\operatorname{Mag}\left(A_{i}\right)=\frac{1}{2}\left\{\begin{array}{l}\alpha\left[\frac{c_{i}+d_{i}}{2}-\left(a_{i}\right)_{\min }\right] \\ +(1-\alpha)\left[\frac{a_{i}+b_{i}}{2}-\left(a_{i}\right)_{\text {min }}\right]\end{array}\right\}$,
where $\left(a_{i}\right)_{\min }=\inf S, S=\bigcup_{i=1}^{N} S_{i}, S_{i}=\left\{a_{i} \mid \mu_{A_{i}}(x)>0\right\}$.
(3) Ranking trapezoidal fuzzy numbers by a unified index

According to Subsection 3.3, we can obtain the unified index $U I\left(A_{i}\right)$ of trapezoidal fuzzy numbers $A_{i}$ as follows:
$U I\left(A_{i}\right)=\left(C V_{i}+\varepsilon_{i}\right)\left[\lambda S_{i}^{R}+(1-\lambda) S_{i}^{L}\right]$,
where $C V_{i}=\frac{-a_{i}^{2}-a_{i} b_{i}-b_{i}^{2}+c_{i}^{2}+c_{i} d_{i}+d_{i}^{2}}{3\left(-a_{i}+b_{i}-c_{i}+d_{i}\right)}, S_{i}^{L}=\left|\frac{a_{i}+b_{i}}{2}\right|$, $S_{i}^{R}=\left|\frac{c_{i}+d_{i}}{2}\right|$ and $\varepsilon_{i}=1 \times 10^{-9}$.
(4) Ranking trapezoidal fuzzy numbers by improved ranking score
According to Subsection 3.4, we can obtain the improved ranking score $\operatorname{IRS}\left(A_{i}\right)$ of trapezoidal fuzzy numbers $A_{i}$ as follows:
$\operatorname{IRS}\left(A_{i}\right)=\omega^{T} \cdot V=\omega_{1} x_{A_{i}}+\mu \omega_{2} h_{A_{i}}+\omega_{3} \frac{\mu}{1+S T D_{A_{i}}}$,
$\mu= \begin{cases}1, & x_{A_{i}} \in[0,+\infty) ; \\ -1 & x_{A_{i}} \in(-\infty, 0) .\end{cases}$
where $x_{A_{i}}=\frac{-a_{i}^{2}-a_{i} b_{i}-b_{i}^{2}+c_{i}^{2}+c_{i} d_{i}+d_{i}^{2}}{3\left(-a_{i}+b_{i}-c_{i}+d_{i}\right)}$ and $h_{A_{i}}=1$.

## IV. FUZZY S-V PORTFOLIO MODELS BASED ON METHODS FOR RANKING FUZZY NUMBERS

In Markowitz's classical M-V model, investors trade-off between maximizing investment income and minimizing variance risk. When the stock's return rate is regarded as a fuzzy number, the possibilistic means and variances are considered as important factors to measure the quality of the stock. In Section III, we review four methods for ranking fuzzy number. Among these methods, many characteristics or factors about fuzzy numbers are considered, which contains a lot of information about fuzzy numbers. Therefore, it is meaningful and valuable to consider the portfolio model based on methods for ranking fuzzy numbers. Next, we will construct concrete models to investigate and demonstrate this problem.

## A. The Novel S-V Portfolio Models

Suppose that there are $N$ optional risky securities. The return rate of each security is regarded as a trapezoidal fuzzy number. Let $r_{i}=\left(a_{i}, b_{i}, c_{i}, d_{i}\right)$ be the return rate of $i$-th asset, and $X^{T}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be investment proportion vector. Furthermore, the total return of the portfolio can be expressed as $R(x)=\sum_{i=1}^{N} x_{i} r_{i}$. According to (40)-(43), for each return rate $r_{i}$, we can obtain the four kinds of ranking scores $R S\left(A_{i}\right)$, $\operatorname{Mag}\left(r_{i}\right), U I\left(r_{i}\right)$ and $\operatorname{IRS}\left(A_{i}\right)$. Then, we obtain the ranking scores of trapezoidal fuzzy numbers for each asset. Next, the objective of maximizing expected return is replaced by maximizing ranking score of the portfolio to construct the score-variance (S-V) model. The four S-V models are constructed as follows:

$$
\begin{align*}
& R S-V: \begin{cases}\max & \sum_{i=1}^{n} x_{i} R S\left(A_{i}\right) \\
\min & V(R(x)) \\
\text { s.t. } & \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N .\end{cases}  \tag{45}\\
& M a g-V: \begin{cases}\max & \sum_{i=1}^{n} x_{i} \operatorname{Mag}\left(r_{i}\right) \\
\min & V(R(x)) \\
\text { s.t. } & \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N .\end{cases}  \tag{46}\\
& U I-V: \begin{cases}\max & \sum_{i=1}^{n} x_{i} U I\left(r_{i}\right) \\
\min & V(R(x)) \\
\text { s.t. } & \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N .\end{cases}  \tag{47}\\
& I R S-V: \begin{cases}\max & \sum_{i=1}^{n} x_{i} I R S\left(A_{i}\right) \\
\min & V(R(x)) \\
\text { s.t. } & \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N .\end{cases} \tag{48}
\end{align*}
$$

According to (12), the risk of portfolio $R(x)=\sum_{i=1}^{N} x_{i} r_{i}$ can be derived as follows:

$$
\begin{aligned}
V(R(x)) & =\frac{1}{18}\left(\sum_{i=1}^{n} x_{i}\left(b_{i}-a_{i}\right)\right)^{2}+\frac{1}{4}\left(\sum_{i=1}^{n} x_{i}\left(c_{i}-b_{i}\right)\right)^{2} \\
& +\frac{1}{18}\left(\sum_{i=1}^{n} x_{i}\left(d_{i}-c_{i}\right)\right)^{2} \\
& +\frac{1}{6}\left(\sum_{i=1}^{n} x_{i}\left(b_{i}-a_{i}\right)\right)\left(\sum_{i=1}^{n} x_{i}\left(c_{i}-b_{i}\right)\right) \\
& +\frac{1}{18}\left(\sum_{i=1}^{n} x_{i}\left(b_{i}-a_{i}\right)\right)\left(\sum_{i=1}^{n} x_{i}\left(d_{i}-c_{i}\right)\right) \\
& +\frac{1}{6}\left(\sum_{i=1}^{n} x_{i}\left(c_{i}-b_{i}\right)\right)\left(\sum_{i=1}^{n} x_{i}\left(d_{i}-c_{i}\right)\right)
\end{aligned}
$$

Moreover, the upper limit $u b_{i}$ and lower limit $l b_{i}$ of investment proportion $x_{i}$ are set to prevent investment from being too concentrated.

## B. Classical M-V Portfolio Models

In order to investigate the feasibility and validity of models (45)-(48), we use the M-V model for comparison at the same time. According to (11), (12) and (49), the M-V model is shown in Eq (50):

$$
\mathrm{M}-\mathrm{V}:\left\{\begin{array}{l}
\max \quad \begin{array}{l}
E(R(x))= \\
\frac{1}{6} \sum_{i=1}^{n} x_{i} a_{i}+\frac{1}{3} \sum_{i=1}^{n} x_{i} b_{i} \\
\min \quad V(R(x))
\end{array} \quad+\frac{1}{3} \sum_{i=1}^{n} x_{i} c_{i}+\frac{1}{6} \sum_{i=1}^{n} x_{i} d_{i}  \tag{50}\\
\text { s.t. } \quad \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N .
\end{array}\right.
$$

In this paper, the upper limit of investment proportion $u b_{i}$ and the lower limit of investment proportion $l b_{i}$ are set as 0.5 and 0.05 respectively.

## V. Numerical Example

## A. Data Process

In this section, we will demonstrate our proposed S-V models by applying a real-world portfolio selection based on global stock market index. We consider a sample of 10 investment assets in the market, including NDX, S\&P 500 Index, FTSE 100 Index, GDAXI, S\&P/ASX 200 Index, CAC 40 Index, SSEC Index, SZI, Nikkei 225 Index and KOSPI. We collected the data set of weekly historical return rates for the 10 securities from Jan. 6th 2010 to Jan. 5th 2020. According to the estimation method of Vercher and Bermudez [23], the trapezoidal fuzzy number of the weekly return of each asset can be expressed as $\left(q_{0.1}, q_{0.4}, q_{0.6}, q_{0.9}\right)$, where $q_{\beta}$ represents the sample quantile of weekly historical return data. Therefore, the return rates of securities can be expressed by trapezoidal fuzzy numbers, as shown in Table I.

## B. Models Solving

Since the S-V and M-V models constructed in Section IV are both bi-objective models with linear constraints. The constraint method is applied to solve the models (45)-(48) and (50), and a set of non-inferior solutions of each model can be obtained. By taking the risk objective of minimizing variance as a constraint condition, the bi-objective models (45)-(48) can be converted into single objective models

TABLE I
The Returns of Securities by The Trapezoidal Fuzzy Numbers

| Securities | Weekly return |
| :--- | :--- |
| NDX | $(-2.60,-0.02,0.97,2.95)$ |
| S\&P 500 Index | $(-2.11,-0.01,0.72,2.37)$ |
| FTSE 100 Index | $(-2.39,-0.27,0.64,2.22)$ |
| GDAXI | $(-3.15,-0.09,0.99,3.16)$ |
| S\&P/ASX 200 Index | $(-2.16,-0.17,0.61,2.09)$ |
| CAC 40 Index | $(-3.19,-0.22,0.84,3.04)$ |
| SSEC Index | $(-3.21,-0.45,0.78,3.32)$ |
| SZI | $(-4.05,-0.43,0.97,4.07)$ |
| Nikkei 225 | $(-3.10,-0.38,0.94,3.30)$ |
| KOSPI | $(-2.44,-0.25,0.69,2.19)$ |

(51)-(56). To be concrete:

## Step 1: Decompose the initial models.

In order to determine the value range of each objective function in the feasible region, we first solve the single -objective model (solving the maximum and minimum values of each objective function in the feasible region).
Split four S-V models (45)-(48) into sub-models (51)-(54) and (55) that only consider maximizing ranking score or minimizing variance as an objective.

$$
\begin{align*}
& \left\{\max \sum_{i=1}^{n} x_{i} R S\left(A_{i}\right)\right.  \tag{51}\\
& \text { s.t. } \quad \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N . \\
& \left\{\max \sum_{i=1}^{n} x_{i} \operatorname{Mag}\left(r_{i}\right)\right.  \tag{52}\\
& \text { s.t. } \quad \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N . \\
& \left\{\max \sum_{i=1}^{n} x_{i} U I\left(r_{i}\right)\right.  \tag{53}\\
& \text { s.t. } \quad \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N . \\
& \left\{\max \sum_{i=1}^{n} x_{i} \operatorname{IRS}\left(A_{i}\right)\right.  \tag{54}\\
& \text { s.t. } \quad \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N . \\
& \int \min V(R(x)) \\
& \text { s.t. } \quad \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, N . \tag{55}
\end{align*}
$$

Split model (50) into two sub-models (55) and (56) that only consider minimizing variance or maximizing expected return as an objective.

$$
\left\{\begin{align*}
\max E(R(x))= & \frac{1}{6} \sum_{i=1}^{n} x_{i} a_{i}+\frac{1}{3} \sum_{i=1}^{n} x_{i} b_{i} \\
& +\frac{1}{3} \sum_{i=1}^{n} x_{i} c_{i}+\frac{1}{6} \sum_{i=1}^{n} x_{i} d_{i} \tag{56}
\end{align*}\right.
$$

s.t. $\quad \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, i=1,2, \ldots, n$.

## Step 2: Solve the sub-models.

By solving sub-models (51)-(56), we can obtain the corresponding optimal solutions $X_{i}(i=1, \ldots, 6)$ of each single-objective model, as shown in Table II.

Then, calculate the objective values of ranking score, expected return and variance corresponding to each optimal solution, as shown in Table III.

From Table III we can get the range of each objective by selecting the corresponding maximum and minimum values. Since this paper converts the risk objective function into the risk constraint condition, we only need to consider the value range of variance risk. The risk range for each model is shown in Table IV.

## Step 4: Models transformation.

Through the constraint method, we transform the bi-objective models (45)-(48) and (50) into (57)-(61) by converting the risk objective into a constraint condition.

TABLE III
The Objective Value of Each Optimal Solution

| optimal <br> solution | ranking score | expected <br> return | variance |
| :---: | :---: | :---: | :---: |
| $X_{1}$ | 0.0778 | 0.2830 | 1.8931 |
| $X_{2}$ | 1.1302 | 0.2830 | 1.8931 |
| $X_{3}$ | 0.4828 | 0.2800 | 1.9675 |
| $X_{4}$ | 0.4996 | 0.2830 | 1.8931 |
| $X_{5}$ | $(0.0766,1.1299,0.4807,0.4976)$ | 0.2841 | 1.9567 |
| $X_{6}$ | $(0.0516,1.0779,0.2230,0.4228)$ | 0.1750 | 1.5240 |
| max | $\sum_{i=1}^{n} x_{i} R S\left(A_{i}\right)$ |  |  |
| s.t. | $V(R(x)) \leq \lambda_{k}^{1}$ |  |  |
|  | $\sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}$, |  |  |
|  | $i=1,2, \ldots, N, k=0, \ldots, K$. |  |  |

TABLE IV
The RISk Range for Each Model

| model | risk range |
| :---: | :---: |
| $R S-V \operatorname{model}(45)$ | $[1.5240,1.8931]$ |
| $M a g-V \operatorname{model}(46)$ | $[1.5240,1.8931]$ |
| $U I-V \operatorname{model}(47)$ | $[1.5240,1.9675]$ |
| $I R S-V \operatorname{model}(48)$ | $[1.5240,1.8931]$ |
| $M-V \operatorname{model}(50)$ | $[1.5240,1.9567]$ |

Step 3: Determine the risk range.
TABLE II
The Optimal Solutions of Sub-MOdels (51-56)

|  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Model | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $x_{5}$ | $x_{6}$ | $x_{7}$ | $x_{8}$ | $x_{9}$ |
| $X_{1}$ for model (51) | 0.50 | 0.10 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| $X_{2}$ for model (52) | 0.50 | 0.10 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| $X_{3}$ for model (53) | 0.50 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.10 |
| $X_{4}$ for model (54) | 0.50 | 0.10 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| $X_{5}$ for model (55) | 0.05 | 0.10 | 0.05 | 0.05 | 0.50 | 0.05 | 0.05 | 0.05 | 0.05 |
| $X_{6}$ for model (56) | 0.50 | 0.05 | 0.05 | 0.10 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |

$$
\begin{align*}
& \int \max \sum_{i=1}^{n} x_{i} \operatorname{Mag}\left(r_{i}\right) \\
& \text { s.t. } \quad V(R(x)) \leq \lambda_{k}^{2}  \tag{58}\\
& \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, \\
& i=1,2, \ldots, N, k=0, \ldots, K \text {. } \\
& \left(\max \sum_{i=1}^{n} x_{i} U I\left(r_{i}\right)\right. \\
& \text { s.t. } \quad V(R(x)) \leq \lambda_{k}^{3}  \tag{59}\\
& \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, \\
& i=1,2, \ldots, N, k=0, \ldots, K \text {. } \\
& \int \max \sum_{i=1}^{n} x_{i} \operatorname{IRS}\left(A_{i}\right) \\
& \text { s.t. } \quad V(R(x)) \leq \lambda_{k}^{4}  \tag{60}\\
& \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, \\
& i=1,2, \ldots, N, k=0, \ldots, K \text {. } \\
& \left\{\max \quad E(R(x))=\frac{1}{6} \sum_{i=1}^{n} x_{i} a_{i}+\frac{1}{3} \sum_{i=1}^{n} x_{i} b_{i}\right. \\
& +\frac{1}{3} \sum_{i=1}^{n} x_{i} c_{i}+\frac{1}{6} \sum_{i=1}^{n} x_{i} d_{i} \\
& \text { s.t. } \quad V(R(x)) \leq \lambda_{k}^{5}  \tag{61}\\
& \sum_{i=1}^{n} x_{i}=1 ; l b_{i} \leq x_{i} \leq u b_{i}, \\
& i=1,2, \ldots, N, k=0, \ldots, K \text {. }
\end{align*}
$$

Where $\lambda_{k}^{j}$ is the upper bound of the risk, and different values
of $j$ correspond to different models. Whenever a risk upper bound $\lambda_{k}^{j}$ is selected, we can obtain a non-inferior solution of the corresponding model. $K+1$ represents the number of selected upper bounds of risk.

## Step 5: Divide the risk value ranges.

In order to obtain a set of non-inferior solutions corresponding to each model, we need to select multiple risk upper bounds. That is, by selecting $K+1$ risk upper bounds for each model, we can obtain $K+1$ non-inferior solutions for each model, which constitutes the model's non-inferior solution set. In this paper, we take $K=10$. In other words, we divide the risk interval of each model into ten equal parts, and then take the interval endpoint of each sub-interval as the upper bounds of the risk. For example, we divide the risk interval [1.5240,1.9567] of M-V model (61) into ten equal parts, and then take 11 endpoints $1.5240=\lambda_{0}^{5}<\lambda_{1}^{5}$ $<\cdots<\lambda_{10}^{5}=1.9567$ of the ten sub-intervals as the upper bounds of risk.

## Step 6: Solve models with different upper bounds of risk.

 By selecting different upper bound $\lambda_{k}^{j}, k=0,1, \ldots, 10$, we can obtain sets of non-inferior solutions of model (57)-(61). In order to demonstrate the feasibility and validity of the proposed S-V model, we calculate the expected return, risk and skewness corresponding to the non-inferior solutions of each model. The results are shown as follows in Table V and VI.
## C. Analysis of Models Results

It can be seen from Table V that the results of $R S-V$, Mag $-V$ and IRS $-V$ models are the same, which indicates that the three ranking score formulas have similar forms when the return rate is a trapezoidal fuzzy number. Therefore, we only consider one of them ( $R S-V$ ) when comparing models results.
Remark 1: It should be pointed out that the same results of models (57), (58) and (59) seem to be unreasonable, but there

TABLE V
THE PORTFOLIO RETURNS OF MODELS (57), (58) AND (60) UNDER DIFFERENT RISK UPPER BOUNDS

| $R S-V$ model (57) |  |  |  | Mag $-V$ model (58) |  |  | IRS $-V$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| return | risk | skewness | return | risk | skewness | return | risk | skewness |
| 0.1750 | 1.5240 | 0.1297 | 0.1750 | 1.5240 | 0.1297 | 0.1750 | 1.5240 | 0.1297 |
| 0.2344 | 1.5609 | 0.1522 | 0.2344 | 1.5609 | 0.1522 | 0.2344 | 1.5609 | 0.1522 |
| 0.2453 | 1.5978 | 0.1586 | 0.2453 | 1.5978 | 0.1586 | 0.2453 | 1.5978 | 0.1586 |
| 0.2502 | 1.6347 | 0.1631 | 0.2502 | 1.6347 | 0.1631 | 0.2502 | 1.6347 | 0.1631 |
| 0.2550 | 1.6716 | 0.1675 | 0.2550 | 1.6716 | 0.1675 | 0.2550 | 1.6716 | 0.1675 |
| 0.2598 | 1.7085 | 0.1720 | 0.2598 | 1.7085 | 0.1720 | 0.2598 | 1.7085 | 0.1720 |
| 0.2645 | 1.7455 | 0.1766 | 0.2645 | 1.7455 | 0.1766 | 0.2645 | 1.7455 | 0.1766 |
| 0.2692 | 1.7823 | 0.1811 | 0.2692 | 1.7823 | 0.1811 | 0.2692 | 1.7823 | 0.1811 |
| 0.2739 | 1.8193 | 0.1857 | 0.2739 | 1.8193 | 0.1857 | 0.2739 | 1.8193 | 0.1857 |
| 0.2785 | 1.8562 | 0.1903 | 0.2785 | 1.8562 | 0.1903 | 0.2785 | 1.8562 | 0.1903 |
| 0.2830 | 1.8930 | 0.1949 | 0.2830 | 1.8930 | 0.1949 | 0.2830 | 1.8930 | 0.1949 |

TABLE VI
THE PORTFOLIO RETURNS OF MODELS (57), (59) AND (61) UNDER DIFFERENT RISK UPPER BOUNDS

| $R S-V$ model (57) |  |  | $U I-V$ model (59) |  |  | M-V model (61) |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| return | risk | skewness | return | risk | skewness | return | risk | skewness |
| 0.1750 | 1.5240 | 0.1297 | 0.1750 | 1.5240 | 0.1297 | 0.1750 | 1.5240 | 0.1297 |
| 0.2344 | 1.5609 | 0.1522 | 0.2367 | 1.5683 | 0.1535 | 0.2364 | 1.5673 | 0.1533 |
| 0.2453 | 1.5978 | 0.1586 | 0.2473 | 1.6127 | 0.1604 | 0.2470 | 1.6105 | 0.1601 |
| 0.2502 | 1.6347 | 0.1631 | 0.2531 | 1.6570 | 0.1658 | 0.2527 | 1.6538 | 0.1654 |
| 0.2550 | 1.6716 | 0.1675 | 0.2589 | 1.7014 | 0.1712 | 0.2583 | 1.6971 | 0.1706 |
| 0.2598 | 1.7085 | 0.1720 | 0.2646 | 1.7457 | 0.1766 | 0.2639 | 1.7403 | 0.1759 |
| 0.2645 | 1.7455 | 0.1766 | 0.2702 | 1.7901 | 0.1821 | 0.2694 | 1.7836 | 0.1813 |
| 0.2692 | 1.7823 | 0.1811 | 0.2758 | 1.8344 | 0.1876 | 0.2748 | 1.8269 | 0.1866 |
| 0.2739 | 1.8193 | 0.1857 | 0.2812 | 1.8788 | 0.1931 | 0.2802 | 1.8702 | 0.1920 |
| 0.2785 | 1.8562 | 0.1903 | 0.2818 | 1.9231 | 0.2049 | 0.2833 | 1.9134 | 0.1948 |
| 0.2830 | 1.8930 | 0.1949 | 0.2800 | 1.9675 | 0.2200 | 0.2841 | 1.9567 | 0.1945 |



Fig. 1. The efficient frontiers of $R S-V, U I-V$ and $\mathrm{M}-\mathrm{V}$ models


Fig. 2. The skewness of $R S-V, U I-V$ and M-V models
is a mathematical basis for it. This is mainly because under the three ranking score formulas (40), (41) and (43), although the scores of ten assets are not the same, their ranking is
consistent. This allows investors to choose higher-ranking assets with a higher investment ratio.


Fig. 3. The Sharpe Ratio of $R S-V, U I-V$ and $M-V$ models

The data of $R S-V, U I-V$ and M-V models in Tables VI are visualized to the efficient frontier in Fig. 1. Moreover, the skewness of assets returns plays significant roles in choosing an optimal portfolio. Considering the skewness of the models' results is also an important factor in demonstrating the performance of the portfolios. The data of $R S-V, U I-V$ and $\mathrm{M}-\mathrm{V}$ models in Tables VI is visualized to Fig. 2. In addition, we adopt another widely used measure of portfolio performance, the Sharpe ratio [24], to validate our model. Its formula is shown as follows:
$S R=\frac{E(R(x))-r_{f}}{\sqrt{V(R(x))}}$,
where $r_{f}$ represents the risk-free interest rate, and $r_{f}$ is taken to be 0 in this paper. According to Table VI and (62), we calculate the Sharpe Ratio of each model and visualize the results to Fig. 3.

From above Tables and Figs. 1-3, we can get several points:

- From Fig. 1, the efficient frontier derived from the results of our proposed S-V model is basically consistent with that of the classic M-V model. The maximum returns of both and models are slightly lower than that of M-V model.
- It can be seen from Figs. 1, 2 and 3 that the results of $R S-V$ and $U I-V$ models constructed in this paper are generally consistent with those of the classical M-V model.
- When the risk tolerance is low, the return and skewness of $R S-V$ model are slightly better than models $U I-V$ and $\mathrm{M}-\mathrm{V}$ models. When the risk tolerance is large, compared with $R S-V$ and $U I-V$ models, M-V model can get a better return, but with a smaller skewness.
- The results of $R S-V$ model are better than those of $U I-V$ and the classical M-V models in Sharpe Ratio.


## VI. CONCLUSION

In this paper, a novel portfolio model based on methods for ranking fuzzy numbers is developed. Firstly, we review four ranking methods of fuzzy numbers. Then, considering that the ranking score can measure the characteristics of fuzzy numbers, we adopt the objective of maximizing the ranking score instead of the objective of maximizing the mean value to construct the score-variance $(\mathrm{S}-\mathrm{V})$ models. Moreover, the $\mathrm{M}-\mathrm{V}$ model is also constructed and compared with the model proposed in this paper. Finally, a numerical experiment is presented to illustrate the feasibility and validity of our proposed S-V models. Compared with the M-V model, the return of our proposed model is slightly smaller, but it performs better in terms of skewness and Sharpe Ratio.

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