Soft Rough Lattices (Ideals, Filters) over Lattices

Kuanyun Zhu, Jingru Wang* and Yongwei Yang

Abstract—**In this paper, we discuss the relationships among soft sets, rough sets and lattices. We introduce the concept of soft rough lattices (ideals, filters) of lattices, which is an extended concept of rough lattices (ideals, filters) of lattices. At first, we put forward the concepts of C-soft sets and CC-soft sets over lattices, respectively. And then, we study the roughness in lattices with respect to soft approximation spaces. In addition, we explore some new soft rough operations over lattices. In particular, we also discuss the lower and upper soft rough lattices (ideals, filters) with respect to another soft set.**

Index Terms—**Soft set; Rough set; Soft rough set; Lattice (ideal, filter); Soft rough lattice (ideal, filter)**

I. Introduction

ROUGH set theory was first introduced by Pawlak [25] which is an extension of set theory. As far as known, a rough set can be described by a pair of ordinary sets called lower and upper approximations. Nowadays, some authors have investigated algebraic properties of rough sets. Liu and Zhu [20] studied the algebraic structures of the approximations based on arbitrary binary relations. Kuroki [17] proposed the concept of rough ideals in a semigroup. Xiao and Zhang [26] put forward the concepts of rough prime ideals and rough fuzzy prime ideals in semigroups. Kuroki [18] discussed the rough subgroups with respect to a t-level subset of a fuzzy normal subgroup. Davvaz [8] applied the concept of fuzzy ideal of a ring for definitions of the lower and upper approximations in a ring. In addition, many researchers applied this theory to algebraic structures in many literatures, such as [7], [9], [12].

Nowadays, the mathematical modelling and manipulating of various types of uncertainties has become an increasingly important issue in solving complicated problems arisings in a wide rang of areas such as economy, engineering, environmental science, medicine and social science. As far as known that there were several theories to describe uncertainties, for example, fuzzy set theory [29], rough set theory [25] and other mathematical tools. In 1999, Molodtsov [24] first put forward soft set theory as a new mathematical tool for dealing with uncertainties. So far, the research on soft sets is progressing rapidly. In 2003, Maji et al. [22] proposed some basic operations. Further, Ali et al. [1] revised some operations. In 2011, Ali [2] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications, such as [3], [4], [5], [23], [16]. In particular, Zhan and Zhu [31] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures in many literatures [13], [14]. In 2019, Zhan and Alcantud [34] gave a survey of parameter reduction of soft sets and corresponding algorithms In 2020, Ma et al. [21] studied interval-valued intuitionistic fuzzy soft sets based decision making and parameter reduction. Wang et al. [27] studied hesitant bipolar-valued fuzzy soft sets and their application in decision making.

Soft set theory and rough set theory were all mathematical tools to deal with uncertainty. In 2010, Feng et al. [10], [11] provided a framework to combine rough sets with soft sets, which gives rise to some interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. In 2014, Li and Xie [19] investigated the relationships among soft sets, soft rough sets and topologies. In 2015, Zhan et al. [32] applied rough soft set theory to hemirings. In recent years, Shabir et al. [15] pointed out that there exist some problems on Feng's soft rough set, for more details, see [15]. Based on this reason, Shabir modified the concept of soft rough set. The underlying concepts are very similar to Pawlak rough sets. In 2019, Zhu [35] studied soft fuzzy rough rings (ideals) of rings and their application in decision making. In fact, the relationships among rough sets, fuzzy sets, soft sets and semirings (hemirings) have been considered by many scientists in many papers, such as [30], [33], [36], [37].

Based on the above considerations, it is an interesting work to further study on this topic. This paper aims at providing a framework to combine soft sets, rough sets with lattices all together, which propose the concept of soft rough lattices (ideals, filters) over lattices. This paper is organized as follows. In Section II, some concepts and results on lattices, soft sets and rough sets are recalled. In Section III, we study some new soft rough operations over lattices. Further, the lower and upper soft rough lattices (ideals, filters) are investigated in Section IV. In particular, in Section V, we further discuss soft rough lattices (ideals, filters) based on another soft set. Section VI is conclusion and further works.

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II. Preliminaries

In this section, we recall some basic notions and results which shall be needed in the sequel. At first, we recall the concept of lattices as follows.

A lattice L is a poset in which any two elements have a unique supremum and an infimum, where $x \vee y$= sup$\{x, y\}$

and $x \wedge y$= inf $\{x, y\}$, respectively. Throughout this paper, L is always a lattice.

Definition 2.1: [6] Let $\emptyset \subsetneq X \subseteq L$. Then X is a sublattice of L if $x, y \in L$, $x \vee y \in X$ and $x \wedge y \in X$.

Definition 2.2: [6] Let $\emptyset \subsetneq I \subseteq L$. Then I is called an ideal of L if

(1) $a, b \in I$ implies $x \vee y \in I$,

(2) $a \in L, b \in I$ and $a \leq b$ imply $a \in I$.

Definition 2.3: [6] Let $\emptyset \subsetneq F \subseteq L$. Then F is called a filter of L if

(1) $a, b \in F$ implies $x \wedge y \in F$,

(2) $a \in L, b \in F$ and $a \geq b$ imply $a \in F$.

Let U be an initial universe set and E be a set of parameters. $P(U)$ denotes the power set over U.

Definition 2.4: [24] A pair $\mathfrak{S} = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping.

Definition 2.5: [10] A soft set $\mathfrak{S} = (F, A)$ over U is called a full soft set if $\bigcup\limits_{a \in A} F(a) = U$.

For a soft set $\mathfrak{S} = (F, A)$, the set Supp$(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A).

Definition 2.6: [28] Let $\mathfrak{S} = (F, A)$ be a soft set over L. Then (F, A) is called a soft lattice (ideal, filter) over L if $F(x)$ is a sublattice (ideal, filter) of L for all $x \in$ Supp(F, A).

Next, we give the concept of rough sets as follows.

Definition 2.7: [25] Let R be an equivalence relation on the universe U and (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called definable if $R_* X = R^* X$; in the opposite case, i.e., if $R_* X - R^* X \neq \emptyset$, X is said to be a rough set, where two operators are defined as:

$$R_* X = \{x \in U | [x]_R \subseteq X\},$$

$$R^* X = \{x \in U | [x]_R \cap X \neq \emptyset\}.$$

In what follows, we give the concept of soft rough sets.

Definition 2.8: [11] Let $\mathfrak{S} = (F, A)$ be a soft set over U. Then the pair $P = (U, \mathfrak{S})$ is called a soft approximation space. Based on P, we define the following two operators:

$$\underline{apr}_P(X) = \{u \in U | \exists a \in A[u \in F(a) \subseteq X]\},$$

$$\overline{apr}_P(X) = \{u \in U | \exists a \in A[u \in F(a), F(a) \cap X \neq \emptyset]\},$$

assigning to every subset $X \subseteq U$.

Two sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the lower and upper soft rough approximations of X in P, respectively. If $\underline{apr}_P(X) = \overline{apr}_P(X)$, X is said to be soft definible; otherwise, X is called a soft rough set.

Definition 2.9: [15] Let (F, A) be a soft set over U and $\theta : U \to P(A)$ be a mapping defined as $\theta(x) = \{a | x \in F(a)\}$. Then the pair (U, θ) is called soft approximation s-pace and for any $X \subseteq U$, the lower soft rough approximation and upper soft rough approximation of X are denoted by $\underline{X}_\theta$ and $\overline{X}_\theta$, respectively, which two operators are defined as

$$\underline{X}_\theta = \{x \in X | \theta(x) \neq \theta(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\theta = \{x \in U | \theta(x) = \theta(y) \text{ for some } y \in X\}.$$

If $\underline{X}_\theta = \overline{X}_\theta$, then X is said to be soft definable, otherwise, X is said to be a soft rough set.

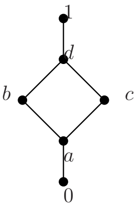

Fig. 1. A lattice L

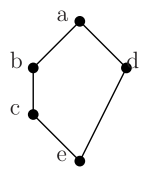

Fig. 2. A lattice L

III. Some operations of soft rough sets over lattices

In this section, firstly, we introduce the concept of soft rough sets over a given lattice. Then, we consider some operations of soft rough sets over a lattice. In particular, two special kinds of soft sets over L are also given. We begin with the definition of soft rough sets over a lattice.

Definition 3.1: Let (F, A) be a soft set over L and $\theta : L \to P(A)$ be a mapping defined as $\theta(x) = \{a | x \in F(a)\}$. Then the pair (L, θ) is called soft approximation space and for any $X \subseteq L$, the lower soft rough approximation and upper soft rough approximation of X are denoted by $\underline{X}_\theta$ and $\overline{X}_\theta$, respectively, which are two operators are defined as

$$\underline{X}_\theta = \{x \in X | \theta(x) \neq \theta(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\theta = \{x \in L | \theta(x) = \theta(y) \text{ for some } y \in X\}$$

If $\underline{X}_\theta = \overline{X}_\theta$, then X is said to be soft definable, otherwise, X is said to be a soft rough set over L.

Remark 3.2: It follows from Definition 3.1 that $\underline{X}_\theta \subseteq X \subseteq \overline{X}_\theta$ for any $X \subseteq L$.

Now, some basic properties of lower and upper soft rough approximations of a subset X of L are investigated. In order to illustrate the roughness in X with respect to soft approximation spaces over lattices, at first, we introduce two special kinds of soft sets over L.

Definition 3.3: Let $\mathfrak{S} = (F, A)$ be a soft set over L and $\theta : L \to P(A)$ be a mapping defined as $\theta(x) = \{a | x \in F(a)\}$. Then $\mathfrak{S}$ is called a C-soft set over L if $\theta(a) = \theta(b)$ and $\theta(c) = \theta(d)$ imply $\theta(a \vee c) = \theta(b \vee d)$ and $\theta(a \wedge c) = \theta(b \wedge d)$ for all $a, b, c, d \in L$.

Definition 3.4: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L and $\theta : L \to P(A)$ be a mapping defined as $\theta(x) = \{a | x \in F(a)\}$. Then $\mathfrak{S}$ is called a CC-soft set over L if for all $c \in L$,

(i) $\theta(c) = \theta(x \vee y)$ for $x, y \in L$, there exist $a, b \in L$ such that $\theta(x) = \theta(a)$ and $\theta(y) = \theta(b)$ satisfying $c = a \vee b$.

(ii) $\theta(c) = \theta(x \wedge y)$ for $x, y \in L$, there exist $a, b \in L$ such that $\theta(x) = \theta(a)$ and $\theta(y) = \theta(b)$ satisfying $c = a \wedge b$.

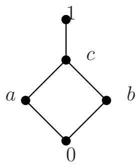

Fig. 3. A lattice L

Example 3.5: Let $L = \{0, a, b, c, d, 1\}$. We define the binary relation $\leq$ in the Figure 1. $\mathfrak{S} = (F, A)$ is a soft set over L which is given by Table 1.

Table 1 Soft set $\mathfrak{S}$

	0	a	b	c	d	1
e_1	1	1	1	1	1	1
e_2	1	0	0	0	1	1
e_3	1	1	1	1	0	0

Then the mapping $\theta : L \to P(A)$ in soft approximation space (L, θ) is given by $\theta(0) = \{e_1, e_2, e_3\}$, $\theta(a) = \theta(b) = \theta(c) = \{e_1, e_3\}$, $\theta(d) = \theta(1) = \{e_1, e_2\}$. Then we can check that $\mathfrak{S}$ is not a C-soft set over L. In fact, $\theta(b) = \theta(c)$ and $\theta(a) = \theta(b)$ but $\theta(b \vee a) = \theta(b) \neq \theta(d) = \theta(c \vee b)$.

Example 3.6: We consider the lattice L in Example 3.5. $\mathfrak{S} = (F, A)$ is a soft set over L which is given by Table 2.

Table 2 Soft set $\mathfrak{S}$

	0	a	b	c	d	1
e_1	1	1	1	0	1	1
e_2	0	0	0	1	1	1
e_3	1	1	0	1	1	1

Then the mapping $\theta : L \to P(A)$ in soft approximation space (L, θ) is given by $\theta(0) = \theta(a) = \{e_1, e_3\}$, $\theta(b) = \{e_1\}$, $\theta(c) = \{e_2, e_3\}$, $\theta(d) = \theta(1) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over L. Nevertheless, $\mathfrak{S}$ is not a CC-soft set over L, because $\theta(1) = \theta(b \vee c)$, $\theta(b) = \theta(b)$ and $\theta(c) = \theta(c)$ but $1 \neq b \vee c$.

Example 3.7: Let $L = \{0, a, b, c, 1\}$. We define the binary relation $\leq$ in the Figure 2. $\mathfrak{S} = (F, A)$ is a soft set over L which is given by Table 3.

Table 3 Soft set $\mathfrak{S}$

	0	a	b	c	1
e_1	1	1	1	1	1
e_2	0	1	0	1	1
e_3	0	0	0	0	1

Then the mapping $\theta : L \to P(A)$ in soft approximation space (L, θ) is given by $\theta(0) = \theta(b) = \{e_1\}$, $\theta(a) = \theta(c) = \{e_1, e_2\}$, $\theta(1) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is a CC-soft set over L.

Now we study some new soft rough operations over lattices. We define the join and meet of two non-empty subsets in a lattice as follows: $A \vee B = \{a \vee b | a \in A, b \in B\}$, $A \wedge B = \{a \wedge b | a \in A, b \in B\}$.

Proposition 3.8: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L and A, B be any two non-empty subsets in L. Then

$$\overline{A}_\theta \vee \overline{B}_\theta \subseteq \overline{A \vee B}_\theta.$$

Proof. Let $c \in \overline{A}_\theta \vee \overline{B}_\theta$. Then $c = a \vee b$, where $a \in \overline{A}_\theta$ and $b \in \overline{B}_\theta$. It follows from Definition 3.1 that $\theta(a) = \theta(y)$ and $\theta(b) = \theta(z)$ for some $y \in A$, $z \in B$. Since $\mathfrak{S}$ is a C-soft set, $\theta(a \vee b) = \theta(y \vee z)$ for some $y \vee z \in A \vee B$. Thus $c = a \vee b \in \overline{A \vee B}_\theta$. Therefore, $\overline{A}_\theta \vee \overline{B}_\theta \subseteq \overline{A \vee B}_\theta$. □

Proposition 3.9: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L and A, B be any two non-empty subsets in L. Then

$$\overline{A}_\theta \wedge \overline{B}_\theta \subseteq \overline{A \wedge B}_\theta.$$

Proof. Let $c \in \overline{A}_\theta \wedge \overline{B}_\theta$. Then $c = a \wedge b$, where $a \in \overline{A}_\theta$ and $b \in \overline{B}_\theta$. It follows from Definition 3.1 that $\theta(a) = \theta(y)$ and $\theta(b) = \theta(z)$ for some $y \in A$, $z \in B$. Since $\mathfrak{S}$ is a C-soft set, $\theta(a \wedge b) = \theta(y \wedge z)$ for $y \wedge z \in A \wedge B$. Hence $c = a \wedge b \in \overline{A \wedge B}_\theta$. Therefore, $\overline{A}_\theta \wedge \overline{B}_\theta \subseteq \overline{A \wedge B}_\theta$. □

The following example shows that the containment in Propositions 3.8 and 3.9 are proper.

Example 3.10: Consider the Example 3.6. Then $\mathfrak{S}$ is a C-soft set over L. If we take $A = \{b\}$ and $Y = \{c\}$, then $\overline{A}_\theta = \{b\}$ and $\overline{B}_\theta = \{c\}$. So $\overline{A}_\theta \vee \overline{B}_\theta = \{d\}$, $\overline{A}_\theta \wedge \overline{B}_\theta = \{a\}$. Also we have, $\overline{A \vee B}_\theta = \overline{\{d\}}_\theta = \{1, d\}$, $\overline{A \wedge B}_\theta = \overline{\{a\}}_\theta = \{0, a\}$. Thus $\overline{A}_\theta \vee \overline{B}_\theta \subsetneqq \overline{A \vee B}_\theta$ and $\overline{A}_\theta \wedge \overline{B}_\theta \subsetneqq \overline{A \wedge B}_\theta$.

If we strength the condition, we can obtain the following result.

Proposition 3.11: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L and A, B be any two non-empty subsets in L. Then

$$\overline{A}_\theta \vee \overline{B}_\theta = \overline{A \vee B}_\theta.$$

Proof. It follows from Proposition 3.8 that we only need to show $\overline{A \vee B}_\theta \subseteq \overline{A}_\theta \vee \overline{B}_\theta$. Now let $c \in \overline{A \vee B}_\theta$. Then $\theta(c) = \theta(x \vee y)$ for some $x \in A$ and $y \in B$. Since $\mathfrak{S}$ is a CC-soft set over L, there exist $a, b \in L$, such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$ satisfying $c = a \vee b$. Thus $a \in \overline{A}_\theta$ and $b \in \overline{B}_\theta$. Hence $c \in \overline{A}_\theta \vee \overline{B}_\theta$. So $\overline{A}_\theta \vee \overline{B}_\theta = \overline{A \vee B}_\theta$. □

Proposition 3.12: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L and A, B be any two non-empty subsets in L. Then

$$\overline{A}_\theta \wedge \overline{B}_\theta = \overline{A \wedge B}_\theta.$$

Proof. It follows from Proposition 3.9 that we only need to show $\overline{A \wedge B}_\theta \subseteq \overline{A}_\theta \wedge \overline{B}_\theta$. Now let $c \in \overline{A \wedge B}_\theta$. Then $\theta(c) = \theta(x \wedge y)$ for some $x \in A$ and $y \in B$. Since $\mathfrak{S}$ is a CC-soft set over R, there exist $a, b \in L$ such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$ satisfying $c = a \wedge b$. Thus $a \in \overline{A}_\theta$ and $b \in \overline{B}_\theta$. Hence $c \in \overline{A}_\theta \wedge \overline{B}_\theta$. So $\overline{A}_\theta \wedge \overline{B}_\theta = \overline{A \wedge B}_\theta$. □

Next, we consider lower soft rough approximations over lattices.

Proposition 3.13: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L and A, B be any two non-empty subsets in L. Then

$$\underline{A}_\theta \vee \underline{B}_\theta \subseteq \underline{A \vee B}_\theta.$$

Proof. Assume that $\underline{A}_\theta \vee \underline{B}_\theta \subseteq \underline{A \vee B}_\theta$ is false, then there exists $c \in \underline{A}_\theta \vee \underline{B}_\theta$ but $c \notin \underline{A \vee B}_\theta$. Then $c = a \vee b$, where $a \in \underline{A}_\theta$ and $b \in \underline{B}_\theta$, and so $\theta(a) \neq \theta(x)$ and $\theta(b) \neq \theta(y)$ for all $x \in A^c$ and $y \in B^c$. $(\star)$

Notice that $c \notin \underline{A \vee B}_\theta$, then we have the following two cases:

Case (i) $c \notin A \vee B$, which contradicts with $c \in \underline{A}_\theta \vee \underline{B}_\theta \subseteq A \vee B$;

Case (ii) $c \in A \vee B$ and $\theta(c) = \theta(x' \vee y')$ for some $x' \vee y' \in (A \vee B)^c$. Thus $x' \in A^c$ or $y' \in B^c$. In fact, if $x' \notin A^c$ and $y' \notin B^c$, we have $x' \vee y' \in A \vee B$, which contradicts with $x' \vee y' \in (A \vee B)^c$. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over L, there exist $a', b' \in L$ such that $\theta(a') = \theta(x')$ and $\theta(b') = \theta(y')$ satisfying $c = a' \vee b'$, for some $x' \in A^c$ or $y' \in B^c$, which contradicts with $(\star)$. Therefore, $\underline{A}_\theta \vee \underline{B}_\theta \subseteq \underline{A \vee B}_\theta$. □

Proposition 3.14: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L and A, B be any two non-empty subsets in L. Then

$$\underline{A}_\theta \wedge \underline{B}_\theta \subseteq \underline{A \wedge B}_\theta.$$

Proof. Assume that $\underline{A}_\theta \wedge \underline{B}_\theta \subseteq \underline{A \wedge B}_\theta$ is false, then there exists $c \in \underline{A}_\theta \wedge \underline{B}_\theta$ but $c \notin \underline{A \wedge B}_\theta$. Then $c = a \wedge b$, where $a \in \underline{A}_\theta$ and $b \in \underline{B}_\theta$. Hence $\theta(a) \neq \theta(x)$ and $\theta(b) \neq \theta(y)$ for all $x \in A^c$ and $y \in B^c$. $(\triangle)$

Notice that $c \notin \underline{A \wedge B}_\theta$, then we have the following two cases:

Case (i) $c \notin A \wedge B$, which contradicts with $c \in \underline{A}_\theta \wedge \underline{B}_\theta \subseteq A \wedge B$;

Case (ii) $c \in A \wedge B$ and $\theta(c) = \theta(x' \wedge y')$ for some $x' \wedge y' \in (A \wedge B)^c$. Thus $x' \in A^c$ or $y' \in B^c$. In fact, if $x' \notin A^c$ and $y' \notin B^c$, we have $x' \wedge y' \in A \wedge B$, which contradicts with $x' \wedge y' \in (A \wedge B)^c$. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over L, there exist $a', b' \in L$, such that $\theta(a') = \theta(x')$ and $\theta(b') = \theta(y')$ satisfying $c = a' \wedge b'$, for some $x' \in A^c$ or $y' \in B^c$, which contradicts with $(\triangle)$. Therefore, $\underline{A}_\theta \wedge \underline{B}_\theta \subseteq \underline{A \wedge B}_\theta$. □

The following example shows that the containment in Propositions 3.13 and 3.14 are proper.

Example 3.15: Consider the lattice L and the soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Then $\mathfrak{S}$ is a CC-soft set over L. If we take $A = \{b\}$ and $B = \{a, c\}$, then $\underline{A}_\theta = \emptyset$ and $\underline{B}_\theta = \{a, c\}$. So $\underline{A}_\theta \vee \underline{B}_\theta = \emptyset$, $\underline{A}_\theta \wedge \underline{B}_\theta = \emptyset$. Also we have $\underline{A \vee B}_\theta = \underline{\{a, c\}}_\theta = \{a, c\}$, $\underline{A \wedge B}_\theta = \underline{\{0, b\}}_\theta = \{0, b\}$. Thus $\underline{A}_\theta \vee \underline{B}_\theta \subsetneqq \underline{A \vee B}_\theta$ and $\underline{A}_\theta \wedge \underline{B}_\theta \subsetneqq \underline{A \wedge B}_\theta$.

IV. Characterizations of Soft Rough Lattices (Ideals, Filters) over Lattices

In this section, at first, we propose the concept of soft rough lattices (ideals, filters) over a lattice. And then, we show some characterizations of soft rough lattices (ideals, filters) over lattices.

Definition 4.1: In Definition 3.1, if $\underline{X}_\theta \neq \overline{X}_\theta$,
(i) X is called a lower (upper) soft rough lattice (ideal, filter) w.r.t. $\mathfrak{S}$ over L, if $\underline{X}_\theta$ ($\overline{X}_\theta$) is a sublattice (ideal, filter) of L;
(ii) X is called an soft rough lattice (ideal, filter) w.r.t. $\mathfrak{S}$ over L, if $\underline{X}_\theta$ and $\overline{X}_\theta$ are sublattices (ideals, filters) of L.

Example 4.2: Consider the lattice L and the soft set $\mathfrak{S} = (F, A)$ in Example 3.7. It follows from Definition 4.1 that for $X = \{0, b, c\} \subseteq L$, we have

$$\underline{X}_\theta = \{0, b\} \text{ and } \overline{X}_\theta = \{0, a, b, c\}.$$

This shows that $\underline{X}_\theta$ and $\overline{X}_\theta$ are sublattices (ideals) of L. In other words, X is a soft rough lattice (ideal) over L.

Theorem 4.3: Let (L, θ) be a soft approximation space. If X and Y are lower soft rough lattices (ideals, filters) over L, then $X \cap Y$ is a lower soft rough lattice (ideal, filter) over L.

Proof. It follows from Definition 4.1 that $\underline{X}_\theta$ and $\underline{Y}_\theta$ are lattices (ideals, filters) of L. So $\underline{X}_\theta \cap \underline{Y}_\theta$ is a sublattice (ideal, filter) of L. It is easy to know that $\underline{X \cap Y}_\theta = \underline{X}_\theta \cap \underline{Y}_\theta$. Thus $\underline{X \cap Y}_\theta$ is also a sublattice (ideal, filter) of L. It follows from Definition 4.1 that $X \cap Y$ is a lower soft rough lattice (ideal, filter) over L. □

Finally, we study the lower and upper soft rough lattice (ideal, filter) over lattices.

Theorem 4.4: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L. If X is a sublattice of L, then X is an upper soft rough lattice over L.

Proof. Let $a, b \in \overline{X}_\theta$. Then there exist $y, z \in X$ such that $\theta(a) = \theta(y)$ and $\theta(b) = \theta(z)$. Since $\mathfrak{S}$ is a C-soft set over L, $\theta(a \vee b) = \theta(y \vee z)$ and $\theta(a \wedge b) = \theta(y \wedge z)$. Since X is a sublattice of L, $y \vee z \in X$ and $y \wedge z \in X$. So $a \vee b \in \overline{X}_\theta$ and $a \wedge b \in \overline{X}_\theta$. Thus $\overline{X}_\theta$ is a sublattice of L. Therefore, X is an upper soft rough lattice over L. □

Theorem 4.5: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L. If X is a sublattice of L, then X is a lower soft rough lattice over L when $\underline{X}_\theta \neq \emptyset$.

Proof. It follows from propositions 3.13 and 3.14 that $\underline{X}_\theta \vee \underline{X}_\theta \subseteq \underline{X \vee X}_\theta$ and $\underline{X}_\theta \wedge \underline{X}_\theta \subseteq \underline{X \wedge X}_\theta$. Further, since $X \vee X \subseteq X$ and $X \wedge X \subseteq X$, $\underline{X \vee X}_\theta \subseteq \underline{X}_\theta$ and $\underline{X \wedge X}_\theta \subseteq \underline{X}_\theta$. So $\underline{X}_\theta \vee \underline{X}_\theta \subseteq \underline{X}_\theta$ and $\underline{X}_\theta \wedge \underline{X}_\theta \subseteq \underline{X}_\theta$. Thus $\underline{X}_\theta$ is a sublattice of L. Therefore, X is a lower soft rough lattice over L. □

Theorem 4.6: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L. If X is an ideal of L, then X is an upper soft rough ideal over L.

Proof. Let X be an ideal of L, $a \in L, b \in \overline{X}_\theta$ and $a \leq b$. Then there exists $c \in X$ such that $\theta(b) = \theta(c)$. Since $\theta(a) = \theta(a)$ and $\mathfrak{S} = (F, A)$ is a C-soft set over L, $\theta(a) = \theta(a \wedge b) = \theta(c \wedge a)$. Since X is an ideal of L, $c \wedge a \leq c$, we have $c \wedge a \in X$. Thus $a \in \overline{X}_\theta$. Thus, $\overline{X}_\theta$ is an ideal of L. Therefore, X is an upper soft rough ideal over L. □

Theorem 4.7: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L. If X is an ideal of L, then X is a lower soft rough ideal over L when $\underline{X}_\theta \neq \emptyset$.

Proof. It follows from Theorem 4.5 that $m \vee n \in \underline{X}_\theta$ for $m, n \in \underline{X}_\theta$. Now let $\underline{X}_\theta \neq \emptyset$. Then for all $a \in L, b \in \underline{X}_\theta$ and $a \leq b$, we assume that $a \notin \underline{X}_\theta$. Thus we have $\theta(b) \neq \theta(d)$ for all $d \in X^c$.

Notice that $a \notin \underline{X}_\theta$, then we have the following two cases:

Case (i) If $a \notin X$, since $b \in \underline{X}_\theta \subseteq X$, $a \leq b$ and X is an ideal of L, we have $a \in X$, which contradicts with $a \notin X$;

Case (ii) $a \in X$. Since $a \notin \underline{X}_\theta$, we have $\theta(a) = \theta(c)$ for some $c \in X^c$. That is $\theta(a \wedge b) = \theta(c)$ for some $c \in X^c$. Since $\mathfrak{S} = (F, A)$ is a CC-soft set, there exist $x, y \in L$ such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$ satisfying $x \wedge y = c \in X^c$. Thus, $x \in X^c$ or $y \in X^c$. In fact, if $x \notin X^c$ and $y \notin X^c$, Since X is an ideal over L, X is a sublattice over L, we have $x \wedge y = c \in X \wedge X \subseteq X$, which contradicts with $x \wedge y \in X^c$. Thus there exist $x \in X^c$ such that $\theta(a) = \theta(x)$ or $y \in X^c$ such that $\theta(b) = \theta(y)$. Thus we have $b \notin \underline{X}_\theta$, which contradicts with $b \in \underline{X}_\theta$. This implies $\underline{X}_\theta$ is an ideal of L, that is X is a lower soft rough ideal over L. □

Theorem 4.8: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L. If X is a filter of L, then X is an upper soft rough filter over L.

Proof. Let X be a filter of L, $a \in L, b \in \overline{X}_\theta$ and $a \geq b$. Then there exists $c \in X$ such that $\theta(b) = \theta(c)$. Since $\theta(a) = \theta(a)$ and $\mathfrak{S} = (F, A)$ is a C-soft set over L, $\theta(a) = \theta(a \vee b) = \theta(c \vee a)$. Since X is a filter of L, $c \vee a \geq c$, we have $c \vee a \in X$. Hence, $a \in \overline{X}_\theta$. That is $\overline{X}_\theta$ is a filter of L. Therefore, X is an upper soft rough filter over L. □

Theorem 4.9: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L. If X is a filter of L, then X is a lower soft rough filter over L when $\underline{X}_\theta \neq \emptyset$.

Proof. It follows from Theorem 4.6 that $m \wedge n \in \underline{X}_\theta$ for $m, n \in \underline{X}_\theta$. Now let $\underline{X}_\theta \neq \emptyset$. Then for all $a \in L, b \in \underline{X}_\theta$ and $a \geq b$, we assume that $a \notin \underline{X}_\theta$. Then we have $\theta(b) \neq \theta(d)$ for all $d \in X^c$.

Notice that $a \notin \underline{X}_\theta$, then we have the following two cases:

Cases (i) If $a \notin X$, since $b \in \underline{X}_\theta \subseteq X$, $a \geq b$ and X is a filter of L, we have $a \in X$, which contradicts with $a \notin X$;

Cases (ii) $a \in X$. Since $a \notin \underline{X}_\theta$, we have $\theta(a) = \theta(c)$ for some $c \in X^c$. That is $\theta(a \vee b) = \theta(c)$ for some $c \in X^c$. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over L, there exist $x, y \in L$ such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$ satisfying $x \vee y = c \in X^c$. Thus, $x \in X^c$ or $y \in X^c$. In fact, if $x \notin X^c$ and $y \notin X^c$, since X is a filter of L, X is a sublattice of L, we have $x \vee y = c \in X \vee X \subseteq X$, which contradicts with $x \vee y \in X^c$. Thus there exist $x \in X^c$ such that $\theta(a) = \theta(x)$ or $y \in X^c$ such that $\theta(b) = \theta(y)$. Thus we have $b \notin \underline{X}_\theta$, which contradicts with $b \in \underline{X}_\theta$. This implies $\underline{X}_\theta$ is a filter of L, that is X is a lower soft rough filter over L. □

Remark 4.10: The above theorems show that any soft rough lattice (ideal, filter) is a generalization of a lattice (ideal, filter) of lattices.

V. Soft rough lattices (ideals, filters) over lattices with respect to another soft set

In this section, at first, we introduce the concept of soft rough lattices (ideals, filters) with respect to another soft set over a lattice. And then, we study the operations and characterizations of lower and upper soft rough lattices (ideals, filters), respectively.

Definition 5.1: Let $\mathfrak{S} = (F, A)$ be a soft set over L and $\theta : L \to P(A)$ be a mapping defined as $\theta(x) = \{a : x \in F(a)\}$. Let $\mathfrak{T} = (G, B)$ be another soft set defined over L. The lower and upper soft rough approximations of $\mathfrak{T}$ with respect to $\mathfrak{S}$ are denoted by $\underline{(G, B)}_\theta = (\underline{G}_\theta, B)$ and $\overline{(G, B)}_\theta = (\overline{G}_\theta, B)$, respectively, which are two operators defined as

$$\underline{G(e)}_\theta = \{x \in G(e) | \theta(x) \neq \theta(y) \text{ for all } y \in L - G(e)\}$$

and

$$\overline{G(e)}_\theta = \{x \in L | \theta(x) = \theta(y) \text{ for some } y \in G(e)\}$$

for all $e \in B, x \in L$.

(i) If $\underline{(G, B)}_\theta = \overline{(G, B)}_\theta$, then $\mathfrak{T}$ is called definable.

(ii) If $\underline{(G, B)}_\theta \neq \overline{(G, B)}_\theta$ and $\underline{G(e)}_\theta$ ($\overline{G(e)}_\theta$) is a sublattice (ideal, filter) of L for all $e \in B$, then $\mathfrak{T}$ is called a lower (upper) soft rough lattice (ideal, filter) with respect to $\mathfrak{S}$ over L. Moreover, $\mathfrak{T}$ is called a soft rough lattice (ideal, filter) with respect to $\mathfrak{S}$ over L if $\underline{G(e)}_\theta$ and $\overline{G(e)}_\theta$ and are sublattices (ideals, filters) with respect to $\mathfrak{S}$ of L for all $e \in B$.

Example 5.2: We consider the lattice L and soft set $\mathfrak{S} = (F, A)$ in Example 3.7. Define a soft set $\mathfrak{T} = (G, B)$ as the following Table 8.

Table 4 Soft set $\mathfrak{S}$

	0	a	b	c	1
e_1	1	0	0	0	1
e_2	0	0	1	0	1
e_3	1	1	1	0	0

By calculating, $\underline{G(e_1)}_\theta = \{1\}$, $\overline{G(e_1)}_\theta = \{0, b, 1\}$, $\underline{G(e_2)}_\theta = \{1\}$, $\overline{G(e_1)}_\theta = \{0, b, 1\}$, $\underline{G(e_3)}_\theta = \{0, b\}$, $\overline{G(e_3)}_\theta = \{0, a, b, c\}$. It is easy to check that $\underline{(G, B)}_\theta$ and $\overline{(G, B)}_\theta$ are sublattices over L for all $e \in B$. In other words, $\mathfrak{T}$ is a soft rough lattice with respect to $\mathfrak{S}$ over L.

Definition 5.3: Let $\mathfrak{T} = (G, B)$ and $\mathfrak{I} = (H, C)$ be two soft sets over L with $D = B \cap C \neq \emptyset$. The $\vee$-operation and $\wedge$-operation of $\mathfrak{T} \vee \mathfrak{I}$ and $\mathfrak{T} \wedge \mathfrak{I}$ are defined as $\mathfrak{T} \vee \mathfrak{I} = (G, B) \vee (H, C) = (K, D)$ and $\mathfrak{T} \wedge \mathfrak{I} = (G, B) \vee (H, C) = (L, D)$, where $K(a) = G(a) \vee H(a)$ and $L(a) = G(a) \wedge H(a)$ for all $a \in D$.

Proposition 5.4: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L and (L, θ) be a soft approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over L with $D = B \cap C \neq \emptyset$. Then

$$\overline{(G_1, B)}_\theta \vee \overline{(G_2, C)}_\theta \subseteq \overline{(G_1 \vee G_2, D)}_\theta.$$

Proof. For all $e \in D$, let $c \in \overline{G_1(e)}_\theta \vee \overline{G_2(e)}_\theta$. Then $c = a \vee b$, where $a \in \overline{G_1(e)}_\theta$ and $b \in \overline{G_2(e)}_\theta$. So there exist $x \in \overline{G_1(e)}_\theta$, $y \in \overline{G_2(e)}_\theta$ such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$. Since $\mathfrak{S}$ is a C-soft set, $\theta(a \vee b) = \theta(x \vee y)$ for $x \vee y \in \overline{G_1(e)}_\theta \vee \overline{G_2(e)}_\theta$. Hence $c = a \vee b \in \overline{G_1(e) \vee G_2(e)}_\theta$, i.e., $\overline{(G_1, B)}_\theta \vee \overline{(G_2, C)}_\theta \subseteq \overline{(G_1 \vee G_2, D)}_\theta$. □

Proposition 5.5: Let $\mathfrak{S} = (F, A)$ be a C-soft set over L and (L, θ) be a soft approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over L with $D = B \cap C \neq \emptyset$. Then

$$\overline{(G_1, B)}_\theta \wedge \overline{(G_2, C)}_\theta \subseteq \overline{(G_1 \wedge G_2, D)}_\theta.$$

Proof. For all $e \in D$, let $c \in \overline{G_1(e)}_\theta \wedge \overline{G_2(e)}_\theta$. Then $c = x \wedge y$, where $x \in \overline{G_1(e)}_\theta$ and $y \in \overline{G_2(e)}_\theta$. So there exist $a \in \overline{G_1(e)}_\theta$, $b \in \overline{G_2(e)}_\theta$ such that $\theta(x) = \theta(a)$ and $\theta(y) = \theta(b)$. Since $\mathfrak{S}$ is a C-soft set, $\theta(x \wedge y) = \theta(a \wedge b)$ for $a \wedge b \in \overline{G_1(e)}_\theta \wedge \overline{G_2(e)}_\theta$. Hence $c = x \wedge y \in \overline{G_1(e) \wedge G_2(e)}_\theta$, i.e., $\overline{(G_1, B)}_\theta \wedge \overline{(G_2, C)}_\theta \subseteq \overline{(G_1 \wedge G_2, D)}_\theta$. □

Proposition 5.6: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L and (L, θ) be a soft approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over L with $D = B \cap C \neq \emptyset$. Then

$$\overline{(G_1, B)}_\theta \vee \overline{(G_2, C)}_\theta = \overline{(G_1 \vee G_2, D)}_\theta.$$

Proof. It follows from Proposition 5.4 that we only need to show $\overline{(G_1 \vee G_2, D)}_\theta \subseteq \overline{(G_1, B)}_\theta \vee \overline{(G_2, C)}_\theta$. For all $e \in D$, let $c \in \overline{G_1(e) \vee G_2(e)}_\theta$. So $\theta(c) = \theta(x \vee y)$ for some $x \in G_1(e)$ and $y \in G_2(e)$. Since $\mathfrak{S}$ is a CC-soft set over L, there exist $a, b \in L$, such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$ satisfying $c = a \vee b$. Thus $a \in \overline{G_1(e)}_\theta$ and $b \in \overline{G_2(e)}_\theta$. Hence $c \in \overline{G_1(e)}_\theta \vee \overline{G_1(e)}_\theta$, i.e., $\overline{(G_1, B)}_\theta \vee \overline{(G_2, C)}_\theta = \overline{(G_1 \vee G_2, D)}_\theta$. □

Proposition 5.7: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L and (L, θ) be a soft approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over L with $D = B \cap C \neq \emptyset$. Then

$$\overline{(G_1, B)}_\theta \wedge \overline{(G_2, C)}_\theta = \overline{(G_1 \wedge G_2, D)}_\theta.$$

Proof. It follows from Proposition 5.5 we only need to show $\overline{(G_1 \wedge G_2, D)}_\theta \subseteq \overline{(G_1, B)}_\theta \wedge \overline{(G_2, C)}_\theta$. For all $e \in D$, let $c \in \overline{G_1(e) \wedge G_2(e)}_\theta$. So $\theta(c) = \theta(x \wedge y)$ for some $x \in G_1(e)$ and $y \in G_2(e)$. Since $\mathfrak{S}$ is a CC-soft set over L, there exist $a, b \in L$, such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$ satisfying $c = a \wedge b$. Thus $a \in \overline{G_1(e)}_\theta$ and $b \in \overline{G_2(e)}_\theta$. Hence $c \in \overline{G_1(e)}_\theta \wedge \overline{G_1(e)}_\theta$, i.e., $\overline{(G_1, B)}_\theta \wedge \overline{(G_2, C)}_\theta = \overline{(G_1 \wedge G_2, D)}_\theta$. □

Next, we consider lower soft rough approximations over lattices.

Proposition 5.8: Let $\mathfrak{S}=(F,A)$ be a C-soft set over L and (L,θ) be a soft approximation space. Let $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ be two soft sets over L with $D=B\cap C\neq\emptyset$. Then

$$\underline{(G_1,B)}_\theta \vee \underline{(G_2,C)}_\theta \subseteq \underline{(G_1\vee G_2,D)}_\theta .$$

Proof. Assume that $\underline{(G_1,B)}_\theta \vee \underline{(G_2,C)}_\theta = \underline{(G_1\vee G_2,D)}_\theta$ does not hold. For all $e\in D$, there exists $c\in\underline{G_1(e)}_\theta\vee\underline{G_2(e)}_\theta$ but $c\notin\underline{G_1(e)\vee G_2(e)}_\theta$. Then $c=a\vee b$, where $a\in\underline{G_1(e)}_\theta$ and $b\in\underline{G_2(e)}_\theta$. So $\theta(a)\neq\theta(x)$ and $\theta(b)\neq\theta(y)$ for all $x\in G_1(e)^c$ and $y\in G_2(e)^c$. $(\star)$

Notice that $c\notin\underline{G_1(e)\vee G_2(e)}_\theta$, then we have the following two cases:

Case (i) $c\notin G_1(e)\vee G_2(e)$, which contradicts with $c\in\underline{G_1(e)}_\theta\vee\underline{G_2(e)}_\theta\subseteq G_1(e)\vee G_2(e)$;

Case (ii) $c\in G_1(e)\vee G_2(e)$ and $\theta(c)=\theta(x'\vee y')$ for some $x'\vee y'\in(G_1(e)\vee G_2(e))^c$. Thus $x'_i\in G_1(e)^c$ or $y'_i\in G_2(e)^c$. In fact, if $x'_i\notin G_1(e)^c$ and $y'_i\notin G_2(e)^c$, we have $x'\vee y'\in G_1(e)\vee G_2(e)$, which contradicts with $x'\vee y'\in(G_1(e)\vee G_2(e))^c$. Since $\mathfrak{S}=(F,A)$ is a CC-soft set over L, there exist $a',b'\in L$ such that $\theta(a')=\theta(x')$ and $\theta(b')=\theta(y')$ satisfying $c=a'\vee b'$, for some $x'\in G_1(e)^c$ or $y'\in G_2(e)^c$, which contradicts with $(\star)$. Hence $\underline{(G_1,B)}_\theta\vee\underline{(G_2,C)}_\theta\subseteq\underline{(G_1\vee G_2,D)}_\theta$. $\square$

Proposition 5.9: Let $\mathfrak{S}=(F,A)$ be a C-soft set over L and (L,θ) be a soft approximation space. Let $\mathfrak{T}_1=(G_1,B)$ and $\mathfrak{T}_2=(G_2,C)$ be two soft sets over L with $D=B\cap C\neq\emptyset$. Then

$$\underline{(G_1,B)}_\theta \wedge \underline{(G_2,C)}_\theta \subseteq \underline{(G_1\wedge G_2,D)}_\theta .$$

Proof. Assume that $\underline{(G_1,B)}_\theta \wedge \underline{(G_2,C)}_\theta = \underline{(G_1\wedge G_2,D)}_\theta$ does not hold. For all $e\in D$, there exists $c\in\underline{G_1(e)}_\theta\wedge\underline{G_2(e)}_\theta$ but $c\notin\underline{G_1(e)\wedge G_2(e)}_\theta$. Then $c=a\wedge b$, where $a\in\underline{G_1(e)}_\theta$ and $b\in\underline{G_2(e)}_\theta$. So $\theta(a)\neq\theta(x)$ and $\theta(b)\neq\theta(y)$ for all $x\in G_1(e)^c$ and $y\in G_2(e)^c$. $(\triangle)$

Notice that $c\notin\underline{G_1(e)\wedge G_2(e)}_\theta$, then we have the following two cases:

Case (i) $c\notin G_1(e)\wedge G_2(e)$, which contradicts with $c\in\underline{G_1(e)}_\theta\wedge\underline{G_2(e)}_\theta\subseteq G_1(e)\wedge G_2(e)$;

Caes (ii) $c\in G_1(e)\wedge G_2(e)$ and $\theta(c)=\theta(x'\wedge y')$ for some $x'\wedge y'\in(G_1(e)\wedge G_1(e))^c$. Thus $x'\in G_1(e)^c$ or $y'\in G_2(e)^c$. In fact, if $x'\notin G_1(e)^c$ and $y'\notin G_2(e)^c$, we have $x'\wedge y'\in G_1(e)\wedge G_2(e)$, which contradicts with $x'\wedge y'\in(G_1(e)\wedge G_1(e))^c$. Since $\mathfrak{S}=(F,A)$ is a CC-soft set over L, there exist $a',b'\in L$ such that $\theta(a')=\theta(x')$ and $\theta(b')=\theta(y')$ satisfying $a'\wedge b'=c$, for some $x'\in G_1(e)^c$ or $y'\in G_2(e)^c$, which contradicts with $(\triangle)$. Hence $\underline{(G_1,B)}_\theta\wedge\underline{(G_2,C)}_\theta\subseteq\underline{(G_1\wedge G_2,D)}_\theta$. $\square$

Finally, we investigate the lower and upper soft rough lattices (ideals, filters) with respect to another soft set.

Theorem 5.10: Let $\mathfrak{S}=(F,A)$ be a C-soft set over L. If $\mathfrak{T}=(G,B)$ is a soft lattice over L, then $\mathfrak{T}$ is an upper soft rough lattice over L.

Proof. For all $e\in B$, let $a,b\in\overline{G(e)}_\theta$. Then there exist $y,z\in G(e)$ such that $\theta(a)=\theta(y)$ and $\theta(b)=\theta(z)$. Since $\mathfrak{S}$ is a C-soft set over L, $\theta(a\vee b)=\theta(y\vee z)$ and $\theta(a\wedge b)=\theta(y\wedge z)$. Since $\mathfrak{T}=(G,B)$ is a soft lattice over L, we have $G(e)$ is a sublattice of L for all $e\in B$, $y\vee z\in G(e)$ and $y\wedge z\in G(e)$. So $a\vee b\in\overline{G(e)}_\theta$ and $a\wedge b\in\overline{G(e)}_\theta$. Thus $\overline{G(e)}_\theta$ is a sublattice of L for all $e\in B$. That is $\mathfrak{T}$ is an upper soft rough lattice over L. $\square$

Theorem 5.11: Let $\mathfrak{S}=(F,A)$ be a CC-soft set over L. If $\mathfrak{T}=(G,B)$ is a soft lattice over L, then $\mathfrak{T}$ is a lower soft rough lattice over L when $\underline{\mathfrak{T}}_\theta\neq\emptyset$.

Proof. It follows from Propositions 5.8 and 5.9 that $\underline{(G,B)}_\theta\vee\underline{(G,B)}_\theta\subseteq\underline{(G,B)\vee(G,B)}_\theta$ and $\underline{(G,B)}_\theta\wedge\underline{(G,B)}_\theta\subseteq\underline{(G,B)\wedge(G,B)}_\theta$. Further, since $(G,B)\vee(G,B)\subseteq(G,B)$ and $(G,B)\wedge(G,B)\subseteq(G,B)$, it is easy to know that $\underline{(G,B)\vee(G,B)}_\theta\subseteq\underline{(G,B)}_\theta$ and $\underline{(G,B)\wedge(G,B)}_\theta\subseteq\underline{(G,B)}_\theta$. So $\underline{(G,B)}_\theta\vee\underline{(G,B)}_\theta\subseteq\underline{(G,B)}_\theta$ and $\underline{(G,B)}_\theta\wedge\underline{(G,B)}_\theta\subseteq\underline{(G,B)}_\theta$. Thus $\underline{(G,B)}_\theta$ is a soft lattice over L, that is $\underline{G(e)}_\theta$ is a lattice of L for all $e\in B$. Thus $\mathfrak{T}=(G,B)$ is a lower soft rough lattice over L. $\square$

Theorem 5.12: Let $\mathfrak{S}=(F,A)$ be a C-soft set over L. If $\mathfrak{T}=(G,B)$ is a soft ideal over L, then $\mathfrak{T}$ is an upper soft rough ideal over L.

Proof. Let $\mathfrak{T}=(G,B)$ be a soft ideal over L. Then, for all $e\in B$, $G(e)$ is an ideal of L. Further, for all $e\in B$, let $a\in L, b\in\overline{G(e)}_\theta$ and $a\leq b$. Then there exists $c\in G(e)$ such that $\theta(b)=\theta(c)$. Since $\theta(a)=\theta(a)$ and $\mathfrak{S}=(F,A)$ is a C-soft set over L, $\theta(a)=\theta(a\wedge b)=\theta(c\wedge a)$. Since $G(e)$ is an ideal of L, $c\wedge a\leq c$, we have $c\wedge a\in G(e)$. Thus $a\in\overline{G(e)}_\theta$ for all $e\in B$, that is $\overline{G(e)}_\theta$ is an ideal of L. Therefore, $\mathfrak{T}$ is an upper soft rough ideal over L. $\square$

Theorem 5.13: Let $\mathfrak{S}=(F,A)$ be a CC-soft set over L. If $\mathfrak{T}=(G,B)$ is a soft ideal over L, then $\mathfrak{T}$ is a lower soft rough ideal over L when $\underline{\mathfrak{T}}_\theta\neq\emptyset$.

Proof. Let $\mathfrak{T}=(G,B)$ be a soft ideal over L, Then for all $e\in B$, $G(e)$ is an ideal of L. For for all $e\in B$, it follows from Theorem 5.11 that $m\vee n\in\underline{G(e)}_\theta$ for $m,n\in\underline{G(e)}_\theta$. Now let $\underline{G(e)}_\theta\neq\emptyset$, for all $a\in L, b\in\underline{G(e)}_\theta$ and $a\leq b$. Assume that $a\notin\underline{G(e)}_\theta$. Then we have $\theta(b)\neq\theta(d)$ for all $d\in G(e)^c$.

Notice that $a\notin\underline{G(e)}_\theta$, then we have the following two cases:

Case (i) If $a\notin G(e)$, since $b\in\underline{G(e)}_\theta\subseteq G(e)$, $a\leq b$ and $G(e)$ is an ideal of L, we have $a\in G(e)$, which contradicts with $a\notin G(e)$;

Case (ii) $a\in G(e)$ and $\theta(a)=\theta(c)$ for some $c\in G(e)^c$. That is $\theta(a\wedge b)=\theta(c)$ for some $c\in G(e)^c$. Since $\mathfrak{S}=(F,A)$ is a CC-soft set, there exist $x,y\in L$ such that $\theta(a)=\theta(x)$ and $\theta(b)=\theta(y)$ satisfying $x\wedge y=c\in G(e)^c$. Thus, $x\in G(e)^c$ or $G(e)\in Y^c$. In fact, if $x\notin G(e)^c$ and $y\notin G(e)^c$, Since $G(e)$ is an ideal of L, $G(e)$ is a sublattice of L, we have $x\wedge y=c\in G(e)\wedge G(e)\subseteq G(e)$, which contradicts with $x\wedge y\in G(e)^c$. That is there exist $x\in G(e)^c$ such that $\theta(a)=\theta(x)$ or $y\in G(e)^c$ such that $\theta(b)=\theta(y)$. That is $b\notin\underline{G(e)}_\theta$, which contradicts with $b\in\underline{G(e)}_\theta$. This implies that $\underline{G(e)}_\theta$ is an ideal over L for all $e\in B$, that is $\mathfrak{T}$ is a lower soft rough ideal over L. $\square$

Theorem 5.14: Let $\mathfrak{S}=(F,A)$ be a C-soft set over L. If $\mathfrak{T}=(G,B)$ is a soft filter over L, then $\mathfrak{T}$ is an upper soft rough filter over L.

Proof. Let $\mathfrak{T}=(G,B)$ be a soft filter over L. Then for all $e\in B$, $G(e)$ is a filter of L. For all $e\in B$, $a\in L, b\in\overline{G(e)}_\theta$ and $a\geq b$. Then there exists $c\in G(e)$ such that $\theta(b)=\theta(c)$. Since $\theta(a)=\theta(a)$ and $\mathfrak{S}=(F,A)$ is a C-soft set over L, $\theta(a)=\theta(a\vee b)=\theta(c\vee a)$. Since $G(e)$ is a filter of L,

$c \vee a \geq c$, we have $c \vee a \in G(e)$. Thus $a \in \overline{G(e)}_\theta$. That is $\overline{G(e)}_\theta$ is a filter of L for all $e \in B$. Therefore, $\mathfrak{T} = (G, B)$ is an upper soft rough filter over L. □

Theorem 5.15: Let $\mathfrak{S} = (F, A)$ be a CC-soft set over L. If $\mathfrak{T} = (G, B)$ is a soft filter over L, then $\mathfrak{T}$ is a lower soft rough filter over L when $\underline{\mathfrak{T}}_\theta \neq \emptyset$.

Proof. Let $\mathfrak{T} = (G, B)$ be a soft filter over L, Then for all $e \in B$, $G(e)$ is a filter of L. For all $e \in B$, it follows from Theorem 5.11 that $m \wedge n \in \underline{G(e)}_\theta$ for $m, n \in \underline{G(e)}_\theta$. Now let $\underline{G(e)}_\theta \neq \emptyset$, for all $a \in L, b \in \underline{G(e)}_\theta$ and $a \geq b$. We assume that $a \notin \underline{G(e)}_\theta$. Then we have $\theta(b) \neq \theta(d)$ for all $d \in G(e)^c$.

Notice that $a \notin \underline{G(e)}_\theta$, then we have the following two cases:

Case (i) If $a \notin G(e)$, since $b \in \underline{G(e)}_\theta \subseteq G(e)$, $a \geq b$ and $G(e)$ is a filter of L, we have $a \in G(e)$, which contradicts with $a \notin G(e)$;

Case (ii) $a \in G(e)$ and $\theta(a) = \theta(c)$ for some $c \in G(e)^c$. That is $\theta(a \vee b) = \theta(c)$ for some $c \in G(e)^c$. Since $\mathfrak{S} = (F, A)$ is a CC-soft set, there exist $x, y \in L$ such that $\theta(a) = \theta(x)$ and $\theta(b) = \theta(y)$ satisfying $x \vee y = c \in G(e)^c$. Thus, $x \in G(e)^c$ or $y \in G(e)^c$. In fact, if $x \notin G(e)^c$ and $y \notin G(e)^c$, Since $G(e)$ is a filter of L, $G(e)$ is a sublattice of L, we have $x \vee y = c \in G(e) \vee G(e) \subseteq G(e)$, which contradicts with $x \vee y \in G(e)^c$. That is there exist $x \in G(e)^c$ such that $\theta(a) = \theta(x)$ or $y \in G(e)^c$ such that $\theta(b) = \theta(y)$. That is $b \notin \underline{G(e)}_\theta$, which contradicts with $b \in \underline{G(e)}_\theta$. This implies $\underline{G(e)}_\theta$ is a filter of L for all $e \in B$, that is $\mathfrak{T}$ is a lower soft rough filter over L. □

VI. Conclusions

In this paper, we apply soft rough sets to lattices and explore some characterizations of soft rough lattices (ideals, filters) over lattices. The main conclusions are listed as follows.

(1) We give the concept of soft rough sets of lattices and investigate some operations of soft rough sets over lattices.

(2) We study the roughness in lattices with respect to soft approximation spaces. In addition, we discuss lower and upper soft rough lattices (ideals, filters) over lattices.

(3) We introduce the concepts of lower and upper soft rough lattices (ideals, filters) with respect to another soft set of lattices and investigate some properties of them.

As future works, we will consider the following topics:

(1) Applying soft rough sets to hyperalgebras, such as hyperrings, hyperhemiring, hyper EQ-algebras and so on;

(2) Studying soft fuzzy rough lattices (ideals, filters) over lattices;

(3) Investigating decision making methods based on soft (fuzzy) rough sets.

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