Permanence on a Class of Discrete Delayed Population Model with More Than one Equilibrium

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Abstract—**In this paper, we discuss the permanence of a class of discrete single-species population model with time delays. In most of the previous literature, the population models have only one positive equilibrium.While the system we considered has more than one positive equilibrium point and its dynamic behaviors will be more complex. By using the theory of oscillation of difference equation, we have established a new and rather weaker condition to ensure the permanence of the system.**

Index Terms—**Permanence, Oscillation, Semicycle,Discrete population model, Equilibrium point.**

I. Introduction

DISCRETE population models have been extensively studied in the past two decades and successively applied to different areas, especially bio-mathematics.These applications rely heavily on the dynamic behavior of the models. The equation sometimes seems to be simple, but its dynamic behavior may be complex. The variation of some parameters will lead from stability to chaos[1-4]. In previous research, the population models proposed usually have exactly one positive equilibrium except the zero equilibrium.In [6], the author studied a nonlinear discrete population, and obtained sufficient conditions to ensure the global attractiveness of the model. In [8], a single-species discrete population model with stage structure and birth pulses was proposed and the effect of seasonal harvesting was investigated. In [10], the author considered a single-species discrete population model with periodic harvest stock and proved the existence of periodic solution.We can find that the models have only one positive equilibrium also in [5,7,12,13]. Given the scarcity of literature on this topic, we need to consider the population models with a finite number of equilibriums.

The rest of this paper is organized as follows. Section A provides the basic preliminary and hypothesis. The main results are presented in Section B, where we establish a sufficient condition to ensure the permanence of a class discrete single-species population model with delays and multi positive equilibriums. In section C, we give a brief conclusion and some open problems.

Generally speaking, one basic hypothesis of previous studies is that the system has only one equilibrium point. The present paper aims to give a new sufficient condition to ensure the permanence of a class of different equations with more than one equilibrium point. To complete our conclusion, some new concepts such as oscillation about $[a, b]$, and middle cycle about $[a,\ b]$ have been proposed. Making use of these concept, we have obtained some good results.

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A. Preliminaries

In this paper, we will establish the permanence of solutions for the following class discrete population model:

$$t_{n+1} = t_n f(t_n, T_n^r),\ n = 0, 1, 2, \cdots. \tag{3.1}$$

Where, $T_n^r = (t_{n-k_1}, \cdots, t_{n-k_r})$.

Definition 1. *Equation (3.1) is said to be permanent, if there exist constants E and F with $0 < E \leq F < \infty$ such that for any initial values $t_{-k}, \cdots, t_0 \in (0, \infty)$, there exists a positive integer N which only depends on the initial values such that*

$$E \leq t_n \leq F \quad \text{for all} \quad n \geq N.$$

Definition 2. (a). *Sequence $\{t_n\}$ is said to oscillate about $\bar{t}$, if the sequence $\{t_n - \bar{t}\}$ is neither eventually all positive nor eventually all negative.*

(b). *Sequence $\{t_n\}$ is said to strictly oscillate about $\bar{t}$, if for every $n_0 \geq 0$, there exist integers $l, s \geq n_0$ such that $(t_l - \bar{t})(t_s - \bar{t}) < 0$.*

(c). *Sequence $\{t_n\}$ is said to oscillate about $[a, b]$, if the sequence $\{t_n - a\}$ are not eventually all negative or $\{t_n - b\}$ eventually all positive.*

(d). *Sequence $\{t_n\}$ is said to strictly oscillate about $[a, b]$, if for every $n_0 \geq 0$, there exist integers $l, s \geq n_0$ such that $t_l - a < 0$ and $t_s - b > 0$.*

Definition 3. (a) *A positive semicycle about $\bar{t}$ of a solution $\{t_n\}$ of equation (3.1) consists of a "string" of terms $\{t_l, t_{l+1}, \cdots, t_z\}$, all greater than or equal to $\bar{t}$, with $l \geq -k$ and $z \leq \infty$ and such that*

$$\text{either } l = -k \text{ or } l > -k \text{ and } t_{l-1} < \bar{t}$$

and

$$\text{either } z = \infty \text{ or } z < \infty \text{ and } t_{z+1} < \bar{t}.$$

(b) *A negative semicycle about $\bar{t}$ of a solution $\{t_n\}$ of equation (3.1) consists of a "string" of terms $\{t_l, t_{l+1}, \cdots, t_z\}$, all less than or equal to $\bar{t}$, with $l \geq -k$ and $z \leq \infty$ and such that*

$$\text{either } l = -k \text{ or } l > -k \text{ and } t_{l-1} \geq \bar{t}$$

and

$$\textit{either } z = \infty \textit{ or } z < \infty \textit{ and } t_{z+1} \geq \bar{t}.$$

Definition 4. A "string" of terms $\{t_l, t_{l+1}, \cdots, t_m\}$ in $\{t_n\}$ is said to be a positive semicycle about $[a, b]$, if it is a positive semicycle about point b, and is said to be a negative semicycle about $[a, b]$, if it is a negative semicycle about point a, and is said to be a middle cycle about $[a, b]$, if all terms are in $[a, b)$.

In this paper, we will assume that $k_1, k_2, \cdots, k_r$ are positive integers and we will denote by k the maximum of $k_1, k_2, \cdots, k_r$. We will also assume that the function $f(q_0, q_1, \cdots, q_r)$ satisfies the following hypotheses:

(H_1) $f \in C[(0,\infty) \times [0,\infty)^r, (0,\infty)]$, $g \in C[[0,\infty)^{r+1}, (0,\infty)]$, and

$$g(q_0, Q^r) = q_0 f(q_0, Q^r),$$
$$g(0, Q^r) = \lim_{q_0 \to 0+} g(q_0, Q^r).$$

here, $Q^r = (q_1, \cdots, q_r) \in [0,\infty)^r$.

(H_2) The equation $f(t, t, \cdots, t) = 1$ has N positive solutions with $0 < \bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_N < \infty$.

(H_3) $f(q_0, Q^r)$ is non-increasing in q_i on $[0, \bar{t}_1]$ and non-increasing in q_i on $[\bar{t}_n, \infty]$ for all $i = 1, 2, \cdots, r$.

Clearly, by the continuity of the function f, it follows that there exist positive number $\alpha \in (0,\ 1]$ and $\beta \geq 1$ such that

$$\alpha \leq f(q_0, Q^r) \leq \beta$$

for all $q_0 \in [\bar{t}_1,\ \bar{t}_N]$ and $Q^r \in \{[\bar{t}_1,\ \bar{t}_N] \cup \{0\}\}^r$.

To be convenient, we first set some constants before proposing the forth hypothesis.

$$\lambda \triangleq \sup_{t \in [0, \bar{t}_1]} \{f(t, Q^r)\}$$
$$\delta \triangleq \max\{f(\bar{t}_N, Q^r)\}$$
$$A \triangleq \lambda \delta^k$$
$$B \triangleq \bar{t}_N \beta^{k+1}$$
$$E \triangleq \max\{A, B\}$$

where, $Q^r \in \{\{0\} \cup [\bar{t}_1,\ \bar{t}_N]\}^r$. Clearly, $\delta \geq 1$.

(H_4) The function $f(q_0, Q^r)$ does not depend on q_0 or

$$f(t, q^r) \geq f(\bar{t}_1, Q^r)$$

for all $t \leq \bar{t}_1$ and $q_\imath \in [\bar{t}_1,\ \bar{t}_N] \cup \{0, E\}$, and

$$f(t, Q^r) \leq f(\bar{t}_N, Q^r)$$

for all $t \geq \bar{t}_N$ and $q_\imath \in [\bar{t}_1, \bar{t}_N] \cup \{0, E\}$.

If $a_{-k}, a_{-k+1}, \cdots, a_{-1} \in [0,\ \infty)$ and $a_0 \in (0,\infty)$ are given, then equation (3.1) has a unique solution $\{t_n\}$ satisfying the initial conditions $t_\jmath = a_\jmath$ for all $\jmath = -k, \cdots, 0$. Clearly, $t_n > 0$ for all $n \geq 0$.

In the sequel, we only consider solution of equation (3.1) with initial values $a_{-k}, a_{-k+1}, \cdots, a_{-1} \in [0,\ \infty)$ and $a_0 \in (0,\ \infty)$.

B. Main Results

Lemma 1. *Let $\{t_n\}$ be a solution of equation (3.1) such that for some $n_0 \geq 0$ either*

$$t_n \geq \bar{t}_N \qquad \textit{for } n \geq n_0 \tag{3.2}$$

or

$$t_n \leq \bar{t}_1 \qquad \textit{for } n \geq n_0. \tag{3.3}$$

Then for all $n \geq n_0 + k$, the sequence $\{t_n\}$ is monotonic, (3.2) implies $\lim_{n\to\infty} t_n = \bar{t}_N$ and (3.3) implies $\lim_{n\to\infty} t_n = \bar{t}_1$.

Proof: Assume that (3.2) holds, then first by using (H_3) and then (H_4) we see that for $n \geq n_0 + k$

$$\begin{aligned} t_{n+1} &= t_n f(x_n, T_n^r) \\ &\leq t_n f(t_n, \bar{t}_N, \cdots, \bar{t}_N) \\ &\leq t_n f(\bar{t}_N, \bar{t}_N, \cdots, \bar{t}_N) \\ &= t_n. \end{aligned}$$

Hence $\{t_n\}$ is nonincreasing for $n \geq n_0 + k$. Let$\ell = \lim\limits_{n\to\infty} t_n \geq \bar{t}_N$, for the sake of contradiction, assume $\ell > \bar{t}_N$, then take the limit on both sides of the equation(3.1), we obtain $f(\ell, \ell, \cdots, \ell) = 1$, which contradicts the hypothesis that $\bar{t}_N$ is the maximum positive solution of equation (3.1).

Assume that (3.3) holds, then first by using (H_3) and then (H_4) we see that for $n \geq n_0 + k$

$$\begin{aligned} t_{n+1} &= t_n f(t_n, T_n^r) \\ &\geq t_n f(t_n, \bar{t}_1, \cdots, \bar{t}_1) \\ &\geq t_n f(\bar{t}_1, \bar{t}_1, \cdots, \bar{t}_1) \\ &= t_n. \end{aligned}$$

Hence $\{t_n\}$ is nondecreasing for $n \geq n_0 + k$. Let $\ell = \lim\limits_{n\to\infty} \leq \bar{t}_1$, for the sake of contradiction, assume $\ell < \bar{t}_1$, then take the limit on both sides of the equation(3.1), we obtain $f(\ell, \ell, \cdots, \ell) = 1$, which contradicts the hypothesis that $\bar{t}_1$ is the minimum positive solution of equation (3.1). The proof is completed.

Lemma 2. *Let $\{t_n\}$ be a solution of equation (3.1). Then the extreme point in any semicycle about $[\bar{t}_1,\ \bar{t}_N]$ occurs in one of the first $k+1$ terms of the semicycles about $[\bar{t}_1,\ \bar{t}_N]$.*

Proof: We assume that $\{t_l, t_{l+1}, \cdots, t_{l+m}\}$ be a positive semicycle about$[\bar{t}_1,\ \bar{t}_N]$. The case of negative semicycles about $[\bar{t}_1,\ \bar{t}_N]$ is similar and hence be omitted.

If $m \leq k$, then the conclusion is obvious.

Now, let $m > k$ and assume that extreme point does not occur in one of the first $k+1$ terms. Let t_{l+k+s} be the extreme point $s \geq 1$, then first by using (H_3) and then (H_4), we see that

$$\begin{aligned} t_{l+k+s} &= t_{l+k+s-1} f(t_{l+k+s-1}, T_{l+k+s}^r) \\ &\leq t_{l+k+s-1} f(t_{l+k+s-1}, \bar{t}_N, \cdots, \bar{t}_N) \\ &\leq t_{l+k+s-1} f(\bar{t}_N, \bar{t}_N, \cdots, \bar{t}_N) \\ &= t_{l+k+s-1}. \end{aligned}$$

Then $t_{l+k+s} = t_{l+k+s-1}$.If $s = 1$, that is, t_{l+k} is the extreme point which is contradiction. Otherwise continue the step above. After finite s_0 steps, we can have $s - s_0 = 0$, that is, $t_{l+k+s-s_0} = t_{l+k}$ is the maximum, which is contradicts the hypothesis.The proof is completed.

Theorem 1. *Let $\{t_n\}$ be a solution of equation (3.1) which is not strictly oscillatory about $[\bar{t}_1,\ \bar{t}_N]$, then $\{t_n\}$ is bounded away from zero and infinity by positive constants.*

Proof: Because $\{t_n\}$ is not strictly oscillatory about $[\bar{t}_1,\ \bar{t}_N]$, then one of the following five cases occurs.

Case 1. For some $n_0 \geq 0, t_n < \bar{t}_1$ for all $n \geq n_0$.

Case 2. For some $n_0 \geq 0, t_n \geq \bar{t}_N$ for all $n \geq n_0$.

Case 3. For some $n_0 \geq 0, \bar{t}_1 \leq t_n < \bar{t}_N$ for all $n \geq n_0$.

Case 4. For some $n_0 \geq 0$, positive semicycle and middle cycle alternates for $n \geq n_0$.

Case 5. For some $n_0 \geq 0$, negative semicycle and middle cycle alternates for $n \geq n_0$.

For cases 1 and 2, making use of lemma 1, we can easily obtain the conclusion of theorem 1. For case 3, the conclusion of theorem 1 is obvious. Now, it remains to prove cases 4 and 5. For case 4, we only need consider the terms in positive semicycles. Let t_M be extreme point with the smallest possible indices in a positive semicycle $\{t_p, t_{p+1}, \cdots, t_q\}$, where $p > n_0 + k + 1$, then by lemma 2, we have $M - p \leq k$.

To be convenient, we define function

$$\psi(t) = \begin{cases} 0, & 0 < t < \bar{t}_1, \\ t, & \bar{t}_1 \leq t < \bar{t}_N, \\ \bar{t}_N, & t \geq \bar{t}_N. \end{cases}$$

and then we set

$$\psi(T_n^r) = (\psi(t_{n-k_1}, \cdots, \psi(t_{n-k_r})).$$

By using (H_3) and then (H_4), we can obtain

$$\begin{aligned}
t_M &= t_{p-1} \prod_{\jmath=p-1}^{M-1} f(t_\jmath, T_\jmath^r) \\
&\leq t_{p-1} \prod_{\jmath=p-1}^{M-1} f(t_\jmath, \psi(T_\jmath^r)) \\
&= t_{p-1} f(t_{p-1}, \psi(T_{p-1}^r)) \prod_{\jmath=p}^{M-1} f(t_\jmath, \psi(T_\jmath^r)) \\
&\leq t_{p-1} f(t_{p-1}, \psi(T_{p-1}^r)) \prod_{\jmath=p}^{M-1} f(\bar{t}_N, \psi(T_\jmath^r)) \\
&\leq t_{p-1} \beta \prod_{\jmath=p}^{M-1} \beta \\
&= t_{p-1} \beta^{M-p+1} \\
&\leq \bar{t}_N \beta^{M-p+1} \\
&\leq \bar{t}_N \beta^{k+1} \\
&= B.
\end{aligned}$$

For case 5, we only need consider the terms in negative semicycles. To be convenient, we define function

$$\phi(t) = \begin{cases} \bar{t}_1, & 0 < t < \bar{t}_1, \\ t, & \bar{t}_1 \leq t < \bar{t}_N. \end{cases}$$

Let t_m be extreme point with the smallest possible indices in a negative semicycle $\{t_q, t_{q+1}, \cdots, t_s\}$, where $q > n_0 + k + 1$, then by lemma 2, we have $M - q \leq k$. By using (H_3) and then (H_4), we can obtain

$$\begin{aligned}
t_m &= t_{q-1} \prod_{\jmath=q-1}^{m-1} f(t_\jmath, T_\jmath^r) \\
&\geq t_{q-1} \prod_{\jmath=q-1}^{m-1} f(t_\jmath, \phi(T_\jmath^r)) \\
&= t_{q-1} f(t_{q-1}, \phi(T_{q-1}^r)) \prod_{\jmath=q}^{m-1} f(t_\jmath, \phi(T_\jmath^r)) \\
&\geq t_{q-1} f(t_{q-1}, \phi(T_{q-1}^r)) \prod_{\jmath=q}^{m-1} f(\bar{t}_1, \phi(T_\jmath^r)) \\
&\geq t_{q-1} \alpha \prod_{\jmath=q}^{m-1} \alpha \\
&= t_{q-1} \alpha^{m-q+1} \\
&\geq \bar{t}_1 \alpha^{m-q+1} \\
&\geq \bar{t}_1 \alpha^{k+1} \\
&= C.
\end{aligned}$$

The proof is completed.

Theorem 2. *Let $\{t_n\}$ be a solution of equation (3.1) which is strictly oscillatory about $[\bar{t}_1,\ \bar{t}_N]$, then $\{t_n\}$ is bounded away from zero and infinity by positive constants.*

Proof: Because $\{t_n\}$ is strictly oscillatory about $[\bar{t}_1,\ \bar{t}_N]$, then one of the following two cases occurs.

Case 6. For some $n_0 \geq 0$, positive semicycle and negative semicycle alternates for $n \geq n_0$.

Case 7. For every $n_0 \geq 0$, besides positive semicycles and negatives semicycle, middle cycles also occur for $n \geq n_0$.

For case 6, let $\{t_p, t_{p+1}, \cdots, t_q\}$, where $p > n_0 + k + 1$, be a positive semicycle about $[\bar{t}_1,\ \bar{t}_N]$ followed by a negative semicycle $\{t_{q+1}, t_{q+2}, \cdots, t_s\}$ about $[\bar{t}_1,\ \bar{t}_N]$. If t_M and t_m are the extreme values in these positive and negative semicycles, respectively, with the smallest possible indices M and m. Then by lemma 2, we have $M - p \leq k$ and $m - q - 1 \leq k$. Then, by using (H_3) and then (H_4) we can obtain that

$$\begin{aligned}
t_M &= t_{p-1} \prod_{\jmath=p-1}^{M-1} f(t_\jmath, T_\jmath^r) \\
&\leq t_{p-1} \prod_{\jmath=p-1}^{M-1} f(t_\jmath, \psi(T_\jmath^r)) \\
&= t_{p-1} f(t_{p-1}, \psi(T_{p-1}^r)) \prod_{\jmath=p}^{M-1} f(t_\jmath, \psi(T_\jmath^r)) \\
&= t_{p-1} f(t_{p-1}, \psi(T_{p-1}^r)) \prod_{\jmath=p}^{M-1} f(\bar{t}_N, \psi(T_\jmath^r)) \\
&\leq \lambda \prod_{\jmath=p}^{M-1} \delta = \lambda \delta^{M-p} \leq \lambda \delta^k = A
\end{aligned}$$

From the proof above, we know that the terms in positive hemicycles are always bounded by E, so we define function

$$\Phi(t)=\begin{cases}\bar{t}_1, & 0<t<\bar{t}_1,\\ t, & \bar{t}_1\le t<\bar{t}_N,\\ E, & t\ge \bar{t}_N.\end{cases}$$

By using (H_3) and then (H_4), we see that

$$\begin{aligned}t_m &= t_q\prod_{j=q}^{m-1} f(t_j,T_j^r)\\ &\ge t_q\prod_{j=q}^{m-1} f(t_j,\Phi(T_j^r))\\ &= t_q f(t_q,\Phi(T_q^r))\prod_{j=q+1}^{m-1} f(t_j,\Phi(T_j^r))\\ &= t_q f(t_q,\Phi(T_q^r))\prod_{j=q+1}^{m-1} f(\bar{t}_N,\Phi(T_j^r))\\ &\ge t_q\omega\prod_{j=q+1}^{m-1}\gamma = t_q\omega^{m-q-1}\\ &\ge \bar{t}_N\omega\gamma^{m-q-1}\\ &\ge \bar{t}_N\omega\gamma^{k}\\ &= D,\end{aligned}$$

where

$$\begin{aligned}\omega &\triangleq \inf_{(t,Q^r)\in[\bar{t}_N,E]\times[\{E\}\cup[\bar{t}_1,\ \bar{t}_N]]^r} f(t,Q^r),\\ \gamma &\triangleq \min_{Q^r\in[\{E\}\cup[\bar{t}_1,\ \bar{t}_N]]^r} f(\bar{t}_1,Q^r).\end{aligned}$$

For case 7, we only need consider terms in positive and negative semicycles. Let us first consider positive semicycles let $\{t_p,t_{p+1},\cdots,t_q\}$ be a positive semicycle, then the following two cases occur.

Case a. A negative semicycle is followed by the positive semicycle.

Case b. A middle cycle is followed by the positive semicycle.

From the proof of cases 4 and 6, we know the terms in positive semicycles are bounded by constants E and $\bar{t}_N$. Then, let us consider terms in negative semicycles. Similarly, we know, for negative semicycle, the following two cases occur.

Case c. A positive semicycle is followed by the negative semicycle.

Case d. A middle cycle is followed by the negative semicycle.

From the proof of the case 6, we know that for case c the terms in negative are bounded by D and $\bar{t}_1$. It remains to prove the case d. To end it, let t_m be extreme point with the smallest possible indices in a negative semicycle $\{t_q,t_{q+1},\cdots,t_s\}$, where $q>n_0+k+1$, then by lemma 2, we have $M-q\le k$. Clearly, by the continuity of the function f, it follows that there exist positive number $\bar{\alpha}$ and $\bar{\beta}$ such that $0<\bar{\alpha}\le f(q_0,q_1,\cdots,q_r)\le\bar{\beta}$, $0<\bar{\alpha}\le 1$ and $1\le\bar{\beta}$ for all $q_0\in[\bar{t}_1,\ \bar{t}_N]$ and $q_1,\cdots,q_r\in[\bar{t}_1,\ \bar{t}_N]\cup\{0,E\}$. Then, we have

$$\begin{aligned}t_m &= t_{q-1}\prod_{j=q-1}^{m-1} f(t_j,T_j^r)\\ &\ge t_{q-1}\prod_{j=q-1}^{m-1} f(t_j,\phi(T_j^r))\\ &= t_{q-1}f(t_{q-1},\phi(T_{q-1}^r))\prod_{j=q}^{m-1} f(t_j,\phi(T_j^r))\\ &\ge t_{q-1}f(t_{q-1},\phi(T_{q-1}^r))\prod_{j=q}^{m-1} f(\bar{t}_N,\phi(T_j^r))\\ &\ge t_{q-1}\bar{\alpha}\prod_{j=q}^{m-1}\bar{\alpha} = t_{q-1}\bar{\alpha}^{m-q+1}\\ &\ge \bar{t}_N\bar{\alpha}^{m-q+1}\\ &\ge \bar{t}_N\bar{\alpha}^{k+1}\triangleq\bar{C}\end{aligned}$$

The proof is completed.

Theorem 3. If assumptions $(H_1)-(H_4)$ hold, then equation (3.1) is permanent.
Proof: From theorems 1 and 2, the conclusion of theorem 3 is obviously.

C. Conclusion

In general, there may be multiple equilibrium points of a discrete population. If a linear difference equation has more than one positive equilibrium points, its dynamic behavior may become complex. In this paper, we have first proposed the concept of oscillation about $[a,\ b]$, and then obtained a sufficient condition to ensure the permanence of a class of discrete population model (3.1) with more than one equilibrium point. In fact, the global attractivity of the model is also important problem. While previous research efforts have largely focused on the dynamic behaviors of a single equilibrium including zero equilibrium. We propose an open problem that is the global attractivity of model (3.1). In fact, dynamical behaviors will be rather complex, so instead of considering the single equilibrium property, we can investigate the property of a certain set.

References

[1] V. L. Kocid and G. Ladas, Global behavior of nonlinear difference quations of higher order oscillatory properties of first order linnear functional differential equations. Mathematics and Its Applications. 1993.

[2] Liz E , Ruiz-Herrera A . The hydra effect, bubbles, and chaos in a simple discrete population model with constant effort harvesting[J]. Journal of Mathematical Biology, 2012, 65(5):997-1016.

[3] Chen F , Wu L , Li Z . Permanence and global attractivity of the discrete Gilpin CAyala type population model[M]. Pergamon Press, Inc. 2007.

[4] Latella D , Loreti M , Massink M . On-the-fly Fluid Model Checking via Discrete Time Population Models[C]. European Workshop on Performance Engineering. 2015.

[5] I. Gyori and G. Ladas, Oscillation thoery of Delay Differential Equations with Applications. Clarendon Press, Oxford, 1991.

[6] Fang N , Chen X . GLOBAL STABILITY OF A NONLINEAR DISCRETE POPULATION MODEL WITH TIME DELAYS[J]. Advances in Complex Systems, 2007, 10(03):315-333.

[7] Chen F , Wu L , Li Z . Permanence and global attractivity of the discrete Gilpin CAyala type population model[M]. Pergamon Press, Inc. 2007.

[8] Gao S , Chen L . The effect of seasonal harvesting on a single-species discrete population model with stage structure and birth pulses[J]. Chaos, Solitons and Fractals, 2005, 24(4):1013-1023.

[9] Gao S , Chen L . Dynamic complexities in a single-species discrete population model with stage structure and birth pulses[J]. Chaos Solitons & Fractals, 2005, 23(2):519-527.
[10] Zhang R Y , Wang Z C , Chen Y M , et al. Periodic solution of a single species discrete population model with periodic harvest stock[J]. Computers & Mathematics with Applications, 2000, 39(1-2):77-90.
[11] Ackleh A S , De Leenheer P . Discrete three-stage population model: persistence and global stability results[J]. Journal of Biological Dynamics, 2008, 2(4):415-427.
[12] Saker S H . Oscillation and attractivity of discrete nonlinear delay population model[J]. Journal of Applied Mathematics and Computing, 2007, 25(1-2):363-374.
[13] Elabbasy, E. M . Periodic solutions and oscillation of discrete nonlinear delay population dynamics model with external force[J]. IMA Journal of Applied Mathematics, 2005, 70(6):753-767.