# New Operations on n-Intuitionistic Polygonal Fuzzy Numbers 

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#### Abstract

Recently, intuitionistic fuzzy sets and numbers are widely interesting in the literature, and many types of Intuitionistic Fuzzy Numbers (IFN's) are studied and applied to different mathematical and real life problems. In this paper, an algorithm to approximate general IFN's by the $n$-Intuitionistic Polygonal Fuzzy Number ( $n$-IPFN) is introduced. The approximation facilitates the calculations due to its linearity, not to mention its realisticity and flexibility. After that, a new method to rank IFN's is introduced and applied to the $n$-IPFN. Based on that, convenient arithmetic operations for $n$-IPFN's that preserve the ranking values are proposed and shown to satisfy the most important properties. As a result, defining a ranking equivalence relation gave a strong algebraic structure that is isomorphic to the real numbers set. Finally, some definitions are proposed for dealing with matrices, functions, equalities and inequalities with $n$-IPFN's.


Index Terms-Intuitionistic Fuzzy Numbers, Polygonal Fuzzy Number, Binary Operations

## I. Introduction

According to Fuzzy Set theory, which was proposed by Zadeh [1] to represent vagueness in real-life, each element $x \in X$ in an ordinary fuzzy set $\tilde{A}=<X, \mu_{\tilde{A}}>$ has a degree of membership in $\tilde{A}$, that is $\mu_{\tilde{A}}(x) \in[0,1]$. This implicitly gives another value, $1-\mu_{\tilde{A}}(x) \in[0,1]$, which represents the degree of non-membership of $x$ in $A$. Atanassov and Krassimir [2] answered an important question here: What if the degree of non-membership of $x$ is not the complementary of $\mu_{\tilde{A}}(x)$ to 1 ? In this case, another function is required to represent the degree of non-membership of $x$, that is the non-membership function, which results in a generalized type of fuzzy sets called the Intuitionistic Fuzzy Sets (IFS). Later, Biswas [3] showed that there are some real life situations where IFS's are more appropriate in representing vagueness than ordinary fuzzy sets. This happens when there is insufficient information to determine the membership value.

A convex IFS whose domain is the set real numbers is an Intuitionistic Fuzzy Number (IFN). In the literature, there are lots of studies and applications of many types of IFN's, especially in decision-making [4], [5], [6]. The type of an IFN determines the shape of its membership and non-membership functions. For instance, each function of a triangular [5], [7] or a trapezoidal IFN [4], [8] consists of

[^0]two or three linear pieces, respectively. Zulkifly et. al [9] introduced Intuitionistic Fuzzy B-Spline Curve interpolation model by defining intuitionistic fuzzy control point relation and blended it with B-spline basis function.

Although linearity is preferable since it facilitates the calculations, there are sudden sharp changes in the functions if the number of linear pieces is small. On the other hand, using an LR IFN [6], whose functions are not necessarily piecewise linear, restricts the usage of this type due to the special relationship assumed on the left and right parts of the functions. In order to make use of the advantages of linearity while having less sharp changes, in this paper, we introduce the $n$-Intuitionistic Polygonal Fuzzy Number ( $n$ IPFN), which generalizes the triangular and the trapezoidal IFN's into an IFN with $n$ linear pieces on the left and right sides of both the membership and non-membership functions. This type can be used to approximate any type of IFN's into an IFN with piecewise linear membership and nonmembership functions, with higher accuracy as $n$ increases. The approximation facilitates applying arithmetic operations on any two IFN's by unifying their type first, then performing the operations.
Recently, ordinary polygonal fuzzy sets and numbers [10] has been getting a big interest in the literature with applications in neural networks and decision-making [11], [12]. In a previous work, Tuffaha and Alrefaei [13] considered the ordinary $n$-polygonal fuzzy number and introduced convenient arithmetic operations on it. These arithmetic operations were then shown to satisfy the most important properties, such as commutativity, associativity, having identities and inverses, and preserving the ranking values [14]. The authors also introduced a ranking equivalence relation on the set of all $n$-polygonal fuzzy numbers and showed that the set of equivalence classes is isomorphic to the real numbers set. In this paper, we generalize that work to IFN's, giving a strong tool to represent vagueness in intuitionistic fuzzy environments.
To the best of our knowledge, intuitionistic polygonal fuzzy numbers are not studied in the literature, except for Wang and Duan [15], who considered such a type of fuzzy numbers, but with a different definition. Although their definition does not require the membership or the nonmembership function to reach one or zero, respectively, our definition is more general in the sense that it does not restrict the endpoints of the linear pieces in the membership and the non-membership functions to be equal as in the definition by Wang and Duan [15]. Giving the freedom to the $x$ 's of the corresponding knots in the two functions provides a more realistic representation, which we seek in this paper.

The paper is organized as follows: the intuitionistic polygonal approximation of IFN's and the definition of the $n$-IPFN
are given in Section II. An extension of the ranking function with total integral value to IFN's is introduced in Section III and then applied to the $n$-IPFN. After that, suitable arithmetic operations on the $n$-IPFN are introduced and shown to satisfy the most important properties in Section IV, while in Section V the ranking equivalence relation is presented and studied. Finally, some definitions are proposed for dealing with matrices, functions, equalities and inequalities with $n$ IPFN's in Section VI followed by concluding remarks and future insights in Section VII.

## II. Intuitionistic Polygonal Approximation

First, the definitions of intuitionistic fuzzy sets and numbers are presented.

Definition 1: [2] An intuitionistic fuzzy set (IFS) $\tilde{A}^{I}$ is a triple $<X, \mu_{\tilde{A}^{I}}, \nu_{\tilde{A}^{I}}>$, where $X$ is a set and $\mu_{\tilde{A}^{I}}, \nu_{\tilde{A}^{I}}: X \rightarrow[0,1]$ are called the membership and the non-membership functions, respectively. The values of $\mu_{\tilde{A}^{I}}(x)$ and $\nu_{\tilde{A}^{I}}(x)$ represent the "degree of membership" and the "degree of non-membership" of $x$ in $\tilde{A}^{I}$, where $0 \leq \mu_{\tilde{A}^{I}}(x)+\nu_{\tilde{A}^{I}}(x) \leq 1$. The complementary of this sum to 1 is called the degree of hesitation, $\pi_{\tilde{A}^{I}}(x)=$ $1-\mu_{\tilde{A}^{I}}(x)-\nu_{\tilde{A}^{I}}(x) \in[0,1]$.
Definition 2: [16] A real intuitionistic fuzzy number (IFN), $\tilde{A}^{I}$, is an intuitionistic fuzzy subset of $\Re$ with a membership $\mu_{\tilde{A}^{I}}$ and a non-membership $\nu_{\tilde{A}^{I}}$ functions, which can be described as:

$$
\begin{aligned}
& \mu_{\tilde{A}^{I}}(x)= \begin{cases}\mu_{\widetilde{A^{I}}}^{L}(x) & ; a<x \leq b \\
1 & ; b<x \leq c \\
\mu_{\widetilde{A^{I}}}^{R}(x) & ; c<x \leq d \\
0 & \text { otherwise }\end{cases} \\
& \nu_{\tilde{A}}(x)= \begin{cases}\nu_{\widetilde{A^{I}}}^{L}(x) & ; a^{\prime}<x \leq b^{\prime} \\
0 & ; b^{\prime}<x \leq c^{\prime} \\
\nu_{\widetilde{A}}^{R}(x) & ; c^{\prime}<x \leq d^{\prime} \\
1 & \text { otherwise }\end{cases}
\end{aligned}
$$

where $a, b, c, d, a^{\prime}, b^{\prime}, c^{\prime}$ and $d^{\prime}$ are real numbers. Moreover, $\mu_{\widetilde{A}^{I}}^{L}:[a, b] \rightarrow[0,1]$ and $\nu_{\widetilde{A}^{I}}^{L}:\left[a^{\prime}, b^{\prime}\right] \rightarrow[0,1]$ are the left membership and non-membership functions of $\widetilde{A}^{I}$, which are continuous, increasing and decreasing, respectively. On the other hand, $\mu_{\widetilde{A}^{I}}^{R}:[c, d] \rightarrow[0,1]$ and $\nu_{\widetilde{A}^{I}}^{R}:\left[c^{\prime}, d^{\prime}\right] \rightarrow[0,1]$ are the right membership and non-membership functions of $\widetilde{A}^{I}$, which are continuous, decreasing and increasing, respectively.

## A. Intuitionistic Polygonal Approximation

Let $\tilde{A}^{I}$ be an IFN defined as in Definition 2. The left and right membership and non-membership functions of $\tilde{A}^{I}$; $\mu_{\tilde{A}^{I}}^{L}, \mu_{\tilde{A}^{I}}^{R}, \nu_{\tilde{A}^{I}}^{L}$ and $\nu_{\tilde{A}^{I}}^{R}$, are approximated by piecewise linear functions $f_{L}, f_{R}, g_{L}$ and $g_{R}$, respectively, as follows:

Step 1: Choose $n \in \mathbb{Z}^{+}$, which indicates the number of linear pieces in each one of $f_{L}, f_{R}, g_{L}$ and $g_{R}$.
Step 2: Divide the interval $[0,1]$ into $n$ equal subintervals with nodes: $0<\frac{1}{n}<\frac{2}{n}<\ldots<\frac{n-1}{n}<1$.
Step 3: Without loss of generality, we assume that $\mu_{\tilde{A}_{R}^{I}}^{L}$ and $\nu_{\tilde{A}^{I}}^{R}$ are strictly increasing functions, while $\mu_{\tilde{A}^{I}}^{R}$ and $\nu_{\tilde{A}^{I}}^{L}$ are strictly decreasing functions. Then, these functions are one-to-one. In addition, each one of them is onto and covers the interval
$[0,1]$ completely. Therefore, their inverse functions, $\left(\mu_{\tilde{A}}^{L}\right)^{-1},\left(\mu_{\tilde{A}}^{R}\right)^{-1},\left(\nu_{\tilde{A}}^{L}\right)^{-1}$ and $\left(\nu_{\tilde{A}}^{R}\right)^{-1}$, exist.
Set $a_{0}=a, a_{n}=b, b_{0}=c, b_{n}=d$,
$a_{i}=\left(\mu_{\tilde{A}}^{L}\right)^{-1}\left(\frac{i}{n}\right): i=1,2, . ., n-1$,
and $b_{i}=\left(\mu_{\tilde{A}}^{R}\right)^{-1}\left(\frac{n-i}{n}\right): i=1, . ., n-1$,
Then, $a_{0}<a_{1}<. .<a_{n-1}<a_{n} \leq b_{0}<b_{1}<$ $. .<b_{n-1}<b_{n}$.
Also, set $a_{0}^{\prime}=a^{\prime}, a_{n}^{\prime}=b, b_{0}^{\prime}=c^{\prime}, b_{n}^{\prime}=d^{\prime}$, $a_{i}^{\prime}=\left(\nu_{\tilde{A}}^{L}\right)^{-1}\left(\frac{n-i}{n}\right): i=1,2, . ., n-1$,
and $b_{i}^{\prime}=\left(\nu_{\hat{A}}^{R}\right)^{-1}\left(\frac{i}{n}\right): i=1, . ., n-1$.
Then $a_{0}^{\prime}<a_{1}^{\prime}<. .<a_{n-1}^{\prime}<a_{n}^{\prime} \leq b_{0}^{\prime}<b_{1}^{\prime}<. .<$ $b_{n-1}^{\prime}<b_{n}^{\prime}$.
Step 4: Perform two piecewise linear interpolations between the points:
$\left\{\left(a_{0}, 0\right),\left(a_{1}, \frac{1}{n}\right), \ldots,\left(a_{i}, \frac{i}{n}\right), \ldots,\left(a_{n-1}, \frac{n-1}{n}\right),\left(a_{n}, 1\right)\right.$, $\left.\left(b_{0}, 1\right),\left(b_{1}, \frac{n-1}{n}\right), \ldots,\left(b_{i}, \frac{n-i}{n}\right), \ldots,\left(b_{n-1}, \frac{1}{n}\right),\left(b_{n}, 0\right)\right\}$ and the points:
$\left\{\left(a_{0}^{\prime}, 1\right),\left(a_{1}^{\prime}, \frac{n-1}{n}\right), \ldots,\left(a_{i}^{\prime}, \frac{n-i}{n}\right), \ldots,\left(a_{n-1}^{\prime}, \frac{1}{n}\right)\right.$,
$\left(a_{n}^{\prime}, 0\right),\left(b_{0}^{\prime}, 0\right),\left(b_{1}^{\prime}, \frac{1}{n}\right), \ldots,\left(b_{i}^{\prime}, \frac{i}{n}\right), \ldots,\left(b_{n-1}^{\prime}, \frac{n-1}{n}\right)$, $\left.\left(b_{n}^{\prime}, 1\right)\right\}$.
The resulting functions, respectively denoted by $f_{\tilde{A}^{I}}$ and $g_{\tilde{A}^{I}}$, are the piecewise linear approximating functions, given by:

$$
\begin{align*}
& f_{\tilde{A}^{I}}(x)= \begin{cases}\frac{1}{n}\left[\frac{x-a_{i}}{a_{i+1}-a_{i}}\right]+\frac{i}{n} & ; a_{i} \leq x \leq a_{i+1}, \\
1 & i=0, . ., n-1 \\
\frac{-1}{n}\left[\frac{x-b_{i}}{b_{i+1}-b_{i}}\right]+\frac{n-i}{n} & ; a_{n} \leq x \leq b_{0} \leq x \leq b_{i+1}, \\
0 & i=0, . ., n-1\end{cases}  \tag{1}\\
& g_{\tilde{A}^{I}}(x)= \begin{cases}\frac{-1}{n}\left[\frac{x-a_{i}^{\prime}}{a_{i+1}^{\prime}-a_{i}^{\prime}}\right]+\frac{n-i}{n} & ; a_{i}^{\prime} \leq x \leq a_{i+1}^{\prime} \\
0 & i=0, . ., n-1\end{cases}  \tag{2}\\
& \frac{1}{n}\left[\frac{x-b_{i}^{\prime}}{\left.b_{i+1}^{\prime-b_{i}^{\prime}}\right]+\frac{i}{n}} \begin{array}{ll} 
& ; a_{n}^{\prime} \leq x \leq b_{0}^{\prime} \leq x \leq b_{i+1}^{\prime} \\
1 & i=0, . ., n-1
\end{array}\right. \\
& 1
\end{align*}
$$

Then, $f_{\tilde{A}^{I}}, g_{\tilde{A}^{I}}: \mathbb{R} \rightarrow[0,1]$ can be seen as a membership and a non-membership functions of an IFN. This gives the definition of the $n$-intuitionistic polygonal fuzzy number.
Definition 3: A real IFN with a membership and a non-membership functions given as in equations (1) and (2), respectively, is called an $n$ Intuitionistic Polygonal Fuzzy Number, denoted by $n$-IPFN, which is represented by its knots: $\left\{\left(a_{0}, a_{1}, . ., a_{n} ; b_{0}, b_{1}, . ., b_{n}\right),\left(a_{0}^{\prime}, a_{1}^{\prime}, . ., a_{n}^{\prime} ; b_{0}^{\prime}, b_{1}^{\prime}, . ., b_{n}^{\prime}\right)\right\}$. If $a_{n}=b_{0}$, then the $n$-IPFN is called non-flat. Otherwise, it is called flat. Moreover, the set of all $n$-IPFN's is denoted by $\mathcal{I} \mathcal{P}_{n}$.

Example 1: Let $\tilde{A}^{I}$ be an IFN with a membership function $\mu_{\tilde{A}^{I}}$ and a non-membership function $\nu_{\tilde{A}^{I}}$ given by:

$$
\begin{gathered}
\mu_{\tilde{A}^{I}}(x)= \begin{cases}\frac{1}{4}(x+1)^{2} & ;-1 \leq x \leq 1 \\
(x-2)^{4} & ; 1 \leq x \leq 2 \\
0 & \text { otherwise }\end{cases} \\
\nu_{\tilde{A}^{I}}(x)= \begin{cases}\frac{1}{4}(x-1)^{2} & ;-1 \leq x \leq 3 \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$

Let $n=3$. Then the piecewise linear membership function approximating $\mu_{\tilde{A}^{I}}$ is:

$$
f_{\tilde{A}^{I}}(x)= \begin{cases}0.289(x+1) & ;-1 \leq x \leq 0.155 \\ 0.697(x-0.155)+\frac{1}{3} & ; 0.155 \leq x \leq 0.633 \\ 0.908(x-0.633)+\frac{2}{3} & ; 0.633 \leq x \leq 1 \\ 1 & ; x=1 \\ -3.458(x-1)+1 & ; 1 \leq x \leq 1.096 \\ -2.318(x-1.096)+\frac{2}{3} & ; 1.096 \leq x \leq 1.24 \\ -0.439(x-1.240)+\frac{1}{3} & ; 1.24 \leq x \leq 2 \\ 0 & \text { otherwise }\end{cases}
$$

Whereas the piecewise linear non-membership function approximating $\nu_{\tilde{A}^{I}}$ is:
$g_{\tilde{A}^{I}}(x)= \begin{cases}-0.908(x+1)+1 & ;-1 \leq x \leq-0.633 \\ -0.697(x+0.633)+\frac{2}{3} & ;-0.633 \leq x \leq-0.155 \\ -0.289(x+0.155)+\frac{1}{3} & ;-0.155 \leq x \leq 1 \\ 1 & ; x=1 \\ 0.288(x-1) & ; 1 \leq x \leq 2.157 \\ 0.7(x-2.157)+\frac{1}{3} & ; 2.157 \leq x \leq 2.633 \\ 0.908(x-2.633)+\frac{2}{3} & ; 2.633 \leq x \leq 3 \\ 0 & \text { otherwise }\end{cases}$
These two functions represent the non-flat 3 -IPFN:
$\{(-1,0.155,0.633,1 ; 1,1.096,1.240,2)$,
$(-1,-0.633,-0.155,1 ; 1,2.157,2.633,3)\}$.
Figure 1 clarifies the approximation of $\tilde{A}^{I}$.


Fig. 1. The piecewise linear approximation of the IFN in Example 1
Remark 1: Any real number $a \in \mathbb{R}$ can be expressed in the $n$-IPFN form:
$a=\{(a, a, . ., a ; a, a . ., a),(a, a, . ., a ; a, a . ., a)\}$, and it is called a crisp real number to indicate that it is not fuzzy. Table II-A shows how the intuitionistic triangular, trapezoidal and hexagonal fuzzy numbers are special cases of the $n$ IPFN:

TABLE I
SOME SPECIAL TYPES OF $n$-IPFN

| $n=1$ | $a_{n}=b_{0}$ | Intuitionistic Triangular | non-flat |
| :---: | :---: | :---: | :---: |
| $n=1$ | $a_{n} \neq b_{0}$ | Intuitionistic Trapezoidal | flat |
| $n=2$ | $a_{n} \neq b_{0}$ | Intuitionistic Hexagonal | flat |

Example 2: Figure 2 shows an example of a flat 2-IPFN given by $\tilde{A}^{I}=\{(-1,2,3 ; 5,7,10),(-2,-1,2 ; 6,8,11)\}$.

## III. Ranking Function

Ranking (comparing) IFN's in the literature takes two different approaches. The first one is to make lexicographic comparisons between values assigned to the compared IFN's,


Fig. 2. The 2-IPFN in Example 2
with priorities related to the importance of each one of these values in determining the relationship between the two IFN's [17], [18], [19]. The second approach is to assign one real value to each IFN, often by taking a convex combination of a value related to the membership function and another related to the non-membership one, and then comparing the IFN's according to the real values assigned to each one of them [20], [5], in this paper, we consider the second approach.
Many of existing approaches depend on generalizing the ranking methods for ordinary fuzzy numbers. The ranking function given in this section for the $n$-IPFN proposes a generalization of the ranking function with total integral value on ordinary fuzzy numbers, which was introduced by Liou \& Wang [21]. The definition is presented first and then extended to IFN's and applied to the $n$-IPFN.
Definition 4: Let $\tilde{N}$ be any fuzzy number. The ranking function with total integral value is given by:

$$
\begin{equation*}
\Re(\tilde{N})=(1-\alpha) J_{L}(\tilde{N})+\alpha J_{R}(\tilde{N}) \tag{3}
\end{equation*}
$$

where: $J_{L}(\tilde{N})=\int_{0}^{1}\left(\mu_{\tilde{N}}^{L}(x)\right)^{-1}(y) d y, \quad J_{R}(\tilde{N})=$ $\int_{0}^{1}\left(\mu_{\tilde{N}}^{R}(x)\right)^{-1}(y) d y$, and $\alpha \in[0,1]$ represents the decision maker's degree of optimism.

This definition can be extended to intuitionistic fuzzy numbers by applying it to both the membership function $\left(\mu_{\tilde{A}^{I}}\right)$ and the complement of the non-membership function to one $\left(1-\nu_{\tilde{A}^{I}}\right)$, which gives two values. Taking a convex combination of these two values gives the ranking value of the given IFN.

Definition 5: Let $\tilde{A}^{I}=<\mathbb{R}, \mu_{\tilde{A}^{I}}, \nu_{\tilde{A}^{I}}>$ be an intuitionistic fuzzy number and let $\alpha_{1}, \alpha_{2}, \lambda \in[0,1]$ be three values that represent the decision maker's degree of optimism. Assume $\nu_{c}=1-\nu_{\tilde{A}^{I}}, \nu_{c}^{L}=1-\nu_{\tilde{A}^{I}}^{L}$ and $\nu_{c}^{R}=1-\nu_{\tilde{A}^{I}}^{R}$. Then, the ranking function with total integral value of $\widetilde{A}^{I}$ is defined as follows:

$$
\begin{equation*}
\Re\left(\tilde{A}^{I}\right)=\lambda \Re_{\mu}\left(\tilde{A}^{I}\right)+(1-\lambda) \Re_{\nu}\left(\tilde{A}^{I}\right), \tag{4}
\end{equation*}
$$

where, $\Re_{\mu}\left(\tilde{A}^{I}\right)=\alpha_{1} I_{R}\left(\tilde{A}^{I}\right)+\left(1-\alpha_{1}\right) I_{L}\left(\tilde{A}^{I}\right)$ and $\Re_{\nu}\left(\tilde{A}^{I}\right)=\alpha_{2} J_{R}\left(\tilde{A}^{I}\right)+\left(1-\alpha_{2}\right) J_{L}\left(\tilde{A}^{I}\right)$, such that

$$
I_{R}\left(\tilde{A}^{I}\right)=\int_{0}^{1}\left(\mu_{\tilde{A}^{I}}^{R}(x)\right)^{-1}(y) d y
$$

$$
\begin{aligned}
& \quad I_{L}\left(\tilde{A}^{I}\right)=\int_{0}^{1}\left(\mu_{\tilde{A}^{I}}^{L}(x)\right)^{-1}(y) d y \\
& \qquad J_{R}\left(\tilde{A}^{I}\right)=\int_{0}^{1}\left(\nu_{c}^{R}(x)\right)^{-1}(y) d y \\
& \text { and } \quad J_{L}\left(\tilde{A}^{I}\right)=\int_{0}^{1}\left(\nu_{c}^{L}(x)\right)^{-1}(y) d y
\end{aligned}
$$

Now, let $\tilde{A}^{I}=\left\{\left(p_{0}, p_{1}, . ., p_{n} ; q_{0}, q_{1}, . ., q_{n}\right)\right.$,
$\left.\left(p_{0}^{\prime}, p_{1}^{\prime}, . ., p_{n}^{\prime} ; q_{0}^{\prime}, q_{1}^{\prime}, . ., q_{n}^{\prime}\right)\right\} \in \mathcal{I} \mathcal{P}_{n}$. Applying Definition 5 to $\tilde{A}^{I}$ and considering $\alpha_{1}=\alpha_{2}=\lambda=0.5$ as is usually done in the literature gives the following ranking function for the $n$-IPFN $\tilde{A}^{I}$ :

$$
\begin{align*}
\Re\left(\tilde{A}^{I}\right)= & \frac{1}{8 n}\left[p_{0}+2 p_{1}+2 p_{2}+\ldots+2 p_{n-1}+p_{n}\right. \\
& +q_{0}+2 q_{1}+2 q_{2}+\ldots+2 q_{n-1}+q_{n} \\
& +p_{0}^{\prime}+2 p_{1}^{\prime}+2 p_{2}^{\prime}+\ldots+2 p_{n-1}^{\prime}+p_{n}^{\prime} \\
& \left.+q_{0}^{\prime}+2 q_{1}^{\prime}+2 q_{2}^{\prime}+\ldots+2 q_{n-1}^{\prime}+q_{n}^{\prime}\right] \tag{5}
\end{align*}
$$

Example 3: The ranking value for the 2-IPFN in Example 2 is $\Re\left(\tilde{A}^{I}\right)=\frac{1}{16}[-1+4+3+5+14+10-2-2+2+$ $6+16+11]=4.125$.

## Comparing Intuitionistic polygonal fuzzy numbers

Here, we propose a way to compare $n$-IPFN's through comparing their ranking values.
Definition 6: Let

$$
\begin{aligned}
& \tilde{A}^{I}=\left\{\left(a_{0}, a_{1}, . ., a_{n} ; b_{0}, b_{1}, . ., b_{n}\right),\left(c_{0}, c_{1}, . ., c_{n} ; d_{0}, d_{1}, . ., d_{n}\right)\right\}, \\
& \tilde{B}^{I}=\left\{\left(a_{0}^{\prime}, a_{1}^{\prime}, . ., a_{n}^{\prime} ; b_{0}^{\prime}, b_{1}^{\prime}, . ., b_{n}^{\prime}\right),\left(c_{0}^{\prime}, c_{1}^{\prime}, . ., c_{n}^{\prime} ; d_{0}^{\prime}, d_{1}^{\prime}, . ., d_{n}^{\prime}\right)\right\}
\end{aligned}
$$

be two IPFN-n's. Then:

- $\tilde{A}^{I}$ and $\tilde{B}^{I}$ are called equivalent, denoted $\tilde{A}^{I} \approx \tilde{B}^{I}$, if $\Re\left(\tilde{A}^{I}\right)=\Re\left(\tilde{B}^{I}\right)$,
- $\tilde{A}^{I}$ and $\tilde{B}^{I}$ are called equal, denoted $\tilde{A}^{I}=\tilde{B}^{I}$, if $a_{i}=$ $a_{i}^{\prime}, b_{i}=b_{i}^{\prime}, c_{i}=c_{i}^{\prime}$ and $d_{i}=d_{i}^{\prime}$ for all $i=0,1, \ldots, n$.
- $\tilde{A}^{I}$ is said to be greater than (greater or equal) $\tilde{B}^{I}$, denoted $\tilde{A}^{I} \succ \tilde{B}^{I}\left(\tilde{A}^{I} \succeq \tilde{B}^{I}\right)$, if $\mathfrak{R}\left(\tilde{A}^{I}\right)>\mathfrak{R}\left(\tilde{B}^{I}\right)$ $\left(\mathfrak{R}\left(\tilde{A}^{I}\right) \geq \mathfrak{R}\left(\tilde{B}^{I}\right)\right)$.
- $\tilde{A}^{I}$ is said to be smaller than (smaller or equal) $\tilde{B}^{I}$, denoted $\tilde{A}^{I} \prec \tilde{B}^{I}\left(\tilde{A}^{I} \preceq \tilde{B}^{I}\right)$, if $\mathfrak{R}\left(\tilde{A}^{I}\right)<\mathfrak{R}\left(\tilde{B}^{I}\right)$ $\left(\mathfrak{R}\left(\tilde{A}^{I}\right) \leq \mathfrak{R}\left(\tilde{B}^{I}\right)\right)$.
Example 4: Consider the following three 3-IPFN's: $\tilde{A}^{I}=\{(-7,-6,-2.5,-2 ;-2,-1,1,4),(-10,-9,-7.5,-4$; $1,3,5,6)\}, \tilde{B}^{I}=\{(-3,-2,-2,-2 ;-2,-2,-2,-1),(-4$, $-3,-2,-2 ;-2,-2,-1,0)\}$, and $\tilde{C}^{I}=\{(1,3,4,5 ; 5,6,8$, 9), $(-2,1,2,4 ; 6,8,9,10)\}$.

Note that $\mathfrak{R}\left(\tilde{A}^{I}\right)=\mathfrak{R}\left(\tilde{B}^{I}\right)=-2$ and $\mathfrak{R}\left(\tilde{C}^{I}\right)=5$. Thus $\tilde{A}^{I}$ and $\tilde{B}^{I}$ are equivalent, but not equal. In addition, $\tilde{A}^{I} \prec \tilde{C}^{I}$ and $\tilde{C}^{I} \succ \tilde{B}^{I}$.

## IV. Arithmetic Operations

In this section, four arithmetic operations are introduced on the $n$-intuitionistic polygonal fuzzy numbers, and many properties of these operations are shown.

## A. Addition on $\mathcal{I} \mathcal{P}_{n}$

In the following three definitions, let $\tilde{A}^{I}$ and $\tilde{B}^{I} \in \mathcal{I} \mathcal{P}_{n}$ given as in Definition 6.

Definition 7: The addition of $\tilde{A}^{I}$ and $\tilde{B}^{I}$ is defined as follows:

$$
\begin{aligned}
& \tilde{A}^{I} \oplus \tilde{B}^{I}= \\
& \left\{\left(a_{0}+a_{0}^{\prime}, a_{1}+a_{1}^{\prime}, . ., a_{n}+a_{n}^{\prime} ; b_{0}+b_{0}^{\prime}, b_{1}+b_{1}^{\prime}, . ., b_{n}+b_{n}^{\prime}\right),\right. \\
& \left.\left(c_{0}+c_{0}^{\prime}, c_{1}+c_{1}^{\prime}, . ., c_{n}+c_{n}^{\prime} ; d_{0}+d_{0}^{\prime}, d_{1}+d_{1}^{\prime}, . ., d_{n}+d_{n}^{\prime}\right)\right\}
\end{aligned}
$$

## Note that:

- $\mathcal{I} \mathcal{P}_{n}$ has an additive identity, that is the crisp number 0 , since for all $\tilde{A}^{I} \in \mathcal{I} \mathcal{P}_{n}$,

$$
\tilde{A}^{I} \oplus\{(0,0, . ., 0 ; 0,0, . ., 0),(0,0, . ., 0 ; 0,0, . ., 0)\}=\tilde{A}^{I}
$$

- It is clear that the addition defined above on $\mathcal{I P}{ }_{n}$ generalizes the conventional addition on the crisp real numbers since for all $p, q \in \mathbb{R}: p \oplus q=p+q$
Now, we have the following theorem.
Theorem 1: For all $\tilde{A}^{I}, \tilde{B}^{I} \in \mathcal{I} \mathcal{P}_{n}, \Re\left(\tilde{A}^{I} \oplus \tilde{B}^{I}\right)=\Re\left(\tilde{A}^{I}\right)+$ $\Re\left(\tilde{B}^{I}\right)$.
$\frac{\text { Proof: }}{\text { Let }}$
$\tilde{A}^{I}=\left\{\left(a_{0}, a_{1}, . ., a_{n} ; b_{0}, b_{1}, . ., b_{n}\right),\left(c_{0}, c_{1}, . ., c_{n} ; d_{0}, d_{1}, . ., d_{n}\right)\right\}$,
$\tilde{B}^{I}=\left\{\left(a_{0}^{\prime}, a_{1}^{\prime}, . ., a_{n}^{\prime} ; b_{0}^{\prime}, b_{1}^{\prime}, . ., b_{n}^{\prime}\right),\left(c_{0}^{\prime}, c_{1}^{\prime}, . ., c_{n}^{\prime} ; d_{0}^{\prime}, d_{1}^{\prime}, . ., d_{n}^{\prime}\right)\right\}$
Then,

$$
\begin{aligned}
\mathfrak{R}\left(\tilde{A}^{I} \oplus \tilde{B}^{I}\right)= & \mathfrak{R}\left[\left\{\left(a_{0}+a_{0}^{\prime}, a_{1}+a_{1}^{\prime}, \ldots, a_{n}+a_{n}^{\prime} ;\right.\right.\right. \\
& \left.b_{0}+b_{0}^{\prime}, b_{1}+b_{1}^{\prime}, ., b_{n}+b_{n}^{\prime}\right) \\
& \left(c_{0}+c_{0}^{\prime}, c_{1}+c_{1}^{\prime}, \ldots, c_{n}+c_{n}^{\prime} ;\right. \\
& \left.\left.\left.d_{0}+d_{0}^{\prime}, d_{1}+d_{1}^{\prime}, \ldots, d_{n}+d_{n}^{\prime}\right)\right\}\right] \\
= & \frac{1}{8 n}\left[\left(a_{0}+a_{0}^{\prime}\right)+2\left(a_{1}+a_{1}^{\prime}\right)+\ldots\right. \\
& +2\left(a_{n-1}+a_{n-1}^{\prime}\right)+\left(a_{n}+a_{n}^{\prime}\right) \\
& +\left(b_{0}+b_{0}^{\prime}\right)+2\left(b_{1}+b_{1}^{\prime}\right)+\ldots \\
& +2\left(b_{n-1}+b_{n-1}^{\prime}\right)+\left(b_{n}+b_{n}^{\prime}\right) \\
& +\left(c_{0}+c_{0}^{\prime}\right)+2\left(c_{1}+c_{1}^{\prime}\right)+\ldots \\
& +2\left(c_{n-1}+c_{n-1}^{\prime}\right)+\left(c_{n}+c_{n}^{\prime}\right) \\
& +\left(d_{0}+d_{0}^{\prime}\right)+2\left(d_{1}+d_{1}^{\prime}\right)+\ldots \\
& \left.+2\left(d_{n-1}+d_{n-1}^{\prime}\right)+\left(d_{n}+d_{n}^{\prime}\right)\right] \\
= & \frac{1}{8 n}\left[a_{0}+2 a_{1}+\ldots+2 a_{n-1}+a_{n}\right. \\
& +b_{0}+2 b_{1}+\ldots+2 b_{n-1}+b_{n} \\
& +c_{0}+2 c_{1}+\ldots+2 c_{n-1}+c_{n} \\
& \left.+d_{0}+2 d_{1}+\ldots+2 d_{n-1}+d_{n}\right] \\
+ & \frac{1}{8 n}\left[a_{0}^{\prime}+2 a_{1}^{\prime}+\ldots+2 a_{n-1}^{\prime}+a_{n}^{\prime}\right. \\
& +b_{0}^{\prime}+2 b_{1}^{\prime}+\ldots+2 b_{n-1}^{\prime}+b_{n}^{\prime} \\
& +c_{0}^{\prime}+2 c_{1}^{\prime}+\ldots+2 c_{n-1}^{\prime}+c_{n}^{\prime} \\
& \left.+d_{0}^{\prime}+2 d_{1}^{\prime}+\ldots+2 d_{n-1}^{\prime}+d_{n}^{\prime}\right] \\
= & \mathfrak{R}(\tilde{P})+\mathfrak{R}(\tilde{Q})
\end{aligned}
$$

Definition 8: The additive inverse of $\tilde{A}^{I}$, in the sense that $\tilde{A}^{I} \oplus$ $\left(-\tilde{A}^{I}\right) \approx 0$, is defined as follows:
$-\tilde{A}^{I}=\left\{\left(-b_{n},-b_{n-1}, . .,-b_{1},-b_{0} ;-a_{n},-a_{n-1}, . .,-a_{1},-a_{0}\right)\right.$,
$\left.\left(-d_{n},-d_{n-1}, . .,-d_{1},-d_{0} ;-c_{n},-c_{n-1}, . .,-c_{1},-c_{0}\right)\right\}$
Definition 9: The subtraction of $\tilde{B}^{I}$ from $\tilde{A}^{I}$ is defined as $\tilde{A}^{I} \ominus$ $\tilde{B}^{I}=\tilde{A}^{I} \oplus\left(-\tilde{B}^{I}\right)$.
Example 5: Consider $\tilde{A}^{I}$ and $\tilde{C}^{I}$ given in Example 4. $\mathfrak{R}\left(\tilde{C}^{I} \oplus \tilde{A}^{I}\right)=\mathfrak{R}(\{(-6,-3,1.5,3 ; 3,5,9,13)$,
$(-12,-8,-5.5,0 ; 7,11,14,16)\})=3=\mathfrak{R}\left(\tilde{C}^{I}\right)+\mathfrak{R}\left(\tilde{A}^{I}\right)$.
$\mathfrak{R}\left(\tilde{C}^{I} \ominus \tilde{A}^{I}\right)=\mathfrak{R}(\{(-3,2,5,7 ; 7,8.5,14,16)$,
$(-8,-4,-1,3 ; 10,15.5,18,20)\})=7=\mathfrak{R}\left(\tilde{C}^{I}\right)-\mathfrak{R}\left(\tilde{A}^{I}\right)$.

## B. Multiplication on $\mathcal{I P}_{n}$ :

A convenient multiplication on $\mathcal{I} \mathcal{P}_{n}$ must satisfy some basic properties, such as:

1) Commutative property.
2) Generalization of the known multiplication on real numbers.
3) $\mathcal{I} \mathcal{P}_{n}$ has a multiplicative identity $\tilde{1}^{I}$.
4) For all $\tilde{A}^{I} \in \mathcal{I} \mathcal{P}_{n}$, where $\Re\left(\tilde{A}^{I}\right) \neq 0, \tilde{A}_{\tilde{B}^{I}}$ has a multiplicative inverse $\tilde{B}^{I} \in \mathcal{I} \mathcal{P}_{n}$, such that: $\tilde{A}^{I} \otimes \tilde{B}^{I} \approx \tilde{1}^{I}$.
5) Moreover, the multiplication must preserve the ranking values of the multiplied fuzzy numbers, i.e.

$$
\begin{equation*}
\Re\left[\tilde{A}^{I}\right] \cdot \Re\left[\tilde{B}^{I}\right]=\Re\left[\tilde{A}^{I} \otimes \tilde{B}^{I}\right] \tag{6}
\end{equation*}
$$

In what follows, our aim is to introduce a multiplication on $\mathcal{I} \mathcal{P}_{n}$ satisfying these properties.
First, we introduce definitions for the $i$-th spreads of a $n$-IPFN which are used in this section.

Definition 10:
Let $\tilde{A}^{I}=\left\{\left(a_{0}, a_{1}, . ., a_{n} ; a_{n+1}, a_{n+2}, . ., a_{2 n+1}\right)\right.$, $\left.\left(b_{0}, b_{1}, . ., b_{n} ; b_{n+1}, b_{n+2}, . ., b_{2 n+1}\right)\right\} \in \mathcal{I} \mathcal{P}_{n}$.
For $i=1,2, . ., 2 n+1$, the interval $I_{i_{\tilde{N}}}=\left[a_{i-1}, a_{i}\right]$ is called the $i$-th membership interval of $\tilde{A}^{I}$, and its length $L_{i}=a_{i}-a_{i-1}$ is called the $i$-th membership spread of $\tilde{A}^{I}$. Moreover, for $i=1,2, . ., 2 n+1$, the interval $I_{i}^{\prime}=\left[b_{i-1}, b_{i}\right]$ is called the $i$-th non-membership interval of $\tilde{A}^{I}$, and its length $L_{i}=b_{i}-b_{i-1}$ is called the $i$-th non-membership spread of $\tilde{A}_{\tilde{A}}$.

Now, let $\tilde{A}^{I} \underset{\tilde{B}^{I}}{=}\left\{\left(a_{0}, a_{1}, . ., a_{n} ; b_{0}, b_{1}, . ., b_{n}\right),\left(c_{0}, c_{1}, . ., c_{n}\right.\right.$; $\left.\left.d_{0}, d_{1}, . ., d_{n}\right)\right\}, \tilde{B}^{I}=\left\{\left(a_{0}^{\prime}, a_{1}^{\prime}, . ., a_{n}^{\prime} ; b_{0}^{\prime}, b_{1}^{\prime}, . ., b_{n}^{\prime}\right),\left(c_{0}^{\prime}, c_{1}^{\prime}, \ldots\right.\right.$ ,$\left.\left.c_{n}^{\prime} ; d_{0}^{\prime}, d_{1}^{\prime}, \ldots, d_{n}^{\prime}\right)\right\}$. We want the product of $\tilde{A}^{I}$ and $\vec{B}^{I}$ to be also a $n$-IPFN, so let it be: $\tilde{C}^{I}=$ $\left\{\left(e_{0}, e_{1}, . ., e_{n} ; f_{0}, f_{1}, . ., f_{n}\right),\left(g_{0}, g_{1}, . ., g_{n} ; h_{0}, h_{1}, . ., h_{n}\right)\right\} \in$ $\mathcal{I} \mathcal{P}_{n}$.
From Equation (6), we want:

$$
\begin{array}{r}
\Re\left(\tilde{A}^{I}\right) \cdot \Re\left(\tilde{B}^{I}\right)=\frac{1}{8 n}\left[e_{0}+2 e_{1}+2 e_{2}+\ldots+2 e_{n-1}+e_{n}\right. \\
+f_{0}+2 f_{1}+2 f_{2}+\ldots+2 f_{n-1}+f_{n}  \tag{7}\\
\\
+g_{0}+2 g_{1}+2 g_{2}+\ldots+2 g_{n-1}+g_{n} \\
\\
\left.+h_{0}+2 h_{1}+2 h_{2}+\ldots+2 h_{n-1}+h_{n}\right]
\end{array}
$$

Let

$$
\begin{align*}
I=\frac{1}{8 n} & {\left[\left(a_{0}+2 a_{1}+2 a_{2}+\ldots+2 a_{n-1}+a_{n}\right.\right.} \\
& +b_{0}+2 b_{1}+2 b_{2}+\ldots+2 b_{n-1}+b_{n} \\
& +c_{0}+2 c_{1}+2 c_{2}+\ldots+2 c_{n-1}+c_{n} \\
& \left.+d_{0}+2 d_{1}+2 d_{2}+\ldots+2 d_{n-1}+d_{n}\right)  \tag{8}\\
\quad & \left(a_{0}^{\prime}+2 a_{1}^{\prime}+2 a_{2}^{\prime}+\ldots+2 a_{n-1}^{\prime}+a_{n}^{\prime}\right. \\
& +b_{0}^{\prime}+2 b_{1}^{\prime}+2 b_{2}^{\prime}+\ldots+2 b_{n-1}^{\prime}+b_{n}^{\prime} \\
& +c_{0}^{\prime}+2 c_{1}^{\prime}+2 c_{2}^{\prime}+\ldots+2 c_{n-1}^{\prime}+c_{n}^{\prime} \\
& \left.\left.+d_{0}^{\prime}+2 d_{1}^{\prime}+2 d_{2}^{\prime}+\ldots+2 d_{n-1}^{\prime}+d_{n}^{\prime}\right)\right]
\end{align*}
$$

Then, from equations (7) and (8), our first condition is:

$$
\begin{align*}
I= & e_{0}+2 e_{1}+2 e_{2}+\ldots+2 e_{n-1}+e_{n} \\
& +f_{0}+2 f_{1}+2 f_{2}+\ldots+2 f_{n-1}+f_{n}  \tag{9}\\
& +g_{0}+2 g_{1}+2 g_{2}+\ldots+2 g_{n-1}+g_{n} \\
& +h_{0}+2 h_{1}+2 h_{2}+\ldots+2 h_{n-1}+h_{n}
\end{align*}
$$

This condition guarantees (6).

Now, for $i=1,2, . ., n$, let

$$
\begin{align*}
X_{i}= & \left(a_{i}-a_{i-1}\right)+\left(a_{i}^{\prime}-a_{i-1}^{\prime}\right), \\
Y_{i} & =\left(c_{i}-c_{i-1}\right)+\left(c_{i}^{\prime}-c_{i-1}^{\prime}\right), \\
X_{n+1} & =\left(b_{0}-a_{n}\right)+\left(b_{0}^{\prime}-a_{n}^{\prime}\right),  \tag{10}\\
Y_{n+1} & =\left(d_{0}-c_{n}\right)+\left(d_{0}^{\prime}-c_{n}^{\prime}\right), \\
X_{n+1+i} & =\left(b_{i}-b_{i-1}\right)+\left(b_{i}^{\prime}-b_{i-1}^{\prime}\right), \\
Y_{n+1+i} & =\left(d_{i}-d_{i-1}\right)+\left(d_{i}^{\prime}-d_{i-1}^{\prime}\right),
\end{align*}
$$

where the $X_{k}$ 's and $Y_{k}$ 's equal the sum of the membership and non-membership $k$-th spreads of $\tilde{A}^{I}$ and $\tilde{B}^{I}$, respectively. We let the spreads of the product $\tilde{C}^{I}$ satisfy:
$e_{i}-e_{i-1}=X_{i}, \quad f_{0}-e_{n}=X_{n+1}, \quad f_{i}-f_{i-1}=X_{n+1+i}$
$g_{i}-g_{i-1}=Y_{i}, \quad h_{0}-g_{n}=Y_{n+1}, \quad h_{i}-h_{i-1}=Y_{n+1+i} \quad(12)$
for all $i=1, . ., n$.
Conditions (9), (11) and (12) are $4 n+3$ equations while we have $4 n+4$ variables, which are $e_{0}, e_{1}, . ., e_{n}, f_{0}, f_{1}, . ., f_{n}, g_{0}, g_{1}, . ., g_{n}, h_{0}, h_{1}, . ., h_{n}$. Thus, we need one more equation. A logical condition is to control the distance between membership and non-membership function of the product such that the distances are proportional to those between the membership and non-membership functions of $\tilde{A}^{I}$ and $\tilde{B}^{I}$. This can be done by adding the equation $h_{0}-f_{0}=d_{0}^{\prime}-b_{0}^{\prime}+d_{0}-b_{0}$, or

$$
\begin{array}{ll} 
& f_{0}-h_{0}=Z \\
\text { where, } & Z=b_{0}+b_{0}^{\prime}-d_{0}-d_{0}^{\prime} \tag{14}
\end{array}
$$

Now, by solving Equations (9), (11), (12) and (13), we get the values of the variables that determine the product $\tilde{C}^{I}$.

Remark 2: Although using the product of the lengths in equations (11) might seem more logical at first glance, we use the sum instead, since the properties 1-5 mentioned at the beginning of this section do not hold together if we use the product, but they all hold with the sum. This is shown later in the following sections.
Theorem 2: The multiplication operation defined above: $\otimes: \mathcal{I} \mathcal{P}_{n} \times \mathcal{I} \mathcal{P}_{n} \longrightarrow \mathcal{I P} \mathcal{D}_{n}$ is well-defined.

Proof: It is easy to show that the system of linear equations (9), (11), (12) and (13) has the unique solution.

$$
\begin{align*}
h_{n} & =J_{n} \\
h_{i-1} & =h_{i}-Y_{n+1+i}, \text { for } i=n, n-1, . ., 1 \\
g_{n} & =h_{0}-Y_{n+1} \\
g_{i-1} & =g_{i}-Y_{i}, \quad \text { for } i=n, n-1, . ., 1  \tag{15}\\
f_{n} & =H_{n}+h_{0} \\
f_{i-1} & =f_{i}-X_{n+1+i}, \text { for } i=n, n-1, . ., 1 \\
e_{n} & =f_{0}-X_{n+1} \\
e_{i-1} & =e_{i}-X_{i}, \quad \text { for } i=n, n-1, . ., 1
\end{align*}
$$

where,

$$
\begin{align*}
H_{n}= & Z+\sum_{i=2}^{n+1} X_{n+i}  \tag{16}\\
J_{n}= & \frac{1}{8 n}\left[I+\sum_{i=1}^{n}(2 i-1) X_{i}+2 n X_{n+1}\right. \\
& +\sum_{i=1}^{n}(2(n+i)-1) X_{n+1+i}+\sum_{i=1}^{n}(2 i-1) Y_{i}+2 n Y_{n+1} \\
& \left.+\sum_{i=1}^{n}(2(3 n+i)-1) Y_{n+1+i}-4 n H_{n}\right] \tag{17}
\end{align*}
$$

Summary of the multiplication algorithm:
Let $\tilde{A}^{I}=\left\{\left(a_{0}, a_{1}, . ., a_{n} ; b_{0}, b_{1}, . ., b_{n}\right),\left(c_{0}, c_{1}, . ., c_{n} ; d_{0}, d_{1}\right.\right.$, ..,$\left.\left.d_{n}\right)\right\}, \tilde{B}^{I}=\left\{\left(a_{0}^{\prime}, a_{1}^{\prime}, . ., a_{n}^{\prime} ; b_{0}^{\prime}, b_{1}^{\prime}, . ., b_{n}^{\prime}\right),\left(c_{0}^{\prime}, c_{1}^{\prime}, . ., c_{n}^{\prime}\right.\right.$; $\left.\left.d_{0}^{\prime}, d_{1}^{\prime}, . ., d_{n}^{\prime}\right)\right\} . \in \mathcal{I} \mathcal{P}_{n}$.
To find the product $\tilde{A}^{I} \otimes \tilde{B}^{I}=\left\{\left(e_{0}, e_{1}, . ., e_{n} ; f_{0}, f_{1} . ., f_{n}\right)\right.$, $\left.\left(g_{0}, g_{1}, . ., g_{n} ; h_{0}, h_{1} . ., h_{n}\right)\right\}$, we first find the values of $I, X_{1}, X_{2}, . ., X_{2 n+1}, Y_{1}, Y_{2}, . ., Y_{2 n+1}, Z, H_{n}$ from equations (8), (11), (14) and (16), then we find the values of $e_{0}, e_{1}, . ., e_{n}, f_{0}, f_{1} . ., f_{n}, g_{0}, g_{1}, . ., g_{n}, h_{0}, h_{1} . ., h_{n}$ from equations (17) and (15).

Example 6: The multiplication of
$\tilde{A}^{I}=\{(-5,-2,-1,1 ; 2,2.5,4,7),(-6,-4,-1.5,0 ;$
$3,3.5,5,9)\}$ and $\tilde{B}^{I}=\{(-10,-7,-4.5,-1 ; 0,2,4,7)$,
$(-11,-7,-5,-2 ; 3,5,8,11)\}$, where $n=3$, is
$\tilde{A}^{I} \otimes \tilde{B}^{I}=\{(-16,-10,-6.5,-1 ; 1,3.5,7,13)$,
$(-81,-12,-7.5,-3 ; 5,7.5,12,19)\}$.
Note that $\Re\left(\tilde{A}^{I}\right) \cdot \Re\left(\tilde{B}^{I}\right)=\Re\left(\tilde{A}^{I} \otimes \tilde{B}^{I}\right)=-0.5$.

1) Multiplication Properties: Now, the properties mentioned at the beginning of this section are discussed. Properties 1 and 2 are clearly satisfied, while property 5 holds by construction. The rest of the properties are discussed in the following theorems.

Theorem 3: $\mathcal{I} \mathcal{P}_{n}$ has a multiplicative identity, that is the crisp number
$1=\{(1,1, \ldots, 1 ; 1,1, \ldots, 1),(1,1, \ldots, 1 ; 1,1, \ldots, 1)\}$.
Proof: Let $\tilde{N}^{I}=\left\{\left(a_{0}, a_{1}, . ., a_{n} ; b_{0}, b_{1}, . ., b_{n}\right)\right.$,
$\left(c_{0}, c_{1}, . ., c_{n} ; d_{0}, d_{1}, . ., d_{n}\right) \in \mathcal{I} \mathcal{P}_{n}$. Suppose that $\tilde{N}^{I} \otimes 1=$ $\left\{\left(e_{0}, e_{1}, . ., e_{n} ; f_{0}, f_{1}, . ., f_{n}\right),\left(g_{0}, g_{1}, . ., g_{n} ; h_{0}, h_{1}, . ., h_{n}\right)\right\}$.
Then: $I=\frac{1}{8 n}\left[a_{0}+2 a_{1}+. .+2 a_{n-1}+a_{n}+b_{0}+2 b_{1}+\ldots+\right.$ $2 b_{n-1}+b_{n}+c_{0}+2 c_{1}+. .+2 c_{n-1}+c_{n}+d_{0}+2 d_{1}+\ldots+$ $\left.2 d_{n-1}+d_{n}\right][8 n]$
$I=a_{0}+2 a_{1}+\ldots+2 a_{n-1}+a_{n}+b_{0}+2 b_{1}+\ldots+2 b_{n-1}+$ $b_{n}+c_{0}+2 c_{1}+. .+2 c_{n-1}+c_{n}+d_{0}+2 d_{1}+\ldots+2 d_{n-1}+d_{n}$. Moreover, $X_{i}=a_{i}-a_{i-1}$ for $i=1, \ldots, n, X_{n+1}=b_{0}-a_{n}$, $X_{n+1+i}=b_{i}-b_{i-1}$ for $i=1, \ldots, n, Y_{i}=c_{i}-c_{i-1}$ for $i=1, \ldots, n, Y_{n+1}=d_{0}-c_{n}$ and $Y_{n+1+i}=d_{i}-d_{i-1}$ for $i=1, \ldots, n$. Thus,

$$
\begin{aligned}
H_{n}= & Z+\sum_{i=2}^{n+1} X_{n+i} \\
= & b_{0}-d_{0}+b_{1}-b_{0}+b_{2}-b_{1}+\ldots+b_{n-1}-b_{n-2}+ \\
& b_{n}-b_{n-1}=-d_{0}+b_{n} \\
J_{n}= & \frac{1}{8 n}\left[I+X_{1}+3 X_{2}+5 X_{3}+\ldots+(2 n-1) X_{n}\right. \\
& +2 n X_{n+1}+(2 n+1) X_{n+1}+(2 n+3) X_{n+3} \\
& +(2 n+5) X_{n+4}+\ldots+(4 n-1) X_{2 n+1} \\
& +4 n d_{0}-4 n b_{n}+Y_{1}+3 Y_{2}+5 Y_{3}+\ldots \\
& +(2 n-1) Y_{n}+2 n Y_{n+1}+(6 n+1) Y_{n+1} \\
& +(6 n+3) Y_{n+3}+(6 n+5) Y_{n+4}+\ldots \\
& +(8 n-1) Y_{2 n+1} \\
= & \frac{1}{8 n}\left[I-a_{0}-2 a_{1}-2 a_{2}-\ldots-2 a_{n-1}-a_{n}\right. \\
& -b_{0}-2 b_{1}-2 b_{2}-\ldots-2 b_{n-1}-b_{n} \\
& -c_{0}-2 c_{1}-2 c_{2}-\ldots-2 c_{n-1}-c_{n}-d_{0} \\
& \left.-2 d_{1}-2 d_{2}-\ldots-2 d_{n-1}-d_{n}\right] \\
= & \frac{1}{8 n}\left[I-I+8 n d_{n}\right]=d_{n}
\end{aligned}
$$

■ Hence, $h_{n}=J_{n}=d_{n}, h_{n-1}=h_{n}-Y_{2 n+1}=d_{n}-\left(d_{n}-\right.$ $\left.d_{n-1}\right)=d_{n-1}$, and similarly we find: $h_{i}=d_{i}$ and $g_{i}=c_{i}$ for all $i=0, . ., n$. Moreover, $f_{n}=H_{n}+h_{0}=-d_{0}+b_{n}+$ $d_{0}=b_{n}, f_{n-1}=f_{n}-X_{2 n+1}=b_{n}-\left(b_{n}-b_{n-1}\right)=b_{n-1}$, and similarly we find: $f_{i}=b_{i}$ and $e_{i}=a_{i}$ for all $i=0, . ., n$. Therefore, $\tilde{N}^{I} \otimes 1=\tilde{N}^{I}$ and 1 is the multiplicative identity in $\mathcal{I} \mathcal{P}_{n}$.

Theorem 4: Let $\tilde{N}^{I}=\left\{\left(a_{0}, a_{1}, . ., a_{n} ; b_{0}, b_{1}, . ., b_{n}\right)\right.$, $\left.\left(c_{0}, c_{1}, . ., c_{n} ; d_{0}, d_{1}, . ., d_{n}\right)\right\} \in \mathcal{I} \mathcal{P}_{n}$ where $\Re\left(\tilde{N}^{I}\right) \neq 0$. Then, $\tilde{N}^{I}$ has a multiplicative inverse, $\tilde{N}^{-1}$, in the sense that $\tilde{N}^{I} \otimes \tilde{N}^{-1} \approx 1$.

## Proof: Let

$$
\begin{aligned}
g_{0} & =\Re\left(\tilde{N}^{I}\right)+\frac{1}{\Re\left(\tilde{N}^{I}\right)}-d_{n} \\
g_{i} & =g_{i-1}+\left(d_{n-i+1}-d_{n-i}\right) \text { for all } i=1, . ., n \\
h_{0} & =g_{n}+\left(d_{0}-c_{n}\right) \\
h_{i} & =h_{i-1}+\left(c_{n-i+1}-c_{n-i}\right) \text { for all } i=1, . ., n \\
e_{0} & =h_{n}+\left(c_{0}-b_{n}\right) \\
e_{i} & =e_{i-1}+\left(b_{n-i+1}-b_{n-i}\right) \text { for all } i=1, . ., n \\
f_{0} & =e_{n}+\left(b_{0}-a_{n}\right) \\
f_{i} & =f_{i-1}+\left(a_{n-i+1}-a_{n-i}\right) \text { for all } i=1, . ., n
\end{aligned}
$$

Then $\tilde{N}^{-1}=\left\{\left(e_{0}, e_{1}, . ., e_{n} ; f_{0}, f_{1}, . ., f_{n}\right),\left(g_{0}, g_{1}, . ., g_{n}\right.\right.$; $\left.\left.h_{0}, h_{1}, . ., h_{n}\right)\right\}_{\tilde{N}}$ is the multiplicative inverse of $\tilde{N}$.
Note that $\Re\left[\tilde{N}^{I} \otimes \tilde{N}^{-1}\right]=\frac{1}{8 n} I$, where it can be seen by simple calculations that $I=8 n$.
Therefore, $\Re\left[\tilde{N}^{I} \otimes \tilde{N}^{-1}\right]=\frac{1}{8 n} \cdot 8 n=1$, which completes the proof.

Definition 11: Let $\tilde{A}^{I}, \tilde{B}^{I} \in \mathcal{I} \mathcal{P}_{n}$, where $\Re\left(\tilde{B}^{I}\right) \neq 0$. We define the division of $\tilde{A}^{I}$ over $\tilde{B}^{I}$ as follows:

$$
\tilde{A}^{I} / \tilde{B}^{I}=\tilde{A}^{I} \otimes\left(\tilde{B}^{-1}\right)
$$

Example 7: Let
$\tilde{A}^{I}=\{(-3,-1,2 ; 5,7,8),(-4,-3,1 ; 7,8.5,9)\}$,
$\tilde{B}^{I}=\{(-5,-3,0 ; 2,8,9),(-7,-2.5,-1 ; 5,7,10)\}$.
Note that $\Re\left(\tilde{B}^{I}\right)=2 \neq 0$. Let $\tilde{B}^{-1}$ be $\left\{\left(e_{0}, e_{1}, . ., e_{n}\right.\right.$;
$\left.\left.f_{0}, f_{1}, . ., f_{n}\right),\left(g_{0}, g_{1}, . ., g_{n} ; h_{0}, h_{1}, . ., h_{n}\right)\right\}$. Then,
$g_{0}=\Re(\tilde{B})+\Re(\tilde{B})-10=-7.5, g_{1}=g_{0}+(10-7)=-4.5$.
Similarly, $g_{2}=-2.5, h_{0}=3.5, h_{1}=5, h_{2}=9.5$,
$e_{0}=-6.5, e_{1}=-5.5, e_{2}=0.5, f_{0}=2.5, f_{1}=5.5$, and $f_{2}=7.5$.

$$
\text { So, } \begin{aligned}
\tilde{X}^{I}= & \tilde{A}^{I} / \tilde{B}^{I} \\
= & \tilde{A}^{I} \otimes\{(-6.5,-5.5,0.5 ; 2.5,5.5,7.5), \\
& (-7.5,-4.5,-2.5 ; 3.5,5,9.5)\} \\
= & \{(-11.5,-8.5,0.5 ; 5.5,10.5,13.5) \\
& (-13.5,-9.5,-3.5 ; 8.5,11.5,16.5)\}
\end{aligned}
$$

We note that $\tilde{X}^{I} \otimes \tilde{B}^{I} \approx \tilde{A}^{I}$ since:
$\tilde{X}^{I} \otimes \tilde{B}^{I}=\{(-17,-12,0 ; 7,18,22),(-21,-12.5,-5 ;$ $13,18,26)\}$ and $\Re\left(\tilde{X}^{I} \otimes \tilde{B}^{I}\right)=3=\Re\left(\tilde{A}^{I}\right)$.

## V. The Ranking Equivalence Relation

Definition 12: Define the ranking relation $\sim$ on $\mathcal{I P}{ }_{n}$ by: For all $\tilde{N}_{1}^{I}, \tilde{N}_{2}^{I} \in \mathcal{I} \mathcal{P}_{n}: \quad \tilde{N}_{1}^{I} \sim \tilde{N}_{2}^{I} \Leftrightarrow \Re\left(\tilde{N}_{1}^{I}\right)=\Re\left(\tilde{N}_{2}^{I}\right)$. It is clear that $\sim$ is an equivalence relation. The set of all equivalence classes in $\mathcal{I P}{ }_{n}$ is denoted by

$$
\mathcal{I F} \mathcal{F}_{n}=\left\{\left[\tilde{N}^{I}\right]: \tilde{N}^{I} \in \mathcal{I} \mathcal{P}_{n}\right\} .
$$

Definition 13: The addition and multiplication on $\mathcal{I} \mathcal{F}_{n}$ are defined as follows. Let $\left[\tilde{N}_{1}^{I}\right],\left[\tilde{N}_{2}^{I}\right] \in \mathcal{I} \mathcal{F}_{n}$. Then:

$$
\begin{aligned}
{\left[\tilde{N}_{1}^{I}\right]+\left[\tilde{N}_{2}^{I}\right]=} & \left\{\tilde{A}^{I} \oplus \tilde{B}^{I}: \tilde{A}^{I}, \tilde{B}^{I} \in \mathcal{I} \mathcal{P}_{n},\right. \\
& \left.\Re\left(\tilde{A}^{I}\right) \Re\left(\tilde{N}_{1}^{I}\right) \text { and } \Re\left(\tilde{B}^{I}\right)=\Re\left(\tilde{N}_{2}^{I}\right)\right\} \\
{\left[\tilde{N}_{1}^{I}\right] \cdot\left[\tilde{N}_{2}^{I}\right]=} & \left\{\tilde{A}^{I} \otimes \tilde{B}^{I}: \tilde{A}^{I}, \tilde{B}^{I} \in \mathcal{I} \mathcal{P}_{n},\right. \\
& \left.\Re\left(\tilde{A}^{I}\right)=\Re\left(\tilde{N}_{1}^{I}\right) \text { and } \Re\left(\tilde{B}^{I}\right)=\Re\left(\tilde{N}_{2}^{I}\right)\right\}
\end{aligned}
$$

The proof of the following theorem is straight forward and similar to that shown in [14].

Theorem 5: The structure $\left(\mathcal{I} \mathcal{F}_{n},+, \cdot\right)$ is a field.
Theorem 6: The field $\left(\mathcal{I} \mathcal{F}_{n},+, \cdot\right)$ and the real numbers set with the conventional binary operations are isomorphic as rings.

Proof: Consider the map $\psi: \mathcal{I F}{ }_{n} \rightarrow \mathbb{R}$ with $\psi\left(\left[\tilde{N}^{I}\right]\right)=\Re\left(\tilde{N}^{I}\right)$. Then, for all $\left[\tilde{A}^{I}\right],\left[\tilde{B}^{I}\right] \in \mathcal{I} \mathcal{F}_{n}$, it can be easily shown that:

$$
\begin{aligned}
\psi\left(\left[\tilde{A}^{I}\right]+\left[\tilde{B}^{I}\right]\right) & =\psi\left(\left[\tilde{A}^{I}\right]\right)+\psi\left(\left[\tilde{B}^{I}\right]\right), \\
\psi\left(\left[\tilde{A}^{I}\right] \cdot\left[\tilde{B}^{I}\right]\right) & =\psi\left(\left[\tilde{A}^{I}\right]\right) \cdot \psi\left(\left[\tilde{B}^{I}\right]\right)
\end{aligned}
$$

Moreover, $\psi$ is obviously onto and one-to-one. Therefore, $\psi$ is an isomorphism and $\mathbb{R} \simeq \mathcal{I} \mathcal{F}_{n}$ as rings.

## VI. Generalization of Some Concepts with polygonal fuzzy Numbers

In this section, some definitions are proposed to generalize some concepts related to the $n$-IPFN's. These concepts are: fuzzy matrices, fuzzy-valued functions and fuzzy equalities and inequalities, which can help to implement $n$-IPFN's in many applications, such as linear systems and optimization.

## A. Matrices with polygonal fuzzy numbers

Definition 14:

- An intuitionistic polygonal fuzzy matrix $\tilde{\mathbf{M}}$ is a matrix whose entries are $n$-IPFN's.
- The set of all intuitionistic polygonal fuzzy matrices is denoted $\mathcal{M}\left(\mathcal{I} \mathcal{P} \mathcal{L}_{n}\right)$.
- The addition and multiplication of intuitionistic polygonal fuzzy matrices are similar to those on real matrices, but using the binary operations introduced in section IV on $\mathcal{I} \mathcal{P}_{n}$.
Definition 15: Let $\tilde{\mathbf{M}}=\left[\tilde{m}_{i j}^{I}\right]_{p \times q}, \tilde{\mathbf{N}}=\left[\tilde{n}_{i j}^{I}\right]_{p \times q} \in$ $\mathcal{M}\left(\mathcal{I} \mathcal{P} \mathcal{L}_{n}\right)$. Then:

1) $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$ are said to be equal, written $\tilde{\mathbf{M}}=\tilde{\mathbf{N}}$, if $\tilde{m}_{i j}^{I}=\tilde{n}_{i j}^{I}$ for all $i=1, . ., p$ and $j=1, . ., q$. Moreover, we call them equivalent, written $\tilde{\mathbf{M}} \approx \tilde{\mathbf{N}}$, if $\tilde{m}_{i j}^{I} \approx \tilde{n}_{i j}^{I}$ (have the same rank) for all $i=1, . ., p$ and $j=1, . ., q$.
2) A set of rows of $\tilde{\mathbf{M}},\left\{\tilde{\mathbf{m}}_{i_{1}}, \tilde{\mathbf{m}}_{i_{2}}, \ldots, \tilde{\mathbf{m}}_{i_{k}}\right\}$, is said to be linearly independent if the equation: $a_{1} * \tilde{\mathbf{m}}_{i_{1}} \oplus a_{2} *$ $\tilde{\mathbf{m}}_{i_{2}} \oplus \ldots \oplus a_{k} * \tilde{\mathbf{m}}_{i_{k}} \approx \tilde{\mathbf{0}}$ with $a_{1}, a_{2}, \ldots, a_{k} \in \mathbb{R}$ can only be satisfied when $a_{i}=0$ for all $i=1, . ., k$.
3) The rank of $\tilde{M}$ is the maximal number of linearly independent rows of $\tilde{M}$.
4) If $p=q$, then $\tilde{\mathbf{M}}$ is a square fuzzy matrix, and we define the determinant of $\tilde{\mathbf{M}}$, denoted $\operatorname{det}(\tilde{\mathbf{M}})$, to be a PLFN- $n$ computed in a similar way to computing the determinant of a real square matrix, but using the binary operations defined in section IV on
$\mathcal{P} \mathcal{L}_{n}$. Furthermore, it can easily be shown that if $\operatorname{det}(\tilde{\mathbf{M}}) \not \approx 0$, then $\tilde{\mathbf{M}}$ has an inverse matrix $\tilde{\mathbf{M}}^{-1}$ such that $\tilde{\mathbf{M}} * \tilde{\mathbf{M}}^{-1} \approx \tilde{\mathbf{I}}$, where the square matrix $\tilde{\mathbf{I}}$ is a fuzzy identity matrix in $\mathcal{M}\left(\mathcal{P} \mathcal{L}_{n}\right)$ which has $n$-IPFN's equivalent to 1 on the main diagonal, and all its other elements are equivalent to 0 .
Example 8: Let $\tilde{\mathbf{M}}=\left[\begin{array}{cc}\tilde{A}^{I} & \tilde{B}^{I} \\ \tilde{C}^{I} & \tilde{D}^{I}\end{array}\right]$, where
$\tilde{A}^{I}=\{(0,2,4 ; 5,6,7),(-2,0,2 ; 6,8,10)\}$,
$\tilde{B}^{I}=\{(5,6,7 ; 7,8,9),(-4,1,5 ; 8,9,11)\}$,
$\tilde{C}^{I}=\{(-2,0,2 ; 3,4,5),(-3,0,1 ; 6,8,12)\}$ and
$\tilde{D}^{I}=\{(1,4,6 ; 6,8,11),(-4,-1,1 ; 7,9,12)\}$.
Then $\operatorname{det}(\tilde{\mathbf{M}})=\left(\tilde{A}^{I} \otimes \tilde{D}^{I}\right) \ominus\left(\tilde{C}^{I} \otimes \tilde{B}^{I}\right)=$ $\{(-11,-4,2 ; 4,10,17),(-27,-16,-9 ; 9,18,31)\} \not \approx 0$.
This means that $\tilde{\mathbf{M}}$ has an inverse:
$\tilde{\mathbf{M}}{ }^{-1}=(\operatorname{det}(\tilde{\mathbf{M}}))^{-1} *\left[\begin{array}{cc}D^{I} & -\tilde{B}^{I} \\ -\tilde{C}^{I} & \tilde{A}^{I}\end{array}\right]=\left[\begin{array}{cc}\tilde{A}^{\prime I} & \tilde{B}^{\prime I} \\ \tilde{C}^{I I} & \tilde{D}^{\prime I}\end{array}\right]$
where $\tilde{A}^{\prime I}=\{(-5.5,-0.5,2.5 ; 2.5,5.5,10.5),(-5.5,-3.5$,
$-0.5 ; 7.5,8.5,8.5)\}, \tilde{B}^{\prime I}=\{(-6,-5,-5 ;-5,-5,-4)$,
$(8,2,-1 ;-2 ;-4 ;-9)\}, \tilde{C}^{\prime I}=\{(-0.5,-0.5,-1.5$;
$-2.5,-2.5,-1.5),(5.5,1.5,1.5 ;-1.5,-4.5,-11.5)\}$ and
$\tilde{D}^{\prime I}=\{(-6,-2,1 ; 2,4,7),(-3,-2,1 ; 7,8,7)\}$.
Note that
$\tilde{\mathbf{M}} * \tilde{\mathbf{M}}^{-1}=$

$$
\left.\begin{array}{c}
\left(\tilde{A}^{I} \otimes \tilde{A}^{\prime I}\right) \oplus\left(\tilde{B}^{I} \otimes \tilde{C}^{\prime I}\right) \\
\left(\tilde{C}^{I} \otimes \tilde{A}^{\prime I}\right) \oplus\left(\tilde{D}^{I} \otimes \tilde{D}^{I} \otimes \tilde{B}^{\prime I}\right) \oplus\left(\tilde{B}^{I} \otimes \tilde{D}^{\prime I}\right) \\
=\left[\begin{array}{cc}
\left.\tilde{C}^{I} \otimes \tilde{B}^{I I}\right) \oplus\left(\tilde{D}^{I} \otimes \tilde{D}^{\prime I}\right)
\end{array}\right] \\
\tilde{G}^{I}
\end{array}\right]
$$

where $\tilde{E}^{I}=\{(-11,-3,2 ; 2,7,15),(-16,-11,-2 ; 10,11,8)\}$, $\tilde{F}^{I}=\{(-16,-8,-2 ; 0,4,10),(-10,-8,-2,10 ; 12,10)\}$, $\tilde{G}^{I}=\{(-16,-6,0 ; 0,6,16),(-16,-12,-6 ; 10,12,12)\}$ and $\tilde{H}^{I}=\{(-19,-9,-2 ; 0,5,13),(-8,-7,-4 ; 12,15,16)\}$.

In fact, if we find the real matrix equivalent to $\tilde{\mathbf{M}}$ by ranking all its fuzzy numbers, we get: $\mathbf{N}=\left[\begin{array}{cc}4 & 6 \\ 3 & 5\end{array}\right]$.
Note that $\operatorname{det}(\mathbf{N})=\mathfrak{R}[\operatorname{det}(\tilde{\mathbf{M}})]=2$ and

$$
\mathbf{N}^{-1}=\left[\begin{array}{cc}
2.5 & -3 \\
-1.5 & 2
\end{array}\right]=\mathfrak{R}\left[\tilde{\mathbf{M}}^{-1}\right]
$$

where

$$
\mathfrak{R}\left[\tilde{\mathbf{M}}^{-1}\right]=\left[\begin{array}{ll}
\mathfrak{R}\left[\tilde{A}^{\prime I}\right] & \mathfrak{R}\left[\tilde{B}^{\prime I}\right] \\
\mathfrak{R}\left[\tilde{C}^{\prime I}\right] & \mathfrak{R}\left[\tilde{D}^{\prime I}\right]
\end{array}\right] .
$$

## B. Fuzzy-Valued Functions

Many applications of fuzzy numbers require dealing with fuzzy-valued functions. For instance, in fuzzy optimization, one seeks the maximum or the minimum of a fuzzy-valued function subject to some constraints. Therefore, it is important to introduce definitions of the maximum or the minimum of a set of fuzzy values or a fuzzy-valued function in a way that is compatible with our definitions in this paper.
Definition 16: Let $I$ be an arbitrary index set, and let $\tilde{S}=\left\{\tilde{a}_{i}^{I}: i \in I\right\}$ be a set of $n$-IPFN's. We define the maximum and $\underset{\tilde{S}}{ }$ minimum fuzzy values of the elements of $\tilde{S}$, denoted $\max (\tilde{S})$ and $\min (\tilde{S})$, to be the elements of $\tilde{S}$ with the maximum and minimum ranking values, respectively. In other words, if $h_{1}=\max \left\{\mathfrak{R}\left(\tilde{a}_{k}^{I}\right): \tilde{a}_{k}^{I} \in \tilde{S}\right\}$ and $h_{2}=\min \left\{\mathfrak{R}\left(\tilde{a}_{k}^{I}\right): \tilde{a}_{k}^{I} \in \tilde{S}\right\}$, then:

$$
\max (\tilde{S})=\left\{\tilde{a}_{i}^{I} \in \tilde{S}: \mathfrak{R}\left(\tilde{a}_{i}^{I}\right)=h_{1}\right\}
$$

$$
\min (\tilde{S})=\left\{\tilde{a}_{i}^{I} \in \tilde{S}: \mathfrak{R}\left(\tilde{a}_{i}^{I}\right)=h_{2}\right\}
$$

Note that $\max (\tilde{S})$ and $\min (\tilde{S})$ may have more than one element of $\tilde{S}$ if it contains more than one $n$-IPFN with the maximum or minimum ranking value.
Definition 17: Let $\tilde{f}:\left(\mathcal{I} \mathcal{P}_{n}\right)^{k} \rightarrow \mathcal{I} \mathcal{P}_{n}$, where $k \in \mathbb{N}$, be a fuzzy-valued function. Then the maximum and minimum of $\tilde{f}$ are defined by:

$$
\begin{aligned}
\max (\tilde{f}) & =\max \left\{\tilde{f}\left(\tilde{\mathbf{x}}^{I}\right): \tilde{\mathbf{x}}^{I} \in\left(\mathcal{I} \mathcal{P}_{n}\right)^{k}\right\} \\
\min (\tilde{f}) & =\min \left\{\tilde{f}\left(\tilde{\mathbf{x}}^{I}\right): \tilde{\mathbf{x}}^{I} \in\left(\mathcal{I} \mathcal{P}_{n}\right)^{k}\right\}
\end{aligned}
$$

Remark 3: $\max (\tilde{f})=-\min (-\tilde{f})$
Proof:

$$
\begin{array}{ll} 
& \tilde{y}^{I} \in \max (\tilde{f}) \\
\Leftrightarrow & \mathfrak{R}\left[\tilde{y}^{I}\right] \geq \mathfrak{R}\left[\tilde{f}\left(\tilde{\mathbf{x}}^{I}\right)\right] \quad \forall \tilde{\mathbf{x}}^{I} \in\left(\mathcal{I} \mathcal{P}_{n}\right)^{k} \\
\Leftrightarrow & -\mathfrak{R}\left[\tilde{y}^{I}\right] \leq-\mathfrak{R}\left[\tilde{f}\left(\tilde{\mathbf{x}}^{I}\right)\right] \quad \forall \tilde{\mathbf{x}}^{I} \in\left(\mathcal{I} \mathcal{P}_{n}\right)^{k} \\
\Leftrightarrow & \mathfrak{R}\left[-\tilde{y}^{I}\right] \leq \mathfrak{R}\left[-\tilde{f}\left(\tilde{\mathbf{x}}^{I}\right)\right] \quad \forall \tilde{\mathbf{x}}^{I} \in\left(\mathcal{I} \mathcal{P}_{n}\right)^{k} \\
\Leftrightarrow & -\tilde{y}^{I} \in \min (-\tilde{f}) \\
\Leftrightarrow & \tilde{y}^{I} \in-\min (-\tilde{f})
\end{array}
$$

## C. Fuzzy Equalities and Inequalities

Here, we give some properties of the equalities and inequalities between $n$-IPFN's. The proof is straightforward.
Theorem 7: For $\tilde{A}^{I}, \tilde{B}^{I}, \tilde{C}^{I} \in \mathcal{I} \mathcal{P}_{n}$, we have:

1) If $\tilde{A}^{I} \oplus \tilde{B}^{I} \approx \tilde{C}^{I}$, then $\tilde{A}^{I} \approx \tilde{C}^{I} \ominus \tilde{B}^{I}$.
2) If $\tilde{A}^{I} \oplus \tilde{B}^{I} \preceq \tilde{C}^{I}$ and $\tilde{B}^{I} \succeq 0$, then $\tilde{A}^{I} \preceq \tilde{C}^{I}$.
3) If $\tilde{A}^{I} \preceq \tilde{C}^{I}$, then $\tilde{C}^{I} \ominus \tilde{A}^{I} \succeq 0$.
4) If $\tilde{A}^{I} \preceq \tilde{C}^{I}$ and $\tilde{B}^{I} \succeq 0$, then $\tilde{A}^{I} \otimes \tilde{B}^{I} \preceq \tilde{C}^{I} \otimes \tilde{B}^{I}$.
5) If $\tilde{A}^{I} \prec 0$, then $\tilde{A}^{I} \oplus \tilde{B}^{I} \prec \tilde{B}^{I}$.
6) $\tilde{A}^{I} \preceq \tilde{B}^{I}$ if and only if $-\tilde{A}^{I} \succeq-\tilde{B}^{I}$.

## VII. Conclusion

In this paper, the $n$-polygonal fuzzy number is extended to a more general form, that is the $n$-IPFN, which is shown to give a realistic piecewise linear approximation of any general IFN in order to facilitate the calculations. A ranking function on the $n$-IPFN is introduced by applying a generalization of the ranking function with total integral value for ordinary fuzzy sets. After that, arithmetic operations on $n$-IPFN's are introduced and shown to satisfy the most important properties, these are associativity, commutativity, having identities and inverses, and preserving the ranking values. Finally, an equivalence relation is proposed on the set of all $n$-IPFN's based on the ranking values. The resulting set of equivalence classes is studied and proved to be a strong algebraic structure isomorphic to the real numbers set.

In the future, many applications of the $n$-IPFN can be studied, such as in neural networks and decision-making. Moreover, the ranking function and the arithmetic operations given in this paper can be extended to a more general type of IFN's by allowing the maximum of the membership and the minimum of the non-membership functions to not necessarily reach one and zero, respectively.

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