Modular Grad-Div Stabilization and Defect-Deferred Correction Method for the Navier-Stokes Equations

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Abstract—**We present the modular grad-div stabilization and defect-deferred correction (MGDDC) method for the incompressible Navier-Stokes equations. The new algorithm does not suffer from either solver breakdown or debilitating slow down for large values of grad-div parameters. Furthermore, it is improving two aspects of the numerical simulations for the problem. In the first step, a deferred correction method lifts the numerical order of accuracy from first order to second order in terms of the time interval. In the second step, a further defect correction is included as well, which is often included to control numerical noise. Stability analysis and optimal-order convergence of the method are proven. Numerical tests are presented to highlight the accuracy and effectiveness of the algorithms.**

Index Terms—**Navier-Stokes equations, Modular grad-div stabilization, Defect-deferred correction, Stability analysis, Error estimates.**

I. Introduction

THE incompressible Navier-Stokes equations(NSE) model Newtonian fluids, such as air flow at low speed and water flow. It is well known that numerical approximation of nonstationary Navier-Stokes equations plays an important role in the transient Navier-Stokes problems. Thus, designing efficient numerical scheme for investigating nonstationary Navier-Stokes problem has practical significance[1], [4], [7], [10], [11], [14], [15], [19], [21], [22], [23], [33], [36].

The time-dependent Navier-Stokes equations is given as follows: For a bounded regular domain $\Omega \subset R^d$ (d=2 or 3). Find $u : \Omega \times [0, T] \to R^d$, $p : \Omega \times [0, T] \to R$, satisfying:

$$\begin{cases} u_t - \mu\Delta u + (u \cdot \nabla)u + \nabla p = f, & \text{in } \Omega \times [0, T] \\ \nabla \cdot u = 0, \quad \text{in } \Omega \times [0, T] \\ u(x, 0) = u_0(x), \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega \times (0, T] \end{cases} \tag{1}$$

where u represents the flow velocity, p denotes the pressure, f is an external force, and $\mu > 0$ is the kinematic viscosity. T is the given final time.

Grad-div stabilization of fluid flow problems has drawn attention in recent years. But the new computational difficulties are introduced. For example, the condition number of the resulting linear system generally grows without bound when the grad-div parameter γ increases. In the application, modular or even large values of γ may be unavoidable[16], [24]. Furthermore, grad-div stabilization increases coupling, decreases sparsity, and makes preconditioning more difficult. Most aspects has addressed, but full resolution is still an open problem.

The classical deferred correction approach could be seen in[17]. A modification of the classical deferred correction approach was introduced by Dutt, Greengard, and Rokhlin [12]. This allowed the construction of stable and high-order accurate spectral deferred correction(SDC) methods[6], [30], [31]. In the deferred correction approach, the formal accuracy is increased to order Δt^k through a series of $k-1$ additional correction steps.

Defect correction has been proven computationally attractive in fluid applications[3], [5], [13], [17], [27], [28], [29] and references therein. The general idea of any defect correction method can be formulated as follows(see, e.g.[5], [35]). Given an operator $\tilde{G}$ to approximate $Gx = 0$, build an iterative procedure:

$$\tilde{G}x_1 = 0,$$
$$\tilde{G}x_{i+1} = \tilde{G}x_i - Gx_i, i \geq 1.$$

The choice of a particular approximation $\tilde{G}$ determines the defect corrrection method in use. In this paper, the "defect" will present viscosity, i.e. we represent using the additional parameter $H > 0$ to obtain an effective viscosity coefficient of $\mu + H$.

In this paper, we develop the MGDDC method for the incompressible Navier-Stokes equations. First, a deferred correction approach lifts the numerical order of accuracy from first order to second order in terms of time interval. Second, a further defect correction is included as well, which is included to control numerical noise or provide a significant reduction of overdiffusive effects. Finally, a modular grad-div

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stabilization method is given to reduce the storage and remain efficiency. The algorithms are simple to implement, retain the benefits of grad-div stabilization, and are resilient to solver breakdown as stabilization parameters increase. Furthermore, the second-order convergence rate can be obtained on the time interval, as well as the improved accuracy is using defect correction to reduce the artificial viscosity effects.

The rest of the paper is organized as follows. In section 2, notations and mathematical preliminaries are given. In section 3, we present the fully-discrete numerical scheme for the NSE. Stability analysis and error estimate are given in section 4. In section 5, we give our computation to verify the accuracy and effectiveness of our new algorithms. Finally, we give the conclusion and future work in section 6.

II. Notations and preliminaries

Assume the domain Ω be a convex polygon or polyhedra and let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the standard $L^2(\Omega)$ inner product and norm, respectively. Define

$$\begin{aligned} X &:= (H_0^1(\Omega))^d \\ &= \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d\times d} \text{ and } v = 0 \text{ on } \partial\Omega\}, \\ Q &:= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}. \end{aligned}$$

Let

$$\begin{aligned} b(u,v,w) :&= ((u\cdot\nabla v), w) + \frac{1}{2}((\nabla\cdot u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v). \end{aligned}$$

$$\begin{aligned} &L^\infty(0,T;X) \\ &= \{v : \{t_1,\cdots,t_M\} \to X | \ \|v\|_{L^\infty(0,T;X)} = \max_{1\le i\le M}\|v(t_i)\|_X < \infty\}. \end{aligned}$$

Then, the weak formulation of problem (1) is as follows:

$$\begin{cases} (u_t, v) + b(u;u,v) + \mu(\nabla u, \nabla v) - (p, \nabla\cdot v) \\ = (f, v), \forall v \in X \\ (q, \nabla\cdot u) = 0, \forall q \in Q. \end{cases} \tag{2}$$

Assume that $(X_h, Q_h) \subset (X, Q)$ are LBB stable finite element spaces defined on a regular mesh τ_h, where h denotes the maximum element diameter. If we consider the Taylor-Hood elements:

$$X_h = \{u_h \in C(\Omega) | u_h|_\kappa \in P_2(\kappa)^d, \ \forall\kappa \in \tau_h\},$$

$$Q_h = \{u_h \in C(\Omega) | u_h|_\kappa \in P_1(\kappa), \ \forall\kappa \in \tau_h\},$$

where $P_k(\kappa), k = 1,2$ is the space of kth-order polynomials on κ.

Next, we introduce some useful lemmas to be used in the sequel.

Lemma 2.1: The Taylor-Hood element pair $(X_h, Q_h) := (P_k, P_{k-1})$ provides the following approximation properties

$$\inf_{v\in X_h}\|u - v\| \le Ch^{k+1}\|u\|_{H^{k+1}(\Omega)^d}. \tag{3}$$

$$\inf_{v\in X_h}\|u - v\|_1 \le Ch^{k}\|u\|_{H^{k+1}(\Omega)^d}. \tag{4}$$

$$\inf_{r\in Q_h}\|p - r\| \le Ch^{k}\|p\|_{H^{k}(\Omega)^d}. \tag{5}$$

Lemma 2.2: If $u, v, w \in (H_0^1(\Omega))^d$, then

$$b(u,v,w) \le C(\Omega)\|\nabla u\| \ \|\nabla v\| \ \|w\|^{\frac{1}{2}} \ \|\nabla w\|^{\frac{1}{2}}. \tag{6}$$

$$b(u,v,w) \le C(\Omega)\|\nabla u\| \ \|\nabla v\| \ \|\nabla w\|. \tag{7}$$

The following discrete Gronwall lemma will also be utilized in the subsequent analysis.

Lemma 2.3: [2] Let l, m, and a_s, b_s, d_s, g_s, for integers $s \ge 0$, be nonnegative numbers such that

$$a_n + l\sum_{s=0}^{n} b_s \le l\sum_{s=0}^{n} d_s a_s + l\sum_{s=0}^{n} g_s + m, \quad \forall n \ge 0. \tag{8}$$

Suppose that $ld_s < 1$ for all s, and set $\rho_s \equiv (1 - ld_s)^{-1}$. Then

$$a_n + l\sum_{s=0}^{n} b_s \le \exp\left(l\sum_{s=0}^{n}\rho_s d_s\right)\left\{l\sum_{s=0}^{n} g_s + m\right\}, \quad \forall n \ge 0. \tag{9}$$

III. Numerical scheme

Denote the fully discrete solutions by u_h^n and p_h^n at time levels $t^n = n\Delta t, n = 0,1,\cdots N$ and $T = N\Delta t$. The fully discrete approximations of (1) are

Algorithm 3.1 (MGDDC scheme) Let $\Delta t > 0, f \in H^{-1}(\Omega)$.

Step 1: Given $\hat{u}_h^n \in X_h$, $n \in \{1,2,\cdots,N-1\}$, find $\hat{u}_h^{n+1} \in X_h$ and $\hat{p}_h^{n+1} \in Q_h$ satisfying

$$\begin{aligned} &(\frac{\hat{u}_h^{n+1} - \hat{u}_h^n}{\Delta t}, v_h) + (\mu + H)(\nabla\hat{u}_h^{n+1}, \nabla v_h) - (\hat{p}_h^{n+1}, \nabla\cdot v_h) \\ &+ b(\hat{u}_h^n, \hat{u}_h^{n+1}, v_h) + (\nabla\cdot\hat{u}_h^{n+1}, q_h) \\ &= (f^{n+1}, v_h) \quad \forall v_h \in X_h, \end{aligned} \tag{10}$$

Step 2: Then, given $\hat{u}_h^{n+1}$ and $\tilde{u}_h^n$, find $\tilde{u}_h^{n+1}$ satisfying

$$\begin{aligned} &(\frac{\tilde{u}_h^{n+1} - \tilde{u}_h^n}{\Delta t}, v_h) + (\mu + H)(\nabla\tilde{u}_h^{n+1}, \nabla v_h) - (\tilde{p}_h^{n+1}, \nabla\cdot v_h) \\ &+ b(\tilde{u}_h^n, \tilde{u}_h^{n+1}, v_h) + (\nabla\cdot\tilde{u}_h^{n+1}, q_h) \\ &= (\frac{f^{n+1} + f^n}{2}, v_h) + \frac{(\mu + H)\Delta t}{2}(\nabla(\frac{\hat{u}_h^{n+1} - \hat{u}^n}{2}), \nabla v_h), \\ &+ H(\nabla(\frac{\hat{u}_h^{n+1} + \hat{u}^n}{2}), \nabla v_h) + \frac{1}{2}b(\hat{u}_h^n, \hat{u}_h^{n+1}, v_h) - \frac{1}{2}b(\hat{u}_h^n, \hat{u}_h^n, v_h), \\ &- (\frac{\hat{p}^{n+1} - \hat{p}^n}{2}, \nabla\cdot v_h) \quad \forall v_h \in X_h, \end{aligned} \tag{11}$$

Step 3: Given $\tilde{u}_h^{n+1}$, find $u_h^{n+1} \in X_h$ satisfying

$$\begin{aligned} &(u_h^{n+1}, v_h) + (\beta + \gamma\Delta t)(\nabla\cdot\hat{u}_h^{n+1}, \nabla\cdot v_h) \\ &= (\tilde{u}_h^{n+1}, v_h) + \beta(\nabla\cdot\hat{u}_h^{n+1}, \nabla\cdot v_h). \end{aligned} \tag{12}$$

where $\hat{u}_h^k$, $\tilde{u}_h^k$ will denote the discrete approximations (defect step and correction step, respectively) to $u(t_k)$.

IV. Stability analysis and error estimate

In this section, we will give the theoretical result of the MGDDC scheme.

Lemma 4.1: (Stability of the defect approximation)

Let $u_h^n \in X_h$ satisfy step 1 for each $n \in \{0,1,\cdots,N-1\}$, Then exists $C > 0$ such that $\hat{u}_h^{n+1}$ satisfies

$$\begin{aligned} &\|\hat{u}_h^{n+1}\|^2 + \sum_{n=1}^{N}\|\hat{u}_h^{n+1} - \hat{u}_h^n\|^2 + \Delta t\sum_{n=1}^{N}(\mu + H)\|\nabla\hat{u}_h^{n+1}\|^2 \\ &\le \|u_h^0\|^2 + C\sum_{n=1}^{N}\frac{\Delta t}{\mu + H}\|f^{n+1}\|^2. \end{aligned} \tag{13}$$

Lemma 4.2: (Accuracy of the defect approximation)

Let the problem data be smooth enough, the discrete velocity-pressure spaces consist of continuous piecewise polynomials of degrees m and $m-1$, respectively ($m \geq 2$). Then exists $C > 0$ such that

$$\|u^{n+1} - \hat{u}_h^{n+1}\|^2 + \frac{\Delta t}{2}\sum_{n=1}^{N}(\mu + H)\|\nabla(u^{n+1} - \hat{u}_h^{n+1})\|^2 \leq C(h^{2m} + \Delta t^2 + H^2). \quad (14)$$

Theorem 4.1: (Accuracy of time derivative of the error in the defect step)

Assume that $u \in H^2(\Omega)$, $\Delta u \in L^2(0,T;L^2(\Omega))$ and $u_{tt}, u_t, u \in L^2(0,T;L^2(\Omega))$. Then exists $C > 0$ such that for any $n \in \{0,1,\cdots,N-1\}$, the discrete time derivative of the error $\frac{e^{n+1}-e^n}{\Delta t}$ satisfies

$$\|\frac{e^{n+1}-e^n}{\Delta t}\|^2 + \Delta t(\mu + H)\sum_{n=1}^{N}\|\nabla\frac{e^{n+1}-e^n}{\Delta t}\|^2 \leq C\Big((\Delta t)^2 + h^{2m} + H^2\Big). \quad (15)$$

where $e^n = u^n - \hat{u}_h^n$.

Proof. Write (2) at time t_{n+1} to obtain

$$(\frac{u^{n+1}-u^n}{\Delta t}, v) + (\mu + H)(\nabla u^{n+1}, \nabla v) + b(u^{n+1};u^{n+1},v) - (p^{n+1}, \nabla\cdot v) = (f^{n+1}, v) + H(\nabla u^{n+1}, \nabla v) + (\frac{u^{n+1}-u^n}{\Delta t}, v). \quad (16)$$

Denote $\frac{u^{n+1}-u^n}{\Delta t} - u_t^{n+1} = \rho^{n+1}$. Subtract (10) from (16) to obtain the equation for the error $e^{n+1} = u^{n+1} - \hat{u}_h^{n+1}$ Then, we have

$$(\frac{e^{n+1}-e^n}{\Delta t}, v) + (\mu + H)(\nabla e^{n+1}, \nabla v) + b(u^{n+1};u^{n+1},v) - b(\hat{u}_h^{n+1};\hat{u}_h^{n+1},v) - (p^{n+1} - \hat{p}_h^{n+1}, \nabla\cdot v) = H(\nabla u^{n+1}, \nabla v) + (\rho^{n+1}, v). \quad (17)$$

Decompose $e^i = u^i - \hat{u}_h^i = (\breve{u}^i - \hat{u}_h^i) - (\breve{u}^i - u^i) = \phi_h^i - \eta^i$ for some $\breve{u}^i \in X_h$.

Taking $v = \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t} \in X_h$ in (17) leads to

$$(\frac{e^{n+1}-e^n}{\Delta t}, \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}) + (\mu + H)(\nabla e^{n+1}, \nabla\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}) + b(u^{n+1};u^{n+1},\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}) - b(\hat{u}_h^{n+1};\hat{u}_h^{n+1},\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}) - (p^{n+1} - \hat{p}_h^{n+1}, \nabla\cdot\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}) = H(\nabla u^{n+1}, \nabla\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}) + (\rho^{n+1}, \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}). \quad (18)$$

Also, take $v = \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t} \in X_h$ in (17) at the previous time level and subtract the resulting equation from (18).

Denoting $s^{n+1} = \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}$, we obtain for $n \geq 1$.

$$\|s^{n+1}\|^2 - (s^{n+1}, s^n) + (\mu + H)\Delta t\|\nabla S^{n+1}\|^2 + b(u^{n+1};u^{n+1},s^{n+1}) - b(u^n;u^n,s^{n+1}) - b(\hat{u}_h^{n+1};\hat{u}_h^{n+1},s^{n+1}) + b(\hat{u}_h^n;\hat{u}_h^n,s^{n+1}) - (\frac{p^{n+1}-p^n}{\Delta t} - \frac{\hat{p}_h^{n+1}-\hat{p}_h^n}{\Delta t}, \nabla\cdot s^{n+1}) = H\Delta t(\nabla(\frac{u^{n+1}-u^n}{\Delta t}), \nabla s^{n+1}) + \Delta t(\frac{\rho^{n+1}-\rho^n}{\Delta t}, s^{n+1}) + \Delta t(\frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{(\Delta t)^2}, s^{n+1}) + (\mu + H)\Delta t(\nabla(\frac{\eta^{n+1}-\eta^n}{\Delta t}), \nabla s^{n+1}). \quad (19)$$

The nonlinear terms are bounded in a manner typical for the deferred correction methods for NSE(See, eg.[2]).

Using the Cauchy-Schwarz and Young's inequalities leads to

$$\|s^{n+1}\|^2 - (s^{n+1}, s^n) + (\mu + H)\Delta t\|\nabla S^{n+1}\|^2 + b(u^{n+1};u^{n+1},s^{n+1}) - b(u^n;u^n,s^{n+1}) - b(\hat{u}_h^{n+1};\hat{u}_h^{n+1},s^{n+1}) + b(\hat{u}_h^n;\hat{u}_h^n,s^{n+1}) - (\frac{p^{n+1}-p^n}{\Delta t} - \frac{\hat{p}_h^{n+1}-\hat{p}_h^n}{\Delta t}, \nabla\cdot s^{n+1}) = H\Delta t(\nabla(\frac{u^{n+1}-u^n}{\Delta t}), \nabla s^{n+1}) + \Delta t(\frac{\rho^{n+1}-\rho^n}{\Delta t}, s^{n+1}) + \Delta t(\frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{(\Delta t)^2}, s^{n+1}) + (\mu + H)\Delta t(\nabla(\frac{\eta^{n+1}-\eta^n}{\Delta t}), \nabla s^{n+1}). \quad (20)$$

Summing over the time levels, multiplying both sides by 2, and using the modified Gronwall lemma gives

$$\|s^{n+1}\|^2 + (\mu + H)\Delta t\sum_{n=1}^{N}\|\nabla s^n\|^2 \leq C(\|s^2\|^2 + O(h^{2m} + (\Delta t)^2 + H^2)). \quad (21)$$

In order to be able to finish the proof using the discrete Gronwall lemma, we will need the following bound:

$$\|s\|^2 + \Delta t\|\nabla s^1\|^2 + \Delta t\|\nabla s^2\|^2 \leq C(h^2 + (\Delta t)^2). \quad (22)$$

The details can be shown in [2]. Using the discrete Gronwall lemma and the triangle inequality, we can derive

$$\|\frac{e^{n+1}-e^n}{\Delta t}\|^2 + \Delta t(\mu + H)\sum_{n=1}^{N}\|\nabla(\frac{e^{n+1}-e^n}{\Delta t})\|^2 \leq C\Big((\Delta t)^2 + h^{2m} + H^2\Big). \quad (23)$$

where $e^i = u^i - \hat{u}^i$.

Theorem 4.2: (Stability of correction step of DDC) Let $\tilde{u}^{n+1} \in X_h$ satisfy (11) for each $n \in \{0,1,2,\cdots,N-1\}$. Then exists $C > 0$ such that $\tilde{u}^{n+1}$ satisfies

$$\begin{aligned}&\|\tilde{u}_h^{n+1}\|^2 + \Delta t(\mu + H)\sum_{n=1}^{N-1}\|\nabla\tilde{u}_h^n\|^2\\ &\leq \frac{5}{2}\Delta t(\mu + H)\sum_{n=1}^{N-1}[\Delta t^2\|\nabla(\frac{e^{n+1} - e^n}{\Delta t})\|^2 + \Delta t^2 C^2_{\nabla u_t}\\ &+ \frac{5\Delta t}{\mu + H}\sum_{n=1}^{N-1}[H^2\|\nabla\hat{u}_h^{n+1}\|^2 + \|\frac{f^{n+1} + f^n}{2}\|^2_{-1}]\\ &+ \frac{5\Delta t}{\mu + H}\sum_{n=1}^{N-1}[\Delta t^2\|\nabla\hat{u}_h^{n+1}\|^2\|(\frac{e^{n+1} - e^n}{\Delta t})^2\|\\ &+ \Delta t\|\nabla\hat{u}_h^{n+1}\|^2\Delta t C^2_{\nabla\hat{u}_t} + \Delta t\|\nabla\hat{u}_h^n\|^2\Delta t\|(\frac{e^{n+1} - e^n}{\Delta t})^2\|\\ &+ \Delta t\|\nabla\hat{u}_h^n\|^2 C^2_{\nabla\hat{u}_t}].\end{aligned} \quad (24)$$

where $C^2_{\nabla\hat{u}_t} = \|\nabla u_t(x,t)\|_{L^\infty(0,t;L^\infty(\Omega)}$.

Proof. Choosing $v = \tilde{u}_h^{n+1}$ in (11) gives

$$\begin{aligned}&(\frac{\tilde{u}_h^{n+1} - \tilde{u}_h^n}{\Delta t}, \tilde{u}_h^{n+1}) + (\mu + H)(\nabla\tilde{u}_h^{n+1}, \nabla\tilde{u}_h^{n+1}) - (\tilde{p}_h^{n+1}, \nabla\cdot\tilde{u}_h^{n+1})\\ &+ b(\tilde{u}_h^n, \tilde{u}_h^{n+1}, \tilde{u}_h^{n+1}) + (\nabla\cdot\tilde{u}_h^{n+1}, q_h)\\ &= (\frac{f^{n+1} + f^n}{2}, \tilde{u}_h^{n+1}) + \frac{(\mu + H)\Delta t}{2}(\nabla(\frac{\hat{u}_h^{n+1} - \hat{u}_h^n}{2}), \nabla\tilde{u}_h^{n+1}),\\ &+ H(\nabla(\frac{\hat{u}_h^{n+1} + \hat{u}^n}{2})\nabla\tilde{u}_h^{n+1}) + \frac{1}{2}b(\hat{u}_h^{n+1}, \hat{u}_h^{n+1}, \tilde{u}_h^{n+1})\\ &- \frac{1}{2}b(\hat{u}_h^n, \hat{u}_h^n, \tilde{u}_h^{n+1}) - (\frac{\hat{p}_h^{n+1} - \hat{p}_h^n}{2}, \nabla\cdot\tilde{u}_h^{n+1}).\end{aligned} \quad (25)$$

The nonlinear terms on the right-hand side are treated as follows

$$\begin{aligned}&\frac{1}{2}b(\hat{u}_h^{n+1}, \hat{u}_h^{n+1}, \tilde{u}_h^{n+1}) - \frac{1}{2}b(\hat{u}_h^n, \hat{u}_h^n, \tilde{u}_h^{n+1})\\ &= \frac{1}{2}b(\hat{u}_h^{n+1}, \hat{u}_h^{n+1}, \tilde{u}_h^{n+1}) - \frac{1}{2}b(\hat{u}_h^n, \hat{u}_h^n, \tilde{u}_h^{n+1})\\ &+ \frac{1}{2}b(\hat{u}_h^{n+1}, \hat{u}_h^n, \tilde{u}_h^{n+1}) - \frac{1}{2}b(\hat{u}_h^{n+1}, \hat{u}_h^n, \tilde{u}_h^{n+1})\\ &= \frac{\Delta t}{2}b(\hat{u}_h^{n+1}, \frac{\hat{u}_h^{n+1} - \hat{u}_H^n}{\Delta t}, \tilde{u}_h^{n+1}) + \frac{\Delta t}{2}b(\frac{\hat{u}_h^{n+1} - \hat{u}_h^n}{\Delta t}, \hat{u}_h^n, \tilde{u}_h^{n+1})\\ &= A + B.\end{aligned} \quad (26)$$

$$\begin{aligned}A &\leq \frac{\Delta t}{2}\|\nabla\hat{u}_h^{n+1}\|\|\nabla(\frac{\hat{u}_h^{n+1} - \hat{u}_H^n}{\Delta t})\|\|\nabla\tilde{u}_h^{n+1}\|\\ &\leq \epsilon(\mu + H)\|\nabla\tilde{u}_h^{n+1}\|^2 + \frac{\Delta t^2}{16\epsilon(\mu + H)}\|\nabla\hat{u}_h^{n+1}\|^2\|\nabla(\frac{\hat{u}_h^{n+1} - \hat{u}_h^n}{\Delta t})\|\\ &\leq \epsilon(\mu + H)\|\nabla\tilde{u}_h^{n+1}\|^2 + \frac{2\Delta t}{16\epsilon(\mu + H)}\|\nabla\hat{u}_h^{n+1}\|^2\Delta t\|\nabla(\frac{e^{n+1} - e^n}{\Delta t})\|\\ &+ \frac{2\Delta t}{16\epsilon(\mu + H)}\|\nabla\hat{u}_h^{n+1}\|^2\Delta t C^2_{\nabla\hat{u}_t}.\end{aligned} \quad (27)$$

Similarly,

$$\begin{aligned}B &\leq \epsilon(\mu + H)\|\nabla\tilde{u}_h^{n+1}\|^2 + \frac{2\Delta t}{16\epsilon(\mu + H)}\|\nabla\hat{u}_h^n\|^2\Delta t\|\nabla(\frac{e^{n+1} - e^n}{\Delta t})\|\\ &+ \frac{2\Delta t}{16\epsilon(\mu + H)}\|\nabla\hat{u}_h^{n+1}\|^2\Delta t C^2_{\nabla\hat{u}_t}.\end{aligned} \quad (28)$$

Note that $\Delta t\|\nabla\hat{u}_h^{N-1}\|^2 \leq \Delta t\sum_{n=1}^{N-1}\|\nabla\hat{u}_h^n\|^2$ and the stability bound for the defect step approximation can be utilized.

Finally, choosing $\epsilon = \frac{1}{10}$, multiplying through by $2\Delta t$ and summing over the time levels gives us the desired result.

We now have all the intermediate results that are needed for proving the accuracy of the correction step solution $\tilde{u}_h^{n+1}$.

Theorem 4.3: (Accuracy of correction step of DDC) Let the assumptions of theorems 7 and 8 be satisfied. Then exists $C > 0$ such that $\tilde{u}^{n+1}$ satisfies

$$\begin{aligned}&\|u^{n+1} - \tilde{u}_h^{n+1}\|^2 + \Delta t(\mu + H)\sum_{n=1}^{N-1}\|\nabla(u^n - \tilde{u}_h^n)\|^2\\ &\leq C(h^4 + h^2\Delta t^2 + h^4 + H^2\Delta t^2 + (\Delta t)^4).\end{aligned} \quad (29)$$

Proof. First, sum (16) at time levels t_n and t_{n+1} and divide by 2 to obtain

$$\begin{aligned}&(\frac{u^{n+1} - u^n}{\Delta t}, v) + (\mu + H)(\nabla(\frac{u^{n+1} + u^n}{2}), \nabla v)\\ &+ \frac{1}{2}b(u^{n+1}; u^{n+1}, v) + \frac{1}{2}b(u^n; u^n, v) - (\frac{p^{n+1} + p^n}{2}, \nabla\cdot v)\\ &= (\frac{f^{n+1} + f^n}{2}, v) + H(\nabla(\frac{u^{n+1} + u^n}{2}, v) - (\frac{u_t^{n+1} + u_t^n}{2}, v)\\ &- (\frac{u^{n+1} - u^n}{\Delta t}, v).\end{aligned} \quad (30)$$

Denoting $\frac{u^{n+1} - u^n}{\Delta t} - \frac{u_t^{n+1} + u_t^n}{2} = r^{n+1}$ and $Ce^{n+1} = u(t_{n+1}) - \tilde{u}^{n+1}$, we obtain

$$\begin{aligned}&(\frac{Ce^{n+1} - Ce^n}{\Delta t}, v) + (\mu + H)(\nabla(Ce^{n+1}), \nabla v) + b(u^{n+1}; u^{n+1}, v)\\ &- \frac{1}{2}b(u^{n+1}; u^{n+1}, v) - b(\tilde{u}^{n+1}; \tilde{u}^{n+1}, v)\\ &+ \frac{1}{2}b(u^n; u^n, v) + \frac{1}{2}b(\hat{u}^{n+1}; \hat{u}^{n+1}, v)\\ &- \frac{1}{2}b(\hat{u}^n; \hat{u}^n, v) - (p^{n+1} - \tilde{p}^n, \nabla\cdot v)\\ &= \frac{\Delta t}{2}(\mu + H)(\nabla(\frac{e^{n+1} - e^n}{\Delta t}), \nabla v) + \frac{H\Delta t}{2}(\nabla(\frac{u^{n+1} - u^n}{\Delta t}, \nabla v)\\ &+ H(\nabla e^{n+1}, \nabla v) + (r^{n+1}, v)\\ &+ \frac{\Delta t}{2}(\frac{p^{n+1} - p^n}{\Delta t} - \frac{\hat{p}^{n+1} - \hat{p}^n}{\Delta t}, \nabla\cdot v).\end{aligned} \quad (31)$$

Decompose $Ce^{n+1} = u^{n+1} - \tilde{u}^{n+1} = \phi_h^{n+1} - \eta^{n+1}$, $\phi_h^{n+1} \in X_h$, we now choose $v = \phi_h^{n+1} \in X_h$.

Notice that after applying the Cauchy-Schwarz and Young inequalities, the expected second order of smallness, $O(\Delta t(h + H + \Delta t))$. This follows from the results of Lemma 4.2 and Theorem 4.1.

The nonlinear terms are treated as follows

$$\begin{aligned}&b(u^{n+1}; u^{n+1}, \phi_h^{n+1}) - b(\tilde{u}^{n+1}; \tilde{u}^{n+1}, \phi_h^{n+1}) - \frac{1}{2}b(u^{n+1}; u^{n+1}, \phi_h^{n+1})\\ &+ \frac{1}{2}b(u^n; u^n, \phi_h^{n+1}) + \frac{1}{2}b(\hat{u}^{n+1}; \hat{u}^{n+1}, \phi_h^{n+1}) - \frac{1}{2}b(\hat{u}^n; \hat{u}^n, \phi_h^{n+1})\\ &= b(u^{n+1}; Ce^{n+1}, \phi_h^{n+1}) + b(Ce^{n+1}; \tilde{u}^{n+1}, \phi_h^{n+1})\\ &- \frac{1}{2}b(u^{n+1}; Ce^{n+1}, \phi_h^{n+1}) - \frac{1}{2}b(e^{n+1}; \hat{u}^{n+1}, \phi_h^{n+1})\\ &+ \frac{1}{2}b(u^n; e^n, \phi_h^{n+1}) + \frac{1}{2}b(e^n; \hat{u}^n, \phi_h^{n+1}).\end{aligned} \quad (32)$$

Adding and subtracting more nonlinear terms and writing $Ce^{n+1} = \phi_h^{n+1} - \eta^{n+1}$, we get

$$\begin{aligned}&- b(u^{n+1}; \phi_h^{n+1}, \phi_h^{n+1}) + b(u^{n+1}; \eta^{n+1}, \phi_h^{n+1}) - b(\phi_h^{n+1}; \tilde{u}^{n+1}, \phi_h^{n+1})\\ &+ b(\eta^{n+1}; \tilde{u}^{n+1}, \phi_h^{n+1}) + \frac{\Delta t}{2} b(\frac{u^{n+1} - u^n}{\Delta t}; e^n, \phi_h^{n+1})\\ &+ \frac{\Delta t}{2} b(u^{n+1}; \frac{e^{n+1} - e^n}{\Delta t}, \phi_h^{n+1}) + \frac{\Delta t}{2} b(e^{n+1}; \frac{u^{n+1} - u^n}{\Delta t}, \phi_h^{n+1})\\ &- \frac{\Delta t}{2} b(e^{n+1}; \frac{e^{n+1} - e^n}{\Delta t}, \phi_h^{n+1}) + \frac{\Delta t}{2} b(\frac{e^{n+1} - e^n}{\Delta t}; \hat{u}^n, \phi_h^{n+1}).\end{aligned} \tag{33}$$

The first of these terms is identical zero; the third term is treated by using the sharper bound by Lemma 2.2, and then it is subsumed using the Gronwall lemma. The remainder of the nonlinear terms provide the necessary second order of smallness. Then we complete the proof of Theorem 4.3.

Lemma 4.3: Consider step 3 of Algorithm MGDDC, then we have the following results

$$\begin{aligned}&||\tilde{u}_h^{n+1}||^2 = ||u_h^{n+1}||^2 + ||\tilde{u}_h^{n+1} - u_h^{n+1}||^2 + 2\gamma\Delta t||\nabla \cdot \hat{u}_h^{n+1}||^2\\ &+ \beta(||\nabla \cdot u_h^{n+1}||^2 - ||\nabla \cdot \hat{u}_h^{n+1}||^2 + ||\nabla \cdot (u_h^{n+1} - \hat{u}_h^{n+1})||^2.\end{aligned} \tag{34}$$

Proof. Let $v_h = u_h^{n+1}$ in step 3 of Algorithm MGDDC, Then, we have

$$\begin{aligned}&(\tilde{u}_h^{n+1}, u_h^{n+1}) = ||u_h^{n+1}||^2 + (\beta + \gamma\Delta t)||\nabla \cdot \hat{u}_h^{n+1}||^2\\ &- \beta(\nabla \cdot \hat{u}_h^{n+1}, \nabla \cdot u_h^{n+1}).\end{aligned} \tag{35}$$

Consider $(\tilde{u}_h^{n+1}, u_h^{n+1})$ and $-\beta(\nabla \cdot \hat{u}_h^{n+1}, \nabla \cdot u_h^{n+1})$. Use the polarization identity on each term. Then,

$$(\tilde{u}_h^{n+1}, u_h^{n+1}) = \frac{1}{2}(||\tilde{u}_h^{n+1}||^2 + ||u_h^{n+1}||^2 - ||\tilde{u}_h^{n+1} - u_h^{n+1}||^2). \tag{36}$$

$$\begin{aligned}&- \beta(\nabla \cdot \hat{u}_h^{n+1}, \nabla \cdot u_h^{n+1})\\ &= -\frac{\beta}{2}(||\nabla \cdot \hat{u}_h^{n+1}||^2 + ||\nabla \cdot u_h^{n+1}||^2 - ||\nabla \cdot (u_h^{n+1} - \hat{u}_h^{n+1})||^2.\end{aligned} \tag{37}$$

Using (36) and (37) and multiply by 2. We can easily derive

$$\begin{aligned}&||\tilde{u}_h^{n+1}||^2 = ||u_h^{n+1}||^2 + ||\tilde{u}_h^{n+1} - u_h^{n+1}||^2 + 2\gamma\Delta t||\nabla \cdot \hat{u}_h^{n+1}||^2\\ &+ \beta(||\nabla \cdot u_h^{n+1}||^2 - ||\nabla \cdot \hat{u}_h^{n+1}||^2 + ||\nabla \cdot (u_h^{n+1} - \hat{u}_h^{n+1})||^2.\end{aligned} \tag{38}$$

as needed.

Using Theorem 4.2 and Lemma 4.3, we can obtain the following theorem.

Theorem 4.4: (Stability of MGDDC) Consider the Algorithm of MGDDC, the following inequality is hold.

$$\begin{aligned}&||u_h^N||^2 + \sum_{n=1}^{N-1}(||\tilde{u}_h^{n+1} - u_h^{n+1}||^2 + ||\tilde{u}_h^{n+1} - \tilde{u}_h^n||^2 + 2\gamma\Delta t||\nabla \cdot \hat{u}_h^{n+1}||^2\\ &+ \beta(||\nabla \cdot u_h^{n+1}||^2 + ||\nabla \cdot (u_h^{n+1} - \hat{u}_h^{n+1})||^2))\\ &\le \frac{5}{2}\Delta t(\mu + H)\sum_{n=1}^{N-1}\{\Delta t^2||\nabla(\frac{e^{n+1} - e^n}{\Delta t})||^2 + \Delta t^2 C^2_{\nabla u_t}\}\\ &+ \frac{5\Delta t}{\mu + H}\sum_{n=1}^{N-1}[H^2||\nabla\hat{u}_h^{n+1}||^2 + ||\frac{f^{n+1} + f^n}{2}||^2_{-1}]\\ &+ \frac{5\Delta t}{\mu + H}\sum_{n=1}^{N-1}[\Delta t^2||\nabla\hat{u}_h^{n+1}||^2||(\frac{e^{n+1} - e^n}{\Delta t})^2||\\ &+ \Delta t||\nabla\hat{u}_h^{n+1}||^2\Delta t C^2_{\nabla\hat{u}_t} + \Delta t||\nabla\hat{u}_h^n||^2\Delta t||(\frac{e^{n+1} - e^n}{\Delta t})^2||\\ &+ \Delta t||\nabla\hat{u}_h^n||^2 C^2_{\nabla\hat{u}_t}] + ||u_h^0||^2.\end{aligned} \tag{39}$$

V. Numerical experiments

In this section, we present some numerical experiments to illustrate the theoretical results in the previous section and show the effectiveness of the new method. The software FreeFem++ developed by Hecht et al. [20] is used in our experiments.

We first consider the experiment to test the convergence rate for the problem $\Omega = [0, 1]\times[0, 1]$, where the true solution is given by

$$\begin{aligned}u_1(x, y, t) &= -\cos(2\pi x)\sin(2\pi y)\exp(\frac{-8\pi^2 t}{\tau}),\\ u_2(x, y, t) &= \sin(2\pi x)\cos(2\pi y)\exp(\frac{-8\pi^2 t}{\tau}),\\ p(x, y, t) &= -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))\exp(\frac{-8\pi^2 t}{\tau}).\end{aligned}$$

we choose $\tau = 100$, $Re = 1$, $\Delta t = 1/m$. The initial condition is given by the exact solution at $t = 0$. We set $\gamma = 1$ and $\beta = 0.2$ for the convergence and vary m between $32, 40, 48, 56$ and 64. Errors are computed for velocity and pressure in the approximate norms. The results are presented in Tables 1-4 for both the MINI and Taylor-Hood elements.

$$\begin{aligned}||u||_{L^2(0,T;L^2(\Omega))} &= \left(\Delta t\sum_{n=0}^{N}||u(t_n) - u_h^n||^2_{L^2(\Omega)}\right)^{1/2}.\\ ||u||_{L^2(0,T;H^1(\Omega))} &= \left(\Delta t\sum_{n=0}^{N}||u(t_n) - u_h^n||^2_{H^1(\Omega)}\right)^{1/2}.\\ ||p||_{L^2(0,T;L^2(\Omega))} &= \left(\Delta t\sum_{n=0}^{N}||p(t_n) - p_h^n||^2_{L^2(\Omega_i)}\right)^{1/2}.\end{aligned}$$

The corresponding error and convergence rates of u_h^n in L^2-norm, H^1-norm and p_h^n in L^2-norm of the classical finite element method are shown in Tables 1 and 2.

Table 1. Errors and rates for the MINI elements

m	$\|\|\hat{u}\|\|_{L^2}$	rate	$\|\|\hat{u}\|\|_{H^1}$	rate	$\|\|\hat{p}\|\|_{L^2}$	rate
32	1.87×10^{-4}	-	5.09×10^{-1}	-	7.08×10^{-2}	-
40	1.24×10^{-4}	1.85	4.55×10^{-1}	0.51	5.47×10^{-2}	1.16
48	8.98×10^{-5}	1.77	4.15×10^{-1}	0.51	4.44×10^{-2}	1.13
56	6.89×10^{-5}	1.71	3.84×10^{-1}	0.51	3.75×10^{-2}	1.11
64	5.51×10^{-5}	1.68	3.59×10^{-1}	0.50	3.24×10^{-2}	1.10

Table 2. Errors and rates for the Taylor-Hood elements

m	$\|\|\hat{u}\|\|_{L^2}$	rate	$\|\|\hat{u}\|\|_{H^1}$	rate	$\|\|\hat{p}\|\|_{L^2}$	rate
32	1.37×10^{-4}	-	5.10×10^{-1}	-	5.94×10^{-2}	-
40	9.85×10^{-5}	1.48	4.55×10^{-1}	0.51	4.74×10^{-2}	1.01
48	7.52×10^{-5}	1.48	4.15×10^{-1}	0.51	3.95×10^{-2}	1.01
56	5.98×10^{-5}	1.49	3.84×10^{-1}	0.51	3.38×10^{-2}	1.01
64	4.90×10^{-5}	1.49	3.59×10^{-1}	0.51	2.96×10^{-2}	1.01

Table 3. Errors and rates for the MINI elements

m	$\|u\|_{L^2}$	rate	$\|u\|_{H^1}$	rate	$\|p\|_{L^2}$	rate
32	1.20×10^{-4}	-	2.56×10^{-2}	-	7.08×10^{-2}	-
40	6.85×10^{-5}	2.52	1.83×10^{-2}	1.50	5.47×10^{-2}	1.16
48	4.33×10^{-5}	2.51	1.39×10^{-2}	1.50	4.44×10^{-2}	1.13
56	2.94×10^{-3}	2.51	1.10×10^{-2}	1.50	3.75×10^{-2}	1.11
64	2.10×10^{-3}	2.51	9.03×10^{-3}	1.50	3.24×10^{-2}	1.10

Table 4. Errors and rates for the Taylor-Hood elements

m	$\|u\|_{L^2}$	rate	$\|u\|_{H^1}$	rate	$\|p\|_{L^2}$	rate
32	1.08×10^{-5}	-	5.69×10^{-3}	-	5.94×10^{-2}	-
40	6.53×10^{-6}	2.26	3.72×10^{-3}	1.90	4.74×10^{-2}	1.01
48	4.31×10^{-6}	2.27	2.62×10^{-3}	1.93	3.95×10^{-2}	1.01
56	3.04×10^{-6}	2.28	1.94×10^{-3}	1.95	3.38×10^{-2}	1.00
64	2.24×10^{-6}	2.29	1.49×10^{-3}	1.97	2.96×10^{-2}	1.01

In tables 1 and 2, we show the errors and convergence rates for MINI elements and Taylor Hood elements respectively, the convergence rates of velocity are deteriorative. In tables 3 and 4, the desired convergence rates of our MGDDC scheme reached to second order, which agree very well with our theoretical results in our numerical analysis. Finally the velocity streamlines and pressure contours with numerical solution and exact solution are presented in Fig. 1 and Fig. 2, respectively. (mesh size $64 \times 64, T = 1, dt = 1/64$). In summary, these experiments confirm the stability, convergence and effectiveness of our MGDDC scheme. Our second test is for Re increasing. A characteristic of grad-div stabilization is an increase in pressure robustness. That is, the effect of the pressure error on the velocity error is reduced by an appropriate choice of γ. For this test, we fix $\Delta t = 1/m = 1/64$ and set $\gamma = 1$ and $\beta = 0.2$. We vary Re such that $1 \leq Re \leq 10^6$. The results are presented for Taylor-Hood elements in table 5. From the results, we can see the MGDDC implementations are in good agreement with insignificant error growth for all computed quantities.

Table 5. Errors of our proposed scheme with increasing Re

Re	$\|u\|_{L^2}$	$\|u\|_{H^1}$	$\|p\|_{L^2}$
1	1.08×10^{-5}	5.69×10^{-3}	5.94×10^{-2}
1e1	2.79×10^{-4}	1.39×10^{-2}	5.92×10^{-2}
1e2	1.92×10^{-3}	3.58×10^{-2}	5.88×10^{-2}
1e3	2.60×10^{-3}	3.58×10^{-2}	5.88×10^{-2}
1e4	2.69×10^{-3}	4.40×10^{-2}	5.88×10^{-2}
1e5	2.69×10^{-3}	4.40×10^{-2}	5.87×10^{-2}
1e6	2.69×10^{-3}	4.40×10^{-2}	5.87×10^{-2}

VI. Conclusion

In this work, We present a modular grad-div stabilization and defect-deferred correction method for the incompressible Navier-Stokes equations. The new algorithm does not suffer from either solver breakdown or debilitating slow down for large values of grad-div parameters. Furthermore,

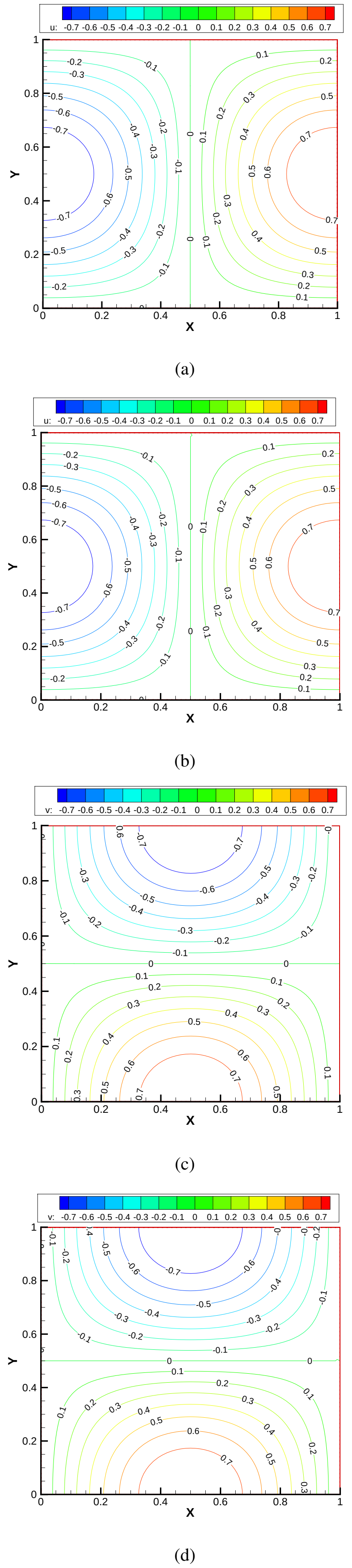
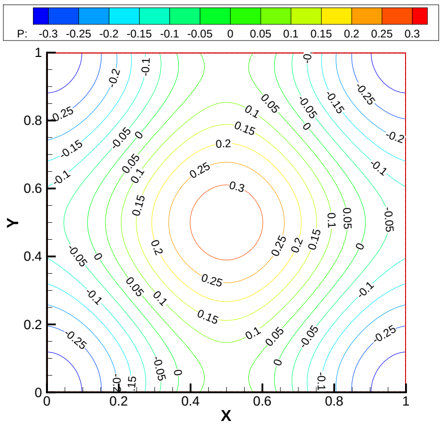
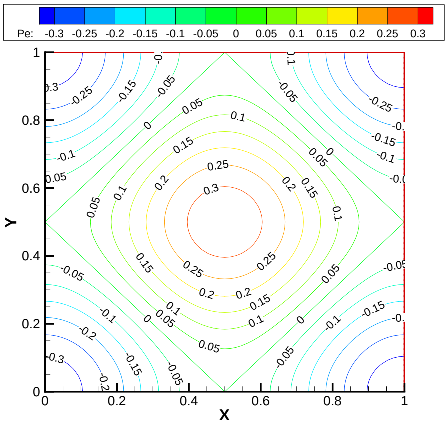


Fig. 1: Velocity contours -(a) numerical solution of u, (b) exact solution of u; (c) numerical solution of v, (d) exact solution of v.



(a)



(b)

Fig. 2: Pressure contours -(a) numerical solution of p, (b) exact solution of p.

it is improving two aspects of the numerical simulations for the problem. Modular grad-div stabilization and defect-deferred correction approaches are combined into a so-called MG-DDC method. Stability and optimal-order convergence of methods are proven. Numerical tests are presented to highlight the accuracy and effectiveness of the algorithms. Moreover, the modular grad-div stabilization and defect-deferred correction method for fluid-fluid interaction are need to further study.

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