An Approximation Method for Solving Fixed Points of General System of Variational Inequalities with Convergence Theorem and Application

Phannipa Worapun, Atid Kangtunyakarn

Abstract—In this paper, we introduce an iterative scheme $\{x_n\}$ for finding a common element of the set of fixed points of quasi-nonexpansive mapping and the solution set of general system of variational inequality problems. We prove strong convergence theorem without condition $T_\omega = (1-\omega)I + \omega T$, and demiclosed condition, where T is a quasi-nonexpansive mapping on Hilbert space. Strong convergence theorems are established in the framework of Hilbert spaces.

Index Terms—system of variational inequalities, quasi-nonexpansive mapping, nonspreading mapping, fixed point.

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I. Introduction

THROUGHOUT this paper, let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Then, mapping $T : C \to C$ is called contraction if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \le \alpha \|x - y\|, \forall x, y \in C,$$

and T is called nonexpansive if the last inequality holds for $\alpha = 1$. The set of fixed points of a mapping $T : C \to C$ is denoted by $F(T)$, that is $F(T) = \{x \in C : Tx = x\}$. Recall the mapping $T : C \to C$ is said to be quasi-nonexpansive if $F(T) \ne \emptyset$ and

$$\|Tx - p\| \le \|x - p\|,$$

for all $x \in C$ and $p \in F(T)$. Fixed point problems have been investigated in the following literature; see [1], [2] and [3].

A mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \ge \alpha \|Ax - Ay\|^2, \forall x, y \in C.$$

The variational inequality problem for a monotone mapping B is the problem of finding a point $u \in C$ satisfying

$$\langle Bu, v - u \rangle \ge 0, \forall v \in C. \tag{1}$$

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We denote the solution set of (1) by $VI(C, B)$. If B is a continuous monotone mapping then the solution set $VI(C, B)$ is always closed and convex.

The variational inequalities theory has shown itself to be an important and fascinating aspect of applied mathematics with a wide scope of uses in pure and applied sciences, etc.

F. Ma et al. [4] researched, created, and developed a new prediction-correction method for monotone variational inequalities with separable structures. Each iteration of the algorithm also permits the involved subvariational inequalities to be solved simultaneously.

A. Barbagallo and P. Mauro [5] deals with a dynamic oligopolistic market equilibrium situations true-to-life case in which capacity constraints and production excesses are permitted, moreover, the production function relies not only on the time but also on the equilibrium distribution. The authors demonstrate the equivalence between this equilibrium definition and a suitable evolutionary quasi-variational inequality. And they pursue the analysis of existence, regularity, and sensitivity of solutions.

In 1976, G.M. Korpelevich [6] introduced the extragradient method for solving the variational inequality problem in the Euclidean space R^n, as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \forall n \ge 0, \end{cases} \tag{2}$$

where A is a monotone and κ- Lipschitz continuous mapping of C into R^n and $\lambda \in \left(0, \frac{1}{\kappa}\right)$. If $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (2) converges weakly to an element in $VI(C, A)$.

Later, motivated by G.M. Korpelevich [6], N. Nadezhkina and W. Takahashi [7] and L.C. Zeng and J.C. Yao [8] proposed some iterative schemes for finding the common elements in $F(T) \cap VI(C, A)$. After that, Y. Yao and J.C. Yao [9] proposed a new iterative scheme for finding an element in $F(T) \cap VI(C, A)$ under some suitable conditions and shown the strong convergence theorem in a real Hilbert space.

The general system of variational inequalities has been widely studied and developed in the following literature (see [10]-[14]).

Recently in 2017, K. Siriyan and A. Kangtanyakarn [15] introduced a new problem about system of variational inequalities in a real Hilbert space, which involves finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \forall x \in C, \end{cases} \tag{3}$$

where $D_1, D_2, D_3 : C \to H$ be three mappings, $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0,1]$. If putting $a = 0$ in (3), we have

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)y^*, x - x^* \rangle \geq 0, \forall x \in C, \\ \langle y^* - (I - \lambda_2 D_2)z^*, x - y^* \rangle \geq 0, \forall x \in C, \\ \langle z^* - (I - \lambda_3 D_3)x^*, x - z^* \rangle \geq 0, \forall x \in C. \end{cases} \tag{4}$$

If we put $D_3 = 0$ and $x^* = z^*$, then the problem (4) reduces to the general system of variational inequality problem introduced by L.C. Ceng et al. [10].

Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping T by assuming the following conditions:

1) $T_\omega = (1-\omega)I + \omega T, \omega \in (0, \frac{1}{2})$,

2) T is demiclosed on H.

For example,M. Tian and X. Jin [16]-[17] and P.E. Mainge [18].

In 2014, K. Cheawchan and A. Kangtunyakarn [19] introduced the new method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without the conditions 1) and 2) in a framework of Hilbert space.

Inspired and motivated by K. Siriyan and A. Kangtanyakarn [15] and K. Cheawchan and A. Kangtunyakarn [19], we introduce a new method for finding a common element of the set of fixed point of system (3) in a real Hilbert space without the conditions 1) and 2). Then, we establish and prove the strong convergence theorem under some proper conditions.

II. Preliminaries

In this section, we provide some useful lemmas and remarks that will be needed to prove our main result.

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. We denote strong and weak convergence by notations "$\to$" and "$\rightharpoonup$", respectively. For every $x \in H$, there exists a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

P_C is called the metric projection of H onto C.

It is well known that metric projection P_C has the following properties:

1) P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

2) For each $x \in H$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \forall y \in C.$$

Lemma 1. [20]. Each Hilbert space H satisfies Opial's condition , i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

holds for every $y \in H$ with $x \neq y$.

Lemma 2. Let H be a real Hilbert space. Then the following results are well known:

1) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2$,

2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, $\forall x, y \in H$.

Lemma 3. [21]. Let $(E, \langle .,. \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have
$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2$.

Lemma 4. [22]. Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 1$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that :

1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 5. [23]. Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively, which $VI(C, D_1) \cap VI(C, D_2) \neq \emptyset$. Define a mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)x),$$

for every $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$ and $a \in (0,1)$. Then $F(G) = VI(C, D_1) \cap VI(C, D_2)$.

Lemma 6. [24]. Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 7. [19]. Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Then $VI(C, I - T) = F(T)$.

Remark 1. from Lemma 6 and Lemma 7, we have $F(T) = VI(C, I - T) = F(P_C(I - \lambda(I - T)))$, $\forall \lambda > 0$.

Lemma 8. [15]. Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2, D_3 : C \to H$ be three mappings. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ and $a \in [0,1]$. The following statements are equivalent

1) $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of the problem (3),

2) x^* is a fixed point of the mapping G, i.e. $x^* \in F(G)$, defined the mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)(ax + (1-a)P_C(I - \lambda_3 D_3)x)), \forall x \in C,$$

where $y^* = P_C(I - \lambda_2 D_2)(ax^* + (1-a)z^*)$ and $z^* = P_C(I - \lambda_3 D_3)x^*$.

III. Main Results

Theorem 1. Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ is a quasi-nonexpansive mapping. Let $D_1, D_2, D_3 : C \to H$ be d_1, d_2, d_3-inverse-strongly monotone mappings and $a \in [0,1]$. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ defined the mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1-a)P_C(I - \lambda_2 D_2)(ax + (1-a)P_C(I - \lambda_3 D_3)x)), \forall x \in C.$$

Assume $F = F(T) \cap F(G) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = P_C(I - \lambda_n(I-T))(\alpha_n u + (1-\alpha_n)Gx_n),$$

where $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$, $\lambda_3 \in (0, 2d_3)$.
Suppose the following condition holds:
1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}$ converges strongly to $z_0 = P_F u$.

Proof: We will present our proof in five separate steps.

Step 1. We show that $\{x_n\}$ is bounded. Let $x, y \in C$. Since D_1 is d_1-inverse strongly monotone and $\lambda_1 \in (0, 2d_1)$, we have

$$\begin{aligned} & \|(I - \lambda_1 D_1)x - (I - \lambda_1 D_1)y\|^2 \\ = \; & \|x-y\|^2 - 2\lambda_1\langle x - y, D_1 x - D_1 y\rangle \\ & + \lambda_1^2 \|D_1 x - D_1 y\|^2 \\ \leq \; & \|x-y\|^2 - 2d_1\lambda_1\|D_1 x - D_1 y\|^2 + \lambda_1^2\|D_1 x - D_1 y\|^2 \\ \leq \; & \|x-y\|^2. \end{aligned}$$

Therefore $(I - \lambda_1 D_1)$, $(I - \lambda_2 D_2)$ and $(I - \lambda_3 D_3)$ are a nonexpansive mapping. Hence $P_C(I - \lambda_1 D_1)$, $P_C(I - \lambda_2 D_2)$, and $P_C(I - \lambda_3 D_3)$ are nonexpansive mappings. From definition of the mapping G, we have G is a nonexpansive mapping. Let $x^* \in F$ and remark 1, we have

$$x^* = F(P_C(I - \lambda_n(I-T))) = F(T).$$

By defining G, we have

$$x^* = G(x^*) = P_C(I - \lambda_1 D_1)(ax^* + (1-a)P_C(I - \lambda_2 D_2)(ax^* + (1-a)P_C(I - \lambda_3 D_3)x^*)).$$

Let $M_n = \alpha_n u + (1-\alpha_n)Gx_n$. Since T is a quasi-nonexpansive mapping, we have

$$\begin{aligned} & \|TM_n - Tx^*\|^2 \\ = \; & \|TM_n - x^*\|^2 \\ = \; & \|(M_n - x^*) - (I-T)M_n\|^2 \\ = \; & \|M_n - x^*\|^2 - 2\langle M_n - x^*, (I-T)M_n\rangle \\ & + \|(I-T)M_n\|^2 \\ \leq \; & \|M_n - x^*\|^2, \end{aligned}$$

then

$$\|(I-T)M_n\|^2 \leq 2\langle M_n - x^*, (I-T)M_n\rangle. \tag{5}$$

We will show that $\{x_n\}$ is bounded by induction that is

$$\|x_n - x^*\| \leq M, \tag{6}$$

when $M = max\{\|u - x^*\|, \|x_1 - x^*\|\}$.
From definition of x_n and (5), we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ = \; & \|P_C(I - \lambda_n(I-T))M_n - x^*\|^2 \\ \leq \; & \|(I - \lambda_n(I-T))M_n - P_C x^*\|^2 \\ = \; & \|M_n - x^* - \lambda_n(I-T))M_n\|^2 \\ = \; & \|M_n - x^*\|^2 - 2\lambda_n\langle M_n - x^*, (I-T)M_n\rangle \\ & + \lambda_n^2\|(I-T)M_n\|^2. \\ \leq \; & \|M_n - x^*\|^2 - \lambda_n\|(I-T)M_n\|^2 \\ & + \lambda_n^2\|(I-T)M_n\|^2 \\ \leq \; & \|M_n - x^*\|^2 \end{aligned}$$

$$\begin{aligned} & \|x_{n+1} - x^*\| \\ \leq \; & \|M_n - x^*\| \\ = \; & \|\alpha_n(u - x^*) + (1-\alpha_n)(Gx_n - x^*)\| \\ \leq \; & \alpha_n\|u - x^*\| + (1-\alpha_n)\|Gx_n - x^*\| \\ \leq \; & \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq \; & M. \end{aligned}$$

From induction, we have (6). This implies that the sequence $\{x_n\}$ is bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ = \; & \|P_C(I - \lambda_n(I-T))M_n \\ & - P_C(I - \lambda_{n-1}(I-T))M_{n-1}\| \\ \leq \; & \|(I - \lambda_n(I-T))M_n - (I - \lambda_{n-1}(I-T))M_{n-1}\| \\ = \; & \|(M_n - M_{n-1}) - \lambda_n(I-T)M_n \\ & + \lambda_{n-1}(I-T)M_{n-1}\| \\ = \; & \|(M_n - M_{n-1}) - \lambda_n((I-T)M_n \\ & - (I-T)M_{n-1}) - (\lambda_n - \lambda_{n-1})(I-T))M_{n-1}\| \\ \leq \; & \|(M_n - M_{n-1})\| + \lambda_n\|(I-T)M_n \\ & - (I-T)M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-T)M_{n-1}\| \\ = \; & \|(\alpha_n u - \alpha_{n-1})u + (1-\alpha_n)(Gx_n - Gx_{n-1}) \\ & - (\alpha_n - \alpha_{n-1})Gx_{n-1})\| + \lambda_n\|(I-T)M_n \\ & - (I-T)M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-T)M_{n-1}\| \\ \leq \; & |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|Gx_n - Gx_{n-1}\| \\ & + |\alpha_n - \alpha_{n-1}|\|Gx_{n-1}\| + \lambda_n\|(I-T)M_n \\ & - (I-T)M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-T)M_{n-1}\| \\ \leq \; & |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\ & + |\alpha_n - \alpha_{n-1}|\|Gx_{n-1}\| + \lambda_n\|(I-T)M_n \\ & - (I-T)M_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I-T)M_{n-1}\| \end{aligned}$$

From conditions $1), 2), 3)$ and Lemma 4, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{7}$$

Step 3 We show that $\lim_{n\to\infty}\|P_C(I - \lambda_n(I-T))M_n -$

$M_n\| = 0$. From definition of x_n, we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
= & \|P_C(I - \lambda_n(I-T))M_n - x^*\|^2 \\
= & \|P_C(I - \lambda_n(I-T))M_n - P_C(I - \lambda_n(I-T))x^*\|^2 \\
\leq & \langle x_{n+1} - x^*, (I - \lambda_n(I-T))M_n - (I - \lambda_n(I-T))x^*\rangle \\
= & \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|(I - \lambda_n(I-T))M_n - x^*\|^2 \\
& -\|x_{n+1} - x^* - ((I - \lambda_n(I-T))M_n \\
& -(I - \lambda_n(I-T))x^*)\|^2) \\
= & \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 \\
& -2\lambda_n\langle M_n - x^*, (I-T)M_n\rangle + \lambda_n^2\|(I-T)M_n\|^2 \\
& -\|x_{n+1} - x^* - ((I - \lambda_n(I-T))M_n \\
& -(I - \lambda_n(I-T))x^*)\|^2) \\
\leq & \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 - \lambda_n\|(I-T)M_n\|^2 \\
& +\lambda_n^2\|(I-T)M_n\|^2 - \|x_{n+1} - x^* \\
& -((I - \lambda_n(I-T))M_n - (I - \lambda_n(I-T))x^*)\|^2) \\
\leq & \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 \\
& -\|x_{n+1} - x^* - ((I - \lambda_n(I-T))M_n \\
& -(I - \lambda_n(I-T))x^*)\|^2) \\
= & \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 \\
& -\|x_{n+1} - M_n - \lambda_n((I-T)x^* - (I-T)M_n)\|^2) \\
= & \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 - \|x_{n+1} - M_n\|^2 \\
& +2\lambda_n\langle x_{n+1} - M_n, (I-T)x^* - (I-T)M_n\rangle \\
& -\lambda_n^2\|(I-T)x^* - (I-T)M_n\|^2),
\end{aligned}$$

implies that

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
= & \|M_n - x^*\|^2 - \|x_{n+1} - M_n\|^2 \\
& +2\lambda_n\langle x_{n+1} - M_n, (I-T)x^* - (I-T)M_n\rangle \\
& -\lambda_n^2\|(I-T)x^* - (I-T)M_n\|^2.
\end{aligned}$$

From definition of M_n, then we have

$$\begin{aligned}
& \|x_{n+1} - M_n\|^2 \\
= & \|M_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& +2\lambda_n\langle x_{n+1} - M_n, (I-T)x^* - (I-T)M_n\rangle \\
& -\lambda_n^2\|(I-T)x^* - (I-T)M_n\|^2 \\
\leq & \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
& +2\lambda_n\langle x_{n+1} - M_n, (I-T)x^* - (I-T)M_n\rangle \\
& -\lambda_n^2\|(I-T)x^* - (I-T)M_n\|^2 \\
\leq & (\|x_n - x_{n+1}\|)(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
& +\alpha_n\|(u - x^*)\|^2 \\
& +2\lambda_n\langle x_{n+1} - M_n, (I-T)x^* - (I-T)M_n\rangle \\
& -\lambda_n^2\|(I-T)x^* - (I-T)M_n\|^2.
\end{aligned}$$

From condition $1), 2)$ and (7), we have $\lim_{n\to\infty}\|x_{n+1} - M_n\| = 0$. This implies that

$$\lim_{n\to\infty}\|P_C(I - \lambda_n(I-T))M_n - M_n\| = 0. \tag{8}$$

Since

$$\|M_n - x_n\| \leq \|M_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

(7) and (8), we have

$$\lim_{n\to\infty}\|M_n - x_n\| = 0. \tag{9}$$

From (9) and condition 1), we have

$$M_n - x_n = \alpha_n(u - x_n) + (1-\alpha_n)(Gx_n - x_n)$$

$$\lim_{n\to\infty}\|Gx_n - x_n\| = 0. \tag{10}$$

Step 4 We show that $\limsup_{n\to\infty}\langle u - z_0, M_n - z_0\rangle \leq 0$, where $z_0 = P_F u$. To show this inequality, take a subsequence M_{n_j} of M_n such that

$$\limsup_{n\to\infty}\langle u - z_0, M_n - z_0\rangle = \limsup_{j\to\infty}\langle u - z_0, M_{n_j} - z_0\rangle$$

Since $\{M_n\}$ is bounded sequence, without loss of generality, we may assume that $M_{n_j} \to \omega$ as $j \to \infty$, where $\omega \in C$. First, we show that $\omega \in F(T)$. From remark 7, we have $F(T) = F(P_C(I - \lambda_{n_j}(I-T)))$. We also assume that $\omega \notin F(T)$, therefore $\omega \neq P_C(I - \lambda_{n_j}(I-T))\omega$. By $M_{n_j} \to \omega$ as $j \to \infty$,(7) and Opial's property, we have

$$\begin{aligned}
& \liminf_{j\to\infty}\|M_{n_j} - \omega\| \\
< & \liminf_{j\to\infty}\|M_{n_j} - P_C(I - \lambda_{n_j}(I-T))\omega\| \\
\leq & \liminf_{j\to\infty}(\|M_{n_j} - P_C(I - \lambda_{n_j}(I-T))M_{n_j}\| + \\
& \|P_C(I - \lambda_{n_j}(I-T))M_{n_j} - P_C(I - \lambda_{n_j}(I-T))\omega\|) \\
\leq & \liminf_{j\to\infty}(\|M_{n_j} - P_C(I - \lambda_{n_j}(I-T))M_{n_j}\| + \\
& \|M_{n_j} - \omega\| + \lambda_{n_j}\|(I-T))M_{n_j} - (I-T))\omega\|) \\
= & \liminf_{j\to\infty}\|M_{n_j} - \omega\|.
\end{aligned}$$

This is a contradiction. We have

$$\omega \in F(T). \tag{11}$$

Assume that $\omega \notin F(G)$. By Opial's property and (10), we have

$$\begin{aligned}
& \liminf_{j\to\infty}\|x_{n_j} - \omega\| \\
< & \liminf_{j\to\infty}\|x_{n_j} - G(\omega)\| \\
\leq & \liminf_{j\to\infty}(\|x_{n_j} - G(x_{n_j})\| + \|G(x_{n_j}) - G(\omega)\|) \\
\leq & \liminf_{j\to\infty}(\|x_{n_j} - G(x_{n_j})\| + \|x_{n_j} - \omega\|) \\
\leq & \liminf_{j\to\infty}\|x_{n_j} - \omega\|.
\end{aligned}$$

This is a contradiction. We have

$$\omega \in F(G). \tag{12}$$

From (11) and (12), we have $\omega \in F$. Since $M_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned}
& \limsup_{n\to\infty}\langle u - z_0, M_n - z_0\rangle \\
= & \lim_{j\to\infty}\langle u - z_0, M_{n_j} - z_0\rangle \\
= & \langle u - z_0, \omega - z_0\rangle \leq 0,
\end{aligned} \tag{13}$$

where $z_0 = P_F u$.

Step 5 Finally, we show that the sequence $\{x_n\}$

converges strongly to $z_0 = P_F u$. From the definition of x_n, we have

$$\begin{aligned} & \|x_{n+1} - z_0\|^2 \\ \leq & \|M_n - z_0\|^2 \\ = & \|\alpha_n(u - z_0) + (1 - \alpha_n)(Gx_n - z_0)\|^2 \\ \leq & (1 - \alpha_n)\|Gx_n - z_0\|^2 + 2\alpha_n\langle u - z_0, M_n - z_0\rangle \\ \leq & (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, M_n - z_0\rangle. \end{aligned}$$

From (13), condition 1) and Lemma 4, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_F u$. This completes the proof.

■

IV. Application

In this section, applying our main result Theorem 1, we prove strong convergence theorems involving the set of fixed point of a nonspreading mapping. Recall the definition of such mapping. A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \forall x, y \in C.$$

Clearly, if $F(T) \neq \emptyset$ then a nonspreading is quasi-nonexpansive mapping. The above mapping is defined by F. Kohsaka and W. Takahashi [25].

The following lemma is needed to prove in application.

Lemma 9. [25]. Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.

Let $\{T_i\}_{i=1}^N$ be a finite family of (nonlinear) mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows

$$\begin{aligned} U_0 &= I, \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_3 + \alpha_3^3 I, \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\ S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned}$$

This mapping is called S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. If $\alpha_3^j = 0$ for all $i = 1, 2, \ldots, N$, then S-mapping reduced to K-mapping. S-mapping and K-mapping introduced by A. Kangtunyakarn and S. Suantai [26].

A. Kangtunyakarn proved the following result using S-mapping

Lemma 10. [27] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \ldots, N - 1$ and $\alpha_1^N \in [0, 1)$, $\alpha_2^N \in [0, 1)$, $\alpha_3^N \in [0, 1)$ for all $j = 1, 2, \ldots, N$.
Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.

Lemma 11 is a spacial case of Lemma 10.

Lemma 11. [28] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots, N - 1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$ and K is a quasi-nonexpansive mapping.

Remark 2. By using Lemma 10 and 11, we have the following theorem.

Theorem 2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \ldots, N - 1$ and $\alpha_1^N \in [0, 1)$, $\alpha_2^N \in [0, 1)$, $\alpha_3^N \in [0, 1)$ for all $j = 1, 2, \ldots, N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $D_1, D_2, D_3 : C \to H$ be d_1, d_2, d_3-inverse strongly monotone mappings and $a \in [0, 1]$. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ defined the mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x)), \forall x \in C.$$

Assume $F = \bigcap_{i=1}^N F(T_i) \bigcap F(G) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = P_C(I - \lambda_n(I - S))(\alpha_n u + (1 - \alpha_n)Gx_n),$$

where $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$, $\lambda_3 \in (0, 2d_3)$.
Suppose the following condition holds:
1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$,
2) $\sum_{n=1}^\infty \lambda_n < \infty$ and $0 < \lambda_n < 1$,
3) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}$ converges strongly to $z_0 = P_F u$.

Proof: By using Theorem 1 and Lemma 10, we obtain the conclusion. ■

Theorem 3. Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots, N - 1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Let $D_1, D_2, D_3 : C \to H$ be d_1, d_2, d_3-inverse strongly monotone mappings and $a \in [0, 1]$. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ defined the mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x)), \forall x \in C.$$

Assume $F = \cap_{i=1}^{N} F(T_i) \cap F(G) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = P_C(I - \lambda_n(I - K))(\alpha_n u + (1 - \alpha_n)Gx_n),$$

where $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$, $\lambda_3 \in (0, 2d_3)$.
Suppose the following condition holds:
1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}$ converges strongly to $z_0 = P_F u$.

Proof: By using Theorem 1 and Lemma 11, we obtain the conclusion. ∎

References

[1] H. Iiduka, Fixed point optimization algorithm and its application to network bandwidth allocation, J. Comp. Appl. Math. 236 (2012) 1733–1742.

[2] G.T. Herman, Fundamentals of Computerized Tomography: Image Reconstruction from Projections, Springer, New York, 2009.

[3] A. Beck, M. Teboulle, Fast gradient-based algorithms for constrained total variation de-noising and deblurring problems, IEEE Trans Image Process 18(2009) 2419–2434.

[4] F. Ma, M. Ni, Z Yu, A New Implementable Prediction-Correction Method for MonotoneVariational Inequalities with Separable Structure, Abstr. Appl. Anal. 2013 (SI54) 1 - 8, 2013.

[5] A. Barbagallo, P. Mauro, A Quasi-Variational Approach for the Dynamic Oligopolistic Market Equilibrium Problem, Abstract and Applied Analysis, vol. 2013.

[6] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekonomika i Mathematicheskie Metody, Inequal. Appl. 12, 747–756 (1976).

[7] N. Nadezhkina, W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz continuous monotone mappings, SIAM J. Optim. 16, 1230–1241 (2006).

[8] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fxed point problems and variational inequality problems, Taiwan. J. Math. 10, 1293–1303 (2006).

[9] Y. Yao, J.C. Yao, On modifed iterative method for nonexpansive mappings and monotone mappings, Appl. Math. Comput, 186, 1551–1558 (2006).

[10] L.C. Ceng, C.Y. Wang, J.C. Yao, Strong convergencetheorems by a relaxed extragradient method for a general system of variational inequalitied, Math. Meth. Oper. Res. 67, 735–390 (2008).

[11] L.C. Ceng, C.F Wen, Three-step Mann iterations for a general system of variational inequalities and an infinite family of nonexpansive mappings in Banach spaces, J. Inequal. Appl. 539, 27 (2013).

[12] L.C. Ceng, S.M. Guu, J.C. Yao, Hybrid viscosity CQ method for finding a common solution od a variational inequality, a general system of variational inequalities, and fixed point problem, Fixed Point Theory Appl. 313, 25 (2013).

[13] L.C. Ceng, H. Gupta, Q.H. Ansari, Implicit and explicit and explicit algorithms for a system of nonlinear variational inequalities in Banach spaces, J. Nonlinear Convex Anal. 16, 965–984 (2015).

[14] L.C. Ceng, S. Plubtieng, M.M. Wong, ,J.C. Yao, System of variational inequalities with constraints of mixed equilibria, variational inequalities, and convex minimization and fixed point problem, J. Nonlinear Convex Anal. 16, 385–421 (2015).

[15] K. Siriyan, A. Kangtunyakarn, A new general system of variational inequalities for convergence theorem and application, Numer Algor 81, 99–123 (2019).

[16] M. Tian, X. Jin, Strong convergent result for quasi-nonexpansive mappings in Hilbert spaces, Fixed Point Theory and Applications 88 (2011).

[17] M. Tian, X. Jin, A general iterative method for quasi-nonexpansive mappings in Hilbert space, Journal of Inequalities and Applications, 38 (2012).

[18] P.E. Mainge, The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, Comp. Math. Appl. 59 (2010) 74–79.

[19] K. Cheawchan, A. Kangtunyakarn, Approximation method for fixed points of nonlinear mapping and variational inequalities with application, Thai J. Math. 13(3), 653–672 (2015).

[20] Z. Opial, Weak convergence of the sequence of successive approximation of nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.

[21] M.O. Osilike, D.I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comp. Math. Appl. 40 (2000) 559–567.

[22] H.K. Xu, An iterative approach to quadric optimization, J. Optim. Theory Appl. 116 (2003) 659–678.

[23] A. Kangtunyakarn, An iterative algorithm to approximate a common element of the set of common fixed points for a finite family of strict pseudocontractions and of the set of solutions for a modified system of variational inequalities, Fixed Point Theory and Applications 143 (2013).

[24] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[25] F. Kohsaka, W.Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. 91 (2008) 166–177

[26] A. Kangtunyakarn, S. Suantai, Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis: Hybrid Systems 3 (2009) 296–309.

[27] A. Kangtunyakarn, Strong convergence of the hybrid method for a finite family of nonspreading mappings and variational inequality problem, Fixed Point Theory and Applications 188 (2012).

[28] A. Kangtunyakarn, S. Suantai, Convergence theorem of common fixed points for a family of nonspreading mapping in Hilbert space, Optim. Lett. 6 (2012) 957–961.