An Approximation Method for Solving Fixed Points of General System of Variational Inequalities with Convergence Theorem and Application

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Abstract—In this paper, we introduce an iterative scheme \( \{x_n\} \) for finding a common element of the set of fixed points of quasi-nonexpansive mapping and the solution set of general system of variational inequality problems. We prove strong convergence theorem without condition \( T_\omega = (1 - \omega)I + \omega T \), and demiclosed condition, where \( T \) is a quasi-nonexpansive mapping on Hilbert space. Strong convergence theorems are established in the framework of Hilbert spaces.

Index Terms—system of variational inequalities, quasi-nonexpansive mapping, nonspreading mapping, fixed point.

I. Introduction

THROUGHOUT this paper, let \( H \) be a real Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \). Let \( C \) be a nonempty closed convex subset of \( H \). Then, mapping \( T : C \rightarrow C \) is called contraction if there exists \( \alpha \in (0, 1) \) such that

\[
\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in C,
\]

and \( T \) is called nonexpansive if the last inequality holds for \( \alpha = 1 \). The set of fixed points of a mapping \( T : C \rightarrow C \) is denoted by \( F(T) \), that is \( F(T) = \{ x \in C : Tx = x \} \).

Recall the mapping \( T : C \rightarrow C \) is said to be quasi-nonexpansive if \( F(T) \neq \emptyset \) and

\[
\|Tx - p\| \leq \|x - p\|,
\]

for all \( x \in C \) and \( p \in F(T) \). Fixed point problems have been investigated in the following literature; see [1], [2] and [3].

A mapping \( A : C \rightarrow H \) is called \( \alpha \)-inverse-strongly monotone if there exists a positive real number \( \alpha \) such that

\[
(Ax - Ay, x - y) \geq \alpha \|Ax - Ay\|^2, \forall x, y \in C.
\]

The variational inequality problem for a monotone mapping \( B \) is the problem of finding a point \( u \in C \) satisfying

\[
\langle Bu, v - u \rangle \geq 0, \forall v \in C.
\] (1)

We denote the solution set of (1) by \( VI(C, B) \). If \( B \) is a continuous monotone mapping then the solution set \( VI(C, B) \) is always closed and convex.

The variational inequalities theory has shown itself to be an important and fascinating aspect of applied mathematics with a wide scope of uses in pure and applied sciences, etc.

F. Ma et al. [4] researched, created, and developed a new prediction-correction method for monotone variational inequalities with separable structures. Each iteration of the algorithm also permits the involved subvariational inequalities to be solved simultaneously.

A. Barbagallo and P. Mauro [5] deals with a dynamic oligopolistic market equilibrium situations true-to-life case in which capacity constraints and production excesses are permitted, moreover, the production function relies not only on the time but also on the equilibrium distribution. The authors demonstrate the equivalence between this equilibrium definition and a suitable evolutionary quasi-variational inequality. And they pursue the analysis of existence, regularity, and sensitivity of solutions.

In 1976, G.M. Korpelevich [6] introduced the extra-gradient method for solving the variational inequality problem in the Euclidean space \( \mathbb{R}^n \), as follows:

\[
\begin{align*}
\{x_0 &= x \in C, \\
y_n &= P_C(x_n - \lambda Ax_n), \\
x_{n+1} &= P_C(x_n - \lambda Ay_n), \quad \forall n \geq 0,
\end{align*}
\] (2)

where \( A \) is a monotone and \( \kappa \)-Lipschitz continuous mapping of \( C \) into \( \mathbb{R}^n \) and \( \lambda \in (0, \frac{1}{\kappa}) \). If \( VI(C, A) \) is nonempty, then the sequence \( \{x_n\} \) generated by (2) converges weakly to an element in \( VI(C, A) \).

Later, motivated by G.M. Korpelevich [6], N. Nadezhkina and W. Takahashi [7] and L.C. Zeng and J.C. Yao [8] proposed some iterative schemes for finding the common elements in \( F(T) \cap VI(C, A) \). After that, Y. Yao and J.C. Yao [9] proposed a new iterative scheme for finding an element in \( F(T) \cap VI(C, A) \) under some suitable conditions and shown the strong convergence theorem in a real Hilbert space.

The general system of variational inequalities has been widely studied and developed in the following literature (see [10]-[14]).

Recently in 2017, K. Siriyun and A. Kangtanyakarn [15] introduced a new problem about system of variational inequalities in a real Hilbert space, which involves finding \( (x^*, y^*, z^*) \in C \times C \times C \) such that
\begin{align*}
\begin{cases}
(x^* - (I - \lambda_1 D_1)(ax^* + (1 - a)y^*), x - x^*) \geq 0, \forall x \in C,
(y^* - (I - \lambda_2 D_2)(ax^* + (1 - a)x^*), x - y^*) \geq 0, \forall x \in C,
(z^* - (I - \lambda_3 D_3)x^*, x - z^*) \geq 0, \forall x \in C,
\end{cases}
\end{align*}

where \( D_1, D_2, D_3 : C \to H \) be three mappings, \( \lambda_1, \lambda_2, \lambda_3 > 0 \) and \( a \in [0, 1] \). If putting \( a = 0 \) in (3), we have
\begin{align*}
\begin{cases}
(x^* - (I - \lambda_1 D_1)y^*, x - x^*) \geq 0, \forall x \in C,
(y^* - (I - \lambda_2 D_2)x^*, y - y^*) \geq 0, \forall x \in C,
(z^* - (I - \lambda_3 D_3)x^*, x - z^*) \geq 0, \forall x \in C.
\end{cases}
\end{align*}

If we put \( D_3 = 0 \) and \( x^* = z^* \), then the problem (4) reduces to the general system of variational inequality problem introduced by L.C. Ceng et al. [10].

Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping \( T \) by assuming the following conditions:
1) \( T_\omega = (1 - \omega)I + \omega T, \omega \in (0, \frac{1}{2}) \),
2) \( T \) is demiclosed on \( H \).

For example, M. Tian and X. Jin [16]-[17] and P.E. Mainge [18].

In 2014, K. Cheawchan and A. Kangtunyakarn [19] introduced the new method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without the conditions 1) and 2) in a framework of Hilbert space.

Inspired and motivated by K. Siripan and A. Kangtunyakarn [15] and K. Cheawchan and A. Kangtunyakarn [19], we introduce a new method for finding a common element of the set of fixed point of system (3) in a real Hilbert space without the conditions 1) and 2). Then, we establish and prove the strong convergence theorem under some proper conditions.

II. Preliminaries

In this section, we provide some useful lemmas and remarks that will be needed to prove our main result.

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). We denote strong and weak convergence by notations \( \rightarrow \) and \( \rightharpoonup \), respectively. For every \( x \in H \), there exists a unique nearest point \( P_C x \in C \) such that
\[
\| x - P_C x \| \leq \| x - y \|, \forall y \in C.
\]

\( P_C \) is called the metric projection of \( H \) onto \( C \).

It is well known that metric projection \( P_C \) has the following properties:
1) \( P_C \) is firmly nonexpansive, i.e.,
\[
\| P_C x - P_C y \| \leq \| x - y \|, \forall x, y \in H.
\]
2) For each \( x \in H \),
\[
z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \geq 0, \forall y \in C.
\]

Lemma 1. [20]. Each Hilbert space \( H \) satisfies Opial's condition, i.e., for any sequence \( \{x_n\} \) with \( x_n \rightharpoonup x \), the inequality
\[
\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \|
\]
holds for every \( y \in H \) with \( x \neq y \).

Lemma 2. Let \( H \) be a real Hilbert space. Then the following results are well known:
1) \( \| x \pm y \|^2 = \| x \|^2 \pm 2 \langle x, y \rangle + \| y \|^2 \),
2) \( \| x + y \|^2 \leq \| x \|^2 + 2 \langle x, y \rangle + \| y \|^2 \), \( \forall x, y, \lambda \in H \).

Lemma 3. [21]. Let \( (E, (\cdot, \cdot)) \) be an inner product space. Then, for all \( x, y, z \in E \) and \( x, y, z \in E \) with \( \alpha, \beta, \gamma \in [0, 1] \) and \( \alpha + \beta + \gamma = 1 \), we have
\[
\| (x + y) + z \|^2 = \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 - \alpha \beta \| x - y \|^2 - \alpha \gamma \| x - z \|^2 - \beta \gamma \| y - z \|^2.
\]

Lemma 4. [22]. Let \( \{s_n\} \) be a sequence of nonnegative real numbers satisfying
\[
s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 1
\]
where \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) and \( \{\delta_n\} \) is a sequence such that:
1) \( \sum_{n=1}^{\infty} \alpha_n = \infty \),
2) \( \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} < 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then, \( \lim_{n \to \infty} s_n = 0 \).

Lemma 5. [23]. Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( D_1, D_2 : C \to H \) be \( d_1, d_2 \)-inverse strongly monotone mappings, respectively, \( \mathcal{W} \mathcal{I}(C, D_1) \cap \mathcal{W} \mathcal{I}(C, D_2) \neq \emptyset \). Define a mapping \( G : C \to C \) by
\[
G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a))P_C(I - \lambda_2 D_2)x,
\]
for every \( \lambda_1 \in (0, 2d_1), \lambda_2 \in (0, 2d_2) \) and \( a \in (0, 1) \). Then \( F(G) = \mathcal{W} \mathcal{I}(C, D_1) \cap \mathcal{W} \mathcal{I}(C, D_2) \).

Lemma 6. [24]. Let \( H \) be a real Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \) and let \( A \) be a mapping of \( C \) into \( H \). Let \( u \in C \). Then for \( \lambda > 0 \),
\[
u = P_C(I - \lambda A)u \Leftrightarrow u \in \mathcal{W} \mathcal{I}(C, A),
\]
where \( P_C \) is the metric projection of \( H \) onto \( C \).

Lemma 7. [19]. Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( T : C \to C \) be a quasi-nonexpansive mapping. Then \( \mathcal{W} \mathcal{I}(C, I - T) = F(T) \).

Remark 1. from Lemma 6 and Lemma 7, we have \( F(T) = \mathcal{W} \mathcal{I}(C, I - T) = F(P_C(I - \lambda(I - T))), \forall \lambda > 0 \).

Lemma 8. [15]. Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( D_1, D_2, D_3 : C \to H \) be three mappings. For every \( \lambda_1, \lambda_2, \lambda_3 > 0 \) and \( a \in [0, 1] \). The following statements are equivalent
1) \( (x^*, y^*, z^*) \in C \times C \times C \) is a solution of the problem (3),
2) \( x^* \) is a fixed point of the mapping \( G \), i.e. \( x^* \in F(G) \), defined the mapping \( G : C \to C \) by
\[
G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a))P_C(I - \lambda_2 D_2)(ax + \lambda_3 D_3)x, \forall x \in C,
\]
where \( y^* = P_C(I - \lambda_2 D_2)(ax^* + (1 - a)z^*) \) and \( z^* = P_C(I - \lambda_3 D_3)x^* \).
III. Main Results

Theorem 1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T : C \to C$ is a quasi-nexpansive mapping. Let $D_1, D_2, D_3 : C \to H$ be $d_1, d_2, d_3$-inverse-strongly monotone mappings and $\alpha \in [0, 1]$. For every $\lambda_1, \lambda_2, \lambda_3 > 0$ defined the mapping $G : C \to C$ by

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x)), \forall x \in C.$$  

Assume $F = F(T) \cap F(G) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = P_C(I - \lambda_n(I - T))(\alpha_n u + (1 - \alpha_n)Gx_n),$$

where $\lambda_1 \in (0, 2d_1), \lambda_2 \in (0, 2d_2), \lambda_3 \in (0, 2d_3).

Suppose the following condition holds:
1) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = PU$.

Proof: We will present our proof in five separate steps.

Step 1. We show that $\{x_n\}$ is bounded. Let $x, y \in C$. Since $D_1$ is $d_1$-inverse strongly monotone and $\lambda_1 \in (0, 2d_1)$, we have

$$\|I - \lambda_1 D_1 x - (I - \lambda_1 D_1)y\|^2$$

$$= \|x - y\|^2 - 2\lambda_1 \langle x - y, D_1 x - D_1 y\rangle$$

$$+ \lambda_1^2 \|D_1 x - D_1 y\|^2$$

$$\leq \|x - y\|^2 - 2d_1 \lambda_1 \|D_1 x - D_1 y\|^2 + \lambda_1^2 \|D_1 x - D_1 y\|^2$$

$$\leq \|x - y\|^2.$$

Therefore $(I - \lambda_1 D_1)$, $(I - \lambda_2 D_2)$ and $(I - \lambda_3 D_3)$ are nonexpansive mappings. Hence $P_C(I - \lambda_1 D_1)$, $P_C(I - \lambda_2 D_2)$, and $P_C(I - \lambda_3 D_3)$ are nonexpansive mappings. From definition of the mapping $G$, we have $G$ is a nonexpansive mapping. Let $x^* \in F$ and remark 1, we have

$$x^* = F(P_C(I - \lambda_n(I - T))) = F(T).$$

By defining $G$, we have

$$x^* = G(x^*) = P_C(I - \lambda_1 D_1)(ax^* + (1 - a)P_C(I - \lambda_2 D_2)(ax^* + (1 - a)P_C(I - \lambda_3 D_3)x^*).$$

Let $M_n = \alpha_n u + (1 - \alpha_n)Gx_n$. Since $T$ is a quasi-nexpansive mapping, we have

$$\|TM_n - Tx^*\|^2$$

$$= \|TM_n - x^*\|^2$$

$$= \|(M_n - x^*) - (I - T)M_n\|^2$$

$$= \|M_n - x^*\|^2 - 2\langle M_n - x^*, (I - T)M_n\rangle$$

$$+ \|(I - T)M_n\|^2$$

$$\leq \|M_n - x^*\|^2,$$

then

$$\|(I - T)M_n\|^2 \leq 2\langle M_n - x^*, (I - T)M_n\rangle.$$ (5)

We will show that $\{x_n\}$ is bounded by induction that is

$$\|x_n - x^*\| \leq M,$$ (6)

when $M = \max\{|\|u - x^*\|, \|x_1 - x^*\|\}$.

From definition of $x_n$ and (5), we have

$$\|x_{n+1} - x^*\|^2$$

$$= \|P_C(I - \lambda_n(I - T))M_n - x^*\|^2$$

$$\leq \|(I - \lambda_n(I - T))M_n - P_Cx^*\|^2$$

$$= \|M_n - x^* - \lambda_n(I - T)M_n\|^2$$

$$= \|M_n - x^*\|^2 - 2\lambda_n \langle M_n - x^*, (I - T)M_n\rangle + \lambda_n^2 \|(I - T)M_n\|^2$$

$$\leq \|M_n - x^*\|^2 - 2\lambda_n \|\|x_n - x^*\| + \lambda_n^2 \|(I - T)M_n\|^2$$

$$\leq \|M_n - x^*\|^2.$$

From induction, we have (6). This implies that the sequence $\{x_n\}$ is bounded.

Step 2. We show that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

$$\|x_{n+1} - x_n\|$$

$$= \|P_C(I - \lambda_n(I - T))M_n - P_C(I - \lambda_{n-1}(I - T))M_{n-1}\|$$

$$\leq \|(I - \lambda_n(I - T))M_n - (I - \lambda_{n-1}(I - T))M_{n-1}\|$$

$$= \|\|M_n - M_{n-1}\| - \lambda_n(I - T)M_n + \lambda_{n-1}(I - T)M_{n-1}\|$$

$$\leq \|\|M_n - M_{n-1}\| + \lambda_n \|I - T\|M_n\| + \lambda_{n-1}(I - T)M_{n-1}\|$$

$$= \|(M_n - M_{n-1})\| + \lambda_n \|I - T\|M_n\|$$

$$\leq \|\|x_n - x_{n-1}\| + (1 - \alpha_n)\|Gx_n - x_{n-1}\| + \lambda_n \|I - T\|M_n\|.$$

$$\leq \alpha_n \|x_n - x_{n-1}\| + (1 - \alpha_n)\|Gx_n - x_{n-1}\| + \lambda_n \|\|x_n - x_{n-1}\| + \lambda_n \|I - T\|M_n\|.$$

From conditions 1, 2, 3) and Lemma 4, we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$ (7)
$M_n\| = 0$. From definition of $x_n$, we have

$$
\|x_{n+1} - x^*\|^2 = \|P_C(I - \lambda_n(I - T))M_n - x^*\|^2
= \|P_C(I - \lambda_n(I - T))M_n - P_C(I - \lambda_n(I - T))x^*\|^2
\leq \langle x_{n+1} - x^*, (I - \lambda_n(I - T))M_n - (I - \lambda_n(I - T))x^* \rangle
= \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 - \|x_{n+1} - x^* - (I - \lambda_n(I - T))M_n - (I - \lambda_n(I - T))x^*\|^2)
= \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 - 2\lambda_n(M_n - x^* - (I - T)x^* + (I - T)M_n) + \lambda_n^2\|I - T\|M_n\|^2)
= \frac{1}{2}(\|x_{n+1} - x^*\|^2 + \|M_n - x^*\|^2 - \|x_{n+1} - x^* - (I - \lambda_n(I - T))M_n - (I - \lambda_n(I - T))x^*\|^2)\]

(7) and (8), we have

$$
\lim_{n \to \infty} \|M_n - x_n\| = 0. \tag{9}
$$

From (9) and condition 1), we have

$$
M_n - x_n = \alpha_n(u - x_n) + (1 - \alpha_n)(Gx_n - x_n)
\lim_{n \to \infty} \|Gx_n - x_n\| = 0. \tag{10}
$$

Step 4 We show that $\limsup_{n \to \infty} \langle u - z_0, M_n - z_0 \rangle \leq 0$, where $z_0 = P_{F\Omega}u$. To show this inequality, take a subsequence $M_{n_j}$ of $M_n$ such that

$$
\limsup_{j \to \infty} (u - z_0, M_{n_j} - z_0) = \limsup_{j \to \infty} (u - z_0, M_n - z_0)
$$

Since $\{M_n\}$ is bounded sequence, without loss of generality, we may assume that $M_{n_j} \to \omega$ as $j \to \infty$, where $\omega \in C$. First, we show that $\omega \in F(T)$. From remark 7, we have $F(T) = F(P_C(I - \lambda_n(I - T)))$. We also assume that $\omega \notin F(T)$, therefore $\omega \notin P_C(I - \lambda_n(I - T))\omega$. By $M_{n_j} \to \omega$ as $j \to \infty$, (7) and Opial’s property, we have

$$
\liminf_{j \to \infty} \|M_{n_j} - \omega\|
< \liminf_{j \to \infty} \|M_{n_j} - P_C(I - \lambda_n(I - T))\omega\|
\leq \liminf_{j \to \infty} \|P_C(I - \lambda_n(I - T))M_{n_j} - P_C(I - \lambda_n(I - T))\omega\|
\leq \liminf_{j \to \infty} \|M_{n_j} - P_C(I - \lambda_n(I - T))\omega\| + \|M_{n_j} - \omega\| + \lambda_n\|I - T\|M_{n_j} - (I - T)\omega\|
= \liminf_{j \to \infty} \|M_{n_j} - \omega\|.
$$

This is a contradiction. We have

$$
\omega \in F(T). \tag{11}
$$

Assume that $\omega \notin F(G)$. By Opial’s property and (10), we have

$$
\liminf_{j \to \infty} \|x_{n_j} - \omega\|
< \liminf_{j \to \infty} \|x_{n_j} - G(\omega)\|
\leq \liminf_{j \to \infty} \|x_{n_j} - G(x_{n_j})\| + \|G(x_{n_j}) - G(\omega)\|
\leq \liminf_{j \to \infty} \|x_{n_j} - G(x_{n_j})\| + \|x_{n_j} - \omega\|
\leq \liminf_{j \to \infty} \|x_{n_j} - \omega\|.
$$

This is a contradiction. We have

$$
\omega \in F(G). \tag{12}
$$

From (11) and (12), we have $\omega \in F$. Since $M_{n_j} \to \omega$ as $j \to \infty$, we have

$$
\sup_{n \to \infty} (u - z_0, M_{n_j} - z_0)
= \lim_{j \to \infty} (u - z_0, M_{n_j} - z_0)
\leq (u - z_0, \omega - z_0) \leq 0, \tag{13}
$$

where $z_0 = P_{F\Omega}u$.

Step 5 Finally, we show that the sequence $\{x_n\}$
converges strongly to \( z_0 = P_F u \). From the definition of \( x_n \), we have
\[
\|x_{n+1} - z_0\|^2 \\
\leq \|M_n - z_0\|^2 \\
= \|\alpha_n(u - z_0) + (1 - \alpha_n)(Gx_n - z_0)\|^2 \\
\leq (1 - \alpha_n)(\|Gx_n - z_0\|^2 + 2\alpha_n\langle u - z_0, M_n - z_0 \rangle) \\
\leq (1 - \alpha_n)(\|x_n - z_0\|^2 + 2\alpha_n(u - z_0, M_n - z_0)).
\]

From (13), condition 1) and Lemma 4, we can conclude that the sequence \( \{x_n\} \) converges strongly to \( z_0 = P_F u \). This completes the proof.

IV. Application

In this section, applying our main result Theorem 1, we prove strong convergence theorems involving the set of fixed point of a nonspreading mapping. Recall the definition of such mapping. A mapping \( T : C \rightarrow C \) is called nonspreading if
\[
\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \forall x, y \in C.
\]
Clearly, if \( F(T) \neq \emptyset \) then a nonspreading is quasi-nonexpansive mapping. The above mapping is defined by F. Kohsaka and W. Takahashi [25].

The following lemma is needed to prove in application.

***Lemma 9.*** [25] Let \( H \) be a Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \) and let \( S \) be a nonspreading mapping of \( C \) into itself. Then \( F(S) \) is closed and convex.

Let \( \{T_i\}_{i=1}^N \) be a finite family of (nonlinear) mappings of \( C \) into itself. For each \( j = 1, 2, \ldots, N \), let \( \alpha_j = (\alpha_j^1, \alpha_j^2, \alpha_j^3) \in I \times I \times I \) where \( I \in [0, 1] \) and \( \alpha_j^1 + \alpha_j^2 + \alpha_j^3 = 1 \). Define the mapping \( S : C \rightarrow C \) as follows
\[
U_0 = I, \\
U_1 = \alpha_1^1 T_1 U_0 + \alpha_1^2 U_0 + \alpha_1^3 I, \\
U_2 = \alpha_2^1 T_2 U_1 + \alpha_2^2 U_1 + \alpha_2^3 I, \\
U_3 = \alpha_3^1 T_3 U_2 + \alpha_3^2 U_2 + \alpha_3^3 I, \\
\cdots \\
U_{N-1} = \alpha_N^{N-1} T_{N-1} U_{N-2} + \alpha_N^{N-2} U_{N-2} + \alpha_N^{N-1} I, \\
S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\]

This mapping is called S-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \alpha_1, \alpha_2, \ldots, \alpha_N \). If \( \alpha_j^1 = 0 \) for all \( i = 1, 2, \ldots, N \), then S-mapping reduced to K-mapping. S-mapping and K-mapping introduced by A. Kangtunyakarn and S. Suantai [26].

A. Kangtunyakarn proved the following result using S-mapping

***Lemma 10.*** [27] Let \( H \) be a Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \) and let \( \{T_i\}_{i=1}^N \) be a finite family of nonspreading mappings of \( C \) into \( C \) with \( \cap_{i=1}^N F(T_i) \neq \emptyset \), and let \( \alpha_j = (\alpha_j^1, \alpha_j^2, \alpha_j^3) \in I \times I \times I \) where \( I \in [0, 1] \), \( \alpha_j^1 + \alpha_j^2 + \alpha_j^3 = 1 \), \( \alpha_j^1, \alpha_j^3 \in (0, 1) \) for all \( j = 1, 2, \ldots, n \) and \( \alpha_j^1 \in [0, 1], \alpha_j^2 \in [0, 1], \alpha_j^3 \in [0, 1] \) for all \( j = 1, 2, \ldots, N \).

Let \( S \) be the mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \alpha_1, \alpha_2, \ldots, \alpha_N \). Then \( F(S) = \cap_{i=1}^N F(T_i) \) and \( S \) is a quasi-nonexpansive mapping.

**Lemma 11.** [28] Let \( H \) be a Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \) and let \( \{T_i\}_{i=1}^N \) be a finite family of nonspreading mappings of \( C \) into \( C \) with \( \cap_{i=1}^N F(T_i) \neq \emptyset \), and let \( \alpha_1, \lambda_2, \ldots, \lambda_N \) be real numbers such that \( 0 < \lambda_i < 1 \) for every \( i = 1, 2, \ldots, N - 1 \) and \( 0 < \lambda_N \leq 1 \). Let \( K \) be the K-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \alpha_1, \lambda_2, \ldots, \lambda_N \). Then \( F(K) = \cap_{i=1}^N F(T_i) \) and \( K \) is a quasi-nonexpansive mapping.

Remark 2. By using Lemma 10 and 11, we have the following theorem.

**Theorem 2.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \{T_i\}_{i=1}^N \) be a finite family of nonspreading mappings of \( C \) into \( C \) with \( \cap_{i=1}^N F(T_i) \neq \emptyset \), and let \( \alpha_0 = (\alpha_0^1, \alpha_0^2, \alpha_0^3) \in I \times I \times I \) where \( I \in [0, 1] \), \( \alpha_0^1 + \alpha_0^2 + \alpha_0^3 = 1 \), \( \alpha_0^1, \alpha_0^2 \in (0, 1) \) for all \( j = 1, 2, \ldots, N - 1 \) and \( \alpha_0^3 \in [0, 1) \) for all \( j = 1, 2, \ldots, N \) for \( a \in [0, 1] \). For every \( \lambda_1, \lambda_2, \lambda_3 > 0 \) defined the mapping \( G : C \rightarrow C \) by
\[
G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x), \forall x \in C.
\]
Assume \( F = \cap_{i=1}^N F(T_i) \cap F(G) \neq \emptyset \). Suppose that \( x_1, u \in C \) and let \( \{x_n\} \) be a sequence generated by
\[
x_{n+1} = P_C(I - \lambda_n (I - S))(\alpha_n u + (1 - \alpha_n) Gx_n),
\]
where \( \lambda_1 \in (0, 2d_1), \lambda_2 \in (0, 2d_2), \lambda_3 \in (0, 2d_3) \).

Suppose the following condition holds:
1) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \),
2) \( \sum_{n=1}^{\infty} \lambda_n < \infty \) and \( 0 < \lambda_n < 1 \),
3) \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \).

Then \( \{x_n\} \) converges strongly to \( z_0 = P_F u \).

Proof: By using Theorem 1 and Lemma 10, we obtain the conclusion.

**Theorem 3.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( \{T_i\}_{i=1}^N \) be a finite family of nonspreading mappings of \( C \) into \( C \) with \( \cap_{i=1}^N F(T_i) \neq \emptyset \), and let \( \lambda_1, \lambda_2, \ldots, \lambda_N \) be real numbers such that \( 0 < \lambda_i < 1 \) for every \( i = 1, 2, \ldots, N - 1 \) and \( 0 < \lambda_N \leq 1 \). Let \( K \) be the K-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \lambda_1, \lambda_2, \ldots, \lambda_N \). Let \( D_1, D_2, D_3 \) : \( C \rightarrow H \) be \( d_1, d_2, d_3 \)-inverse strongly monotone mappings and \( a \in [0, 1] \). For every \( \lambda_1, \lambda_2, \lambda_3 > 0 \) defined the mapping \( G : C \rightarrow C \) by
\[
G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)(ax + (1 - a)P_C(I - \lambda_3 D_3)x), \forall x \in C.
\]
Assume $F = \cap_{i=1}^{n} F(T_i) \cap F(G) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = P_{C}(I - \lambda_n(I - K))(\alpha_n u + (1 - \alpha_n)Gx_n),$$

where $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$, $\lambda_3 \in (0, 2d_3)$.

Suppose the following condition holds:

1) $\lim_{n \to \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$,
2) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{F}u$.

Proof: By using Theorem 1 and Lemma 11, we obtain the conclusion.

References