

The Extinction and Persistence of a Stochastic SIQR Epidemic Model with Vaccination Effect

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Abstract—In this paper, a stochastic SIQR epidemic model with vaccination effect is investigated. Our purpose is to study the dynamical behaviors of the model. First, the existence and uniqueness of the positive solution of the model is proved. Then, the dynamic properties of the model, such as extinction and persistence, are analyzed by regarding certain conditions on the parameters. Some sufficient conditions for extinction and persistence are given by using Itô's formula and martingale methods. The results show that low-intensity white noise may cause disease to spread, while high-intensity white noise may drive infective, quarantined individuals to extinction. Finally, numerical simulations are illustrated to support the theoretical analysis.

Index Terms—Stochastic SIQR epidemic model; Vaccination effect; Extinction; Persistence; Itô's formula

I. INTRODUCTION

INFECTIOUS diseases have ranked with wars and famine as major challenges to human and society for centuries, affecting the social stability and economic development. COVID-19 has spread widely around the world since 2020, causing huge economic losses to countries around the world. Therefore, studying the spreading laws of infectious diseases has important practical significance. Dynamics of infectious diseases is mainly engaged in theoretical research on the spread and development of diseases with the purpose to trace factors that are contribute to their occurrence, so as to more effectively control epidemic of the disease.

Since the concept of component model of infectious disease was proposed, more and more scholars have been interested in the dynamics of the infectious disease model. In the past decades, many epidemic models such as SIR model [1-4], SIRS model [5-10], SIRI model [11], SIS model [12-13], SEIS model [14], SIQS model [15-16] and SEIRS model [17-18] characterizing the spread law of infectious diseases in the real environment were investigated by many scholars. In addition, some mathematical models of HIV/AIDS have also been extensively studied [30-33]. On the other hand, vaccination is one of the important means to

control the spread of epidemics, so it is also necessary to add vaccination effect to infectious disease models [19-25]. Vaccination can enable the vaccinated person to obtain a permanent or temporary immunity. If the immunity is permanent, it means that the susceptible population is restored to health. Literature [26] studied the following infectious disease model with vaccination effect:

$$\begin{cases} \frac{dS(t)}{dt} = (1-q)A - \beta S(t)I(t) - \mu S(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + \varepsilon_1 + \gamma + \delta)I(t), \\ \frac{dQ(t)}{dt} = \delta I(t) - (\mu + \varepsilon_2 + \lambda)Q(t), \\ \frac{dR(t)}{dt} = qA + \gamma I(t) + \lambda Q(t) - \mu R(t), \end{cases} \quad (1.1)$$

where $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ represent the numbers of susceptible, infective, quarantined and recovered individuals at time t , respectively. A is the recruitment rate of the population. q is the vaccination rate and $0 < q < 1$, which means that q is the proportion of susceptible people who recover to health and become R class. β is the constant contact rate between I and S . μ is the natural death rate. ε_1 and ε_2 are the death rates due to the disease of infective and quarantined individuals, respectively. γ and λ are the natural recovery rates of the infective and quarantined individuals, respectively. δ is the removal rate from infective individuals to quarantined individuals. The system (1.1) assumes that the parameters are all positive constants.

System (1.1) is a deterministic model, and its stability was discussed in [26]. In fact, in the real environment, random interference is everywhere, and random factors have an important impact on the outbreak of infectious diseases. May [27] pointed out that the birth rate, the death rate and other parameter involved in the biological system always exhibit stochastic fluctuations due to environmental noise. Therefore, it is necessary to add random disturbance to the deterministic model. In this paper, we construct a stochastic epidemic model by introducing noise term. Essentially the transmission rate can be rewritten as an average transmission rate plus an error term, and the error term follows a Gaussian distribution by the central limit theorem [28]. Based on the above consideration, we suppose the parameter β is disturbed with

$$\beta dt \rightarrow \beta dt + \sigma dB(t),$$

where σ^2 is a continuous bounded function on \mathbb{R} standing for the intensity of the white noise at time t , and $B(t)$ is a

Manuscript received April 20, 2021; revised August 2, 2021. This work was supported by National Natural Science Foundation of China (Grant no. 12001001) and Natural Science Foundation of Bengbu University (Grant no. 2020ZR15).

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standard Brownian motion. Then corresponding to the deterministic epidemic system (1.1), the stochastic SIQR epidemic model takes the following form:

$$\begin{cases} dS(t) = [(1-q)A - \beta S(t)I(t) - \mu S(t)]dt \\ \quad - \sigma S(t)I(t)dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu + \varepsilon_1 + \gamma + \delta)I(t)]dt \\ \quad + \sigma S(t)I(t)dB(t), \\ dQ(t) = [\delta I(t) - (\mu + \varepsilon_2 + \lambda)Q(t)]dt, \\ dR(t) = [qA + \gamma I(t) + \lambda Q(t) - \mu R(t)]dt. \end{cases} \quad (1.2)$$

The rest of this paper is organized as follows. In the next section, existence and uniqueness of global positive solution are obtained by stochastic analysis. In Sect. 3, we present extinction of the model (1.2) under certain assumption. Next, we show sufficient condition for persistence of the system. To verify the theoretical analysis in this paper, some numerical simulations are also presented in Sect. 5.

In this paper, we always set (Ω, F, P) be a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions. Let $B(t)$ be the one-dimensional Brownian motion defined on this probability space.

II. EXISTENCE OF UNIQUE GLOBAL POSITIVE SOLUTION

Our purpose is to study the disease dynamic of the stochastic model (1.2), and whether there is a unique global solution is the basis for studying dynamic properties. The following result shows that the solution of model (1.2) is positive and global.

Theorem 1 For any initial value $(S(0), I(0), Q(0), R(0))$, then model (1.2) has a unique solution $(S(t), I(t), Q(t),$

$R(t))$ defined on $t \in [0, +\infty)$ and the solution will remain in \mathbb{R}_+^4 with probability one.

Proof: Since the coefficients of model (1.2) satisfy locally Lipschitz condition, then for any given initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}_+^4$, there is a unique local solution $(S(t), I(t), Q(t), R(t))$ on $t \in [0, \tau_e]$, where τ_e is the explosion time of the solution. To show the solution $(S(t), I(t), Q(t), R(t))$ is global, we need to show that $\tau_e = \infty$ a.s.

Let $k_0 \geq 1$ be sufficiently large so that for $S(0), I(0), Q(0)$ and $R(0)$ given, they all lie within the interval

$[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e] : \min(S(t), I(t), Q(t), R(t)) \leq \frac{1}{k}$$

$$\text{or } \max(S(t), I(t), Q(t), R(t)) \geq k\},$$

where in this paper, let $\inf \emptyset = \infty$. From the stopping time definition, we know that τ_k is a monotonically increasing function as $k \rightarrow \infty$.

Set $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s.

If we can show that $\tau_\infty = \infty$, then $\tau_e = \infty$ and $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}_+^4$ a.s. for all $t \in [0, +\infty)$. In other words, to complete the proof, we need to show that $\tau_\infty = \infty$. If this

statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} > \varepsilon \quad (2.1)$$

for all $k \geq k_1$.

Besides, for $t \leq \tau_k$, we can obtain

$$\begin{aligned} d(S + I + Q + R) &= [A - \mu(S + I + Q + R) - \varepsilon_1 I - \varepsilon_2 Q]dt \\ &\leq [A - \mu(S + I + Q + R)]dt. \end{aligned}$$

After calculation, we can see

$$\begin{aligned} S(t) + I(t) + Q(t) + R(t) &\leq \begin{cases} \frac{A}{\mu}, & \text{if } S(0) + I(0) + Q(0) + R(0) \leq \frac{A}{\mu} \\ S(0) + I(0) + Q(0) + R(0), & \text{if } S(0) + I(0) + Q(0) + R(0) > \frac{A}{\mu} \end{cases} \\ &:= M. \end{aligned}$$

Define a C^2 -function $V : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ by

$$\begin{aligned} V(S, I, Q, R) &= (S - 1 - \ln S) + (I - 1 - \ln I) \\ &\quad + (Q - 1 - \ln Q) + (R - 1 - \ln R). \end{aligned}$$

The non-negativity of this function can be seen from $u - 1 - \ln u \geq 0, \forall u > 0$. By using Itô's formula, we get

$$\begin{aligned} dV(S, I, Q, R) &= (1 - \frac{1}{S})dS + \frac{1}{2S^2}(dS)^2 + (1 - \frac{1}{I})dI \\ &\quad + \frac{1}{2I^2}(dI)^2 + (1 - \frac{1}{Q})dQ + (1 - \frac{1}{R})dR \\ &= [A - \mu(S + I + Q + R) - \varepsilon_1 I - \varepsilon_2 Q]dt \\ &\quad - \left(\frac{(1-q)A}{S} - \beta I - \mu\right)dt + \sigma IdB(t) \\ &\quad - [\beta S - (\mu + \varepsilon_1 + \gamma + \delta)]dt - \sigma SdB(t) \\ &\quad - \left[\frac{\delta I}{Q} - (\mu + \varepsilon_2 + \lambda)\right]dt \\ &\quad - \left(\frac{qA}{R} + \frac{\gamma I}{R} + \frac{\lambda Q}{R} - \mu\right)dt + \frac{\sigma^2(I^2 + S^2)}{2}dt \\ &= [A - \mu(S + I + Q + R) - \varepsilon_1 I - \varepsilon_2 Q \\ &\quad - \frac{(1-q)A}{S} + \beta(I - S) + (4\mu + \varepsilon_1 + \varepsilon_2 + \gamma + \delta \\ &\quad + \lambda) - \frac{\delta I}{Q} - \frac{qA}{R} - \frac{\gamma I}{R} - \frac{\lambda Q}{R} + \frac{\sigma^2(I^2 + S^2)}{2}]dt \\ &\quad + \sigma(I - S)dB(t). \end{aligned} \quad (2.2)$$

Let

$$\begin{aligned} LV &= A - \mu(S + I + Q + R) - \varepsilon_1 I - \varepsilon_2 Q - \frac{(1-q)A}{S} \\ &\quad + \beta(I - S) + (4\mu + \varepsilon_1 + \varepsilon_2 + \gamma + \delta + \lambda) \\ &\quad - \frac{\delta I}{Q} - \frac{qA}{R} - \frac{\gamma I}{R} - \frac{\lambda Q}{R} + \frac{\sigma^2(I^2 + S^2)}{2}, \end{aligned}$$

then we have

$$\begin{aligned} LV &\leq A + \beta I + (4\mu + \varepsilon_1 + \varepsilon_2 + \gamma + \delta + \lambda) + \frac{\sigma^2(I^2 + S^2)}{2} \\ &\leq A + \beta M + (4\mu + \varepsilon_1 + \varepsilon_2 + \gamma + \delta + \lambda) + \sigma^2 M^2 \\ &:= K. \end{aligned} \quad (2.3)$$

Next, substituting (2.3) into (2.2), we have

$$dV \leq Kdt + \sigma(I - S)dB(t). \tag{2.4}$$

Integrating both sides of (2.4) from 0 to $\tau_k \wedge T$ and then taking the expectations, we have

$$\begin{aligned} EV(S(\tau_k \wedge T), I(\tau_k \wedge T), Q(\tau_k \wedge T), R(\tau_k \wedge T)) \\ \leq V(S(0), I(0), Q(0), R(0)) + KE(\tau_k \wedge T) \\ \leq V(S(0), I(0), Q(0), R(0)) + KT. \end{aligned} \tag{2.5}$$

Let $\Omega_k = \{\tau_k \leq T\}$ for all $k \geq k_1$, and by (2.1), $P\{\Omega_k\} > \varepsilon$. Note that for every $\omega \in \{\tau_k \leq T\}$, At least one of $S(\tau_k, \omega)$, $I(\tau_k, \omega)$, $Q(\tau_k, \omega)$ and $R(\tau_k, \omega)$ equals k or $\frac{1}{k}$, then

$$\begin{aligned} V(S(\tau_k, \omega), I(\tau_k, \omega), Q(\tau_k, \omega), R(\tau_k, \omega)) \\ \geq (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k \right). \end{aligned}$$

It then follows from (2.1) and (2.5) that $V(S(0), I(0), Q(0), R(0)) + KT \geq E[1_{\Omega_k}(\omega)V(S(\tau_k, \omega), I(\tau_k, \omega), Q(\tau_k, \omega), R(\tau_k, \omega))] \geq \varepsilon \left[(k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k \right) \right]$,

where $1_{\Omega_k}(\omega)$ is the indicator function of Ω_k . Letting $k \rightarrow \infty$, leads to the contradiction $\infty > V(S(0), I(0), Q(0), R(0)) + KT = \infty$.

So we have

$$\tau_e = \infty \text{ a.s.}$$

This completes the proof.

Remark 1 Theorem 1 shows that for any given initial value $(S(0), I(0), Q(0), R(0))$, there exists a unique global solution $(S(t), I(t), Q(t), R(t))$ to system(1.2). Noting that

$$d(S + I + Q + R) \leq [A - \mu(S + I + Q + R)]dt,$$

we can get

$$\begin{aligned} S(t) + I(t) + Q(t) + R(t) \\ \leq \frac{A}{\mu} + e^{-\mu t} \left(S(0) + I(0) + Q(0) + R(0) - \frac{A}{\mu} \right). \end{aligned}$$

Clearly, if the condition

$$S(0) + I(0) + Q(0) + R(0) \leq \frac{A}{\mu}$$

holds, then

$$S(t) + I(t) + Q(t) + R(t) \leq \frac{A}{\mu}.$$

Thus, the region

$$\begin{aligned} \Gamma^* = \{(S, I, Q, R) : S > 0, I > 0, Q > 0, R > 0, \\ S(t) + I(t) + Q(t) + R(t) \leq \frac{A}{\mu}\} \end{aligned} \tag{2.6}$$

is a positively invariant set with respect to system (1.2).

III. EXTINCTION OF THE SYSTEM

In the previous section, we have obtained some results on the existence, uniqueness and boundedness of the positive solution. In this section, we investigate certain sufficient conditions that may drive infective and quarantined individuals to extinction.

Theorem 2 Let $(S(t), I(t), Q(t), R(t))$ be the solution of model (1.2) with initial value $(S(0), I(0), Q(0), R(0))$. Then

we have

(a) If $\sigma^2 > \frac{\beta^2}{2(\mu + \varepsilon_1 + \gamma + \delta)}$ holds, then

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq -(\mu + \varepsilon_1 + \gamma + \delta) + \frac{\beta^2}{2\sigma^2} < 0;$$

(b) If $(1 - q)A[2\mu\beta - \sigma^2(1 - q)A] < 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)$

and $\sigma^2 < \frac{\mu\beta}{(1 - q)A}$ hold, then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq \frac{\beta(1 - q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) \\ - \frac{\sigma^2(1 - q)^2 A^2}{2\mu^2} < 0. \end{aligned}$$

Moreover

$$\lim_{t \rightarrow \infty} Q(t) = 0,$$

$$\lim_{t \rightarrow \infty} R(t) = \frac{qA}{\mu},$$

$$\lim_{t \rightarrow \infty} S(t) = \frac{(1 - q)A}{\mu}.$$

Proof: Applying Itô's formula to the I component of system(1.2), we obtain that

$$d(\ln I) = [\beta S - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{1}{2}\sigma^2 S^2]dt + \sigma SdB(t).$$

Integrating both sides of the above equality from 0 to t and dividing t , we can easily show that

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} \\ = \beta \langle S(t) \rangle - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{1}{2}\sigma^2 \langle S^2(t) \rangle + \frac{M(t)}{t} \\ \leq \beta \langle S(t) \rangle - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{1}{2}\sigma^2 \langle S(t) \rangle^2 + \frac{M(t)}{t}. \end{aligned} \tag{3.1}$$

Next, we can get

$$\begin{aligned} \frac{\ln I(t)}{t} \leq -\frac{1}{2}\sigma^2 (\langle S(t) \rangle - \frac{\beta}{\sigma^2})^2 - (\mu + \varepsilon_1 + \gamma + \delta) \\ + \frac{\beta^2}{2\sigma^2} + \frac{M(t)}{t} + \frac{\ln I(0)}{t} \\ \leq -(\mu + \varepsilon_1 + \gamma + \delta) + \frac{\beta^2}{2\sigma^2} + \frac{M(t)}{t} + \frac{\ln I(0)}{t}, \end{aligned} \tag{3.2}$$

where $M(t) = \int_0^t \sigma S(r)dB(r)$ is a continuous local martingale vanishing at $t = 0$, and

$$\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} \leq \frac{\sigma^2 A^2}{\mu^2} < \infty \text{ a.s.}$$

Thus, by the strong law of large numbers, we have

$$\lim_{t \rightarrow \infty} \frac{M_t}{t} = 0.$$

If condition (a) is met, we take the superior limit on both sides of (3.2), and obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq -(\mu + \varepsilon_1 + \gamma + \delta) + \frac{\beta^2}{2\sigma^2} < 0,$$

which implies that

$$\lim_{t \rightarrow \infty} I(t) = 0 \text{ a.s.} \tag{3.3}$$

Now integrating both sides of model (1.2) from 0 to t and dividing t , one can show that

$$\left\{ \begin{aligned} \frac{S(t) - S(0)}{t} &= (1 - q)A - \beta \langle S(t)I(t) \rangle - \mu \langle S(t) \rangle \\ &\quad - \frac{\sigma}{t} \int_0^t S(r)I(r)dB(r), \\ \frac{I(t) - I(0)}{t} &= \beta \langle S(t)I(t) \rangle - (\mu + \varepsilon_1 + \gamma + \delta) \langle I(t) \rangle \\ &\quad + \frac{\sigma}{t} \int_0^t S(r)I(r)dB(r), \\ \frac{Q(t) - Q(0)}{t} &= \delta \langle I(t) \rangle - (\mu + \varepsilon_2 + \lambda) \langle Q(t) \rangle, \\ \frac{R(t) - R(0)}{t} &= qA + \gamma \langle I(t) \rangle + \lambda \langle Q(t) \rangle - \mu \langle R(t) \rangle. \end{aligned} \right. \quad (3.4)$$

Noting that

$$\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} = (1 - q)A - \mu \langle S(t) \rangle - (\mu + \varepsilon_1 + \gamma + \delta) \langle I(t) \rangle,$$

we can easily get that

$$\langle S(t) \rangle = \frac{(1 - q)A}{\mu} - \frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \langle I(t) \rangle + \varphi(t), \quad (3.5)$$

where

$$\varphi(t) = -\frac{1}{\mu} \left[\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} \right].$$

From (2.6), we can easily see that

$$\lim_{t \rightarrow \infty} \varphi(t) = 0 \text{ a.s.} \quad (3.6)$$

On the other hand, substituting (3.5) into (3.1), we have

$$\begin{aligned} &\frac{\ln I(t) - \ln I(0)}{t} \\ &\leq \beta \langle S(t) \rangle - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{1}{2} \sigma^2 \langle S(t) \rangle^2 + \frac{M(t)}{t} \\ &= \beta \left(\frac{(1 - q)A}{\mu} - \frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \langle I(t) \rangle + \varphi(t) \right) \\ &\quad - \frac{1}{2} \sigma^2 \left(\frac{(1 - q)A}{\mu} - \frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \langle I(t) \rangle + \varphi(t) \right)^2 \\ &\quad - (\mu + \varepsilon_1 + \gamma + \delta) + \frac{M(t)}{t} \\ &= \frac{\beta(1 - q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{\sigma^2(1 - q)^2 A^2}{2\mu^2} \\ &\quad - \frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \left(\beta - \frac{\sigma^2(1 - q)A}{\mu} \right) \langle I(t) \rangle \\ &\quad - \frac{1}{2} \sigma^2 \frac{(\mu + \varepsilon_1 + \gamma + \delta)^2}{\mu^2} \langle I(t) \rangle^2 + \frac{M(t)}{t} + \Phi(t) \\ &\leq \frac{\beta(1 - q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{\sigma^2(1 - q)^2 A^2}{2\mu^2} \\ &\quad - \frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \left(\beta - \frac{\sigma^2(1 - q)A}{\mu} \right) \langle I(t) \rangle \\ &\quad + \frac{M(t)}{t} + \Phi(t), \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} \Phi(t) &= \beta \varphi(t) - \frac{1}{2} \sigma^2 \varphi^2(t) - \frac{\sigma^2(1 - q)A}{\mu} \varphi(t) \\ &\quad + \frac{\sigma^2(\mu + \varepsilon_1 + \gamma + \delta)}{\mu} \langle I(t) \rangle \varphi(t). \end{aligned}$$

From (3.6) we can get

$$\lim_{t \rightarrow \infty} \Phi(t) = 0 \text{ a.s.}$$

If condition (b) is met, by taking the superior limit on both sides of (3.7), we see

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} &\leq \frac{\beta(1 - q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) \\ &\quad - \frac{\sigma^2(1 - q)^2 A^2}{2\mu^2} \\ &< 0. \end{aligned}$$

From the third equation of model (1.2) and (3.3), we can get

$$\lim_{t \rightarrow \infty} Q(t) = 0 \text{ a.s.} \quad (3.8)$$

By using (3.3) and (3.8), one can show in the same way that

$$\lim_{t \rightarrow \infty} R(t) = \frac{qA}{\mu} \text{ a.s.} \quad (3.9)$$

Also from system (1.2) we can obtain

$$d(S + I + Q + R) = [A - \mu(S + I + Q + R) - \varepsilon_1 I - \varepsilon_2 Q]dt.$$

By a simple computation, we can get

$$\begin{aligned} S(t) + I(t) + Q(t) + R(t) &= e^{-\mu t} [S(0) + I(0) + Q(0) + R(0)] \\ &\quad + e^{-\mu t} \int_0^t (A - \varepsilon_1 I(r) - \varepsilon_2 Q(r)) e^{\mu r} dr. \end{aligned}$$

Using the well-known L'Hospital's rule, we have

$$\begin{aligned} &\lim_{t \rightarrow \infty} (S(t) + I(t) + Q(t) + R(t)) \\ &= \lim_{t \rightarrow \infty} \frac{S(0) + I(0) + Q(0) + R(0)}{e^{\mu t}} \\ &\quad + \lim_{t \rightarrow \infty} \frac{\int_0^t (A - \varepsilon_1 I(r) - \varepsilon_2 Q(r)) e^{\mu r} dr}{e^{\mu t}} \\ &= \lim_{t \rightarrow \infty} \frac{A - \varepsilon_1 I(t) - \varepsilon_2 Q(t)}{\mu}. \end{aligned} \quad (3.10)$$

Finally, substituting the limit results of $I(t)$, $Q(t)$ and $R(t)$ into (3.10), we have

$$\lim_{t \rightarrow \infty} S(t) = \frac{(1 - q)A}{\mu} \text{ a.s.}$$

The proof is now completed.

Remark 2 Theorem2 show that if the white noise is large enough to satisfy the condition $\sigma^2 > \frac{\beta^2}{2(\mu + \varepsilon_1 + \gamma + \delta)}$, the

disease will be extinct. At the same time, quarantined individuals will also die out. On the other hand, if the intensity of the white noise is not large and the model parameters meet certain conditions, the disease will also be extinct.

IV. PERSISTENCE OF THE SYSTEM

In this section, we discuss the strong persistence in the mean of system (1.2) under certain conditions on the parameters. First we recall the concept of persistent in time average.

Definition 1 The system (1.2) is said to be persistent in time average if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(r)dr > 0 \text{ a.s.}$$

For simplicity, we define

$$\langle f(t) \rangle = \frac{\int_0^t f(s) ds}{t}.$$

The lemma below is used to prove persistence in the mean.

Lemma 1 Let $f \in C[[0, \infty) \times \Omega, (0, \infty)]$ and $F(t) \in C([0, \infty) \times \Omega, \mathbb{R})$. If there exist positive constants λ_0 and λ such that

$$\ln f(t) < \lambda t - \lambda_0 \int_0^t f(s) ds + F(t) \text{ a.s.,}$$

and

$$\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0 \text{ a.s.,}$$

then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds \leq \frac{\lambda}{\lambda_0} \text{ a.s.}$$

Theorem 3 Assume that

$$2\mu\beta(1-q)A - \sigma^2 A > 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)$$

and

$$\sigma^2 < \frac{\mu\beta}{(1-q)A}$$

hold. If $(S(t), I(t), Q(t), R(t))$ is a solution of system (1.2), then

$$\bar{I}_* \leq \liminf_{t \rightarrow \infty} \langle I(t) \rangle \leq \limsup_{t \rightarrow \infty} \langle I(t) \rangle \leq \bar{I}^* \text{ a.s.,}$$

$$\bar{Q}_* \leq \liminf_{t \rightarrow \infty} \langle Q(t) \rangle \leq \limsup_{t \rightarrow \infty} \langle Q(t) \rangle \leq \bar{Q}^* \text{ a.s.,}$$

and

$$\bar{R}_* \leq \liminf_{t \rightarrow \infty} \langle R(t) \rangle \leq \limsup_{t \rightarrow \infty} \langle R(t) \rangle \leq \bar{R}^* \text{ a.s.,}$$

where

$$\bar{I}_* = \frac{2\mu\beta(1-q)A - \sigma^2 A - 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)}{2\beta\mu(\mu + \varepsilon_1 + \gamma + \delta)},$$

$$\bar{I}^* = \frac{(1-q)A[2\mu\beta - \sigma^2(1-q)A] - 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)}{2(\mu + \varepsilon_1 + \gamma + \delta)[\beta\mu - \sigma^2(1-q)A]},$$

$$\bar{Q}_* = \frac{\delta}{\mu + \varepsilon_2 + \lambda} \bar{I}_*,$$

$$\bar{Q}^* = \frac{\delta}{\mu + \varepsilon_2 + \lambda} \bar{I}^*,$$

$$\bar{R}_* = \frac{qA}{\mu} + \frac{\gamma}{\mu} \bar{I}_* + \frac{\lambda}{\mu} \bar{Q}_*,$$

$$\bar{R}^* = \frac{qA}{\mu} + \frac{\gamma}{\mu} \bar{I}^* + \frac{\lambda}{\mu} \bar{Q}^*.$$

Proof From (3.7) yields

$$\begin{aligned} & \frac{\ln I(t) - \ln I(0)}{t} \\ & \leq \frac{\beta(1-q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{\sigma^2(1-q)^2 A^2}{2\mu^2} \\ & \quad - \frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \left(\beta - \frac{\sigma^2(1-q)A}{\mu} \right) \langle I(t) \rangle \\ & \quad + \frac{M(t)}{t} + \Phi(t). \end{aligned} \tag{4.1}$$

Next we can rearrange (4.1) to the following inequality

$$\begin{aligned} \langle I(t) \rangle & \leq \frac{1}{m} \left[\frac{\beta(1-q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{\sigma^2(1-q)^2 A^2}{2\mu^2} \right. \\ & \quad \left. + \frac{M(t)}{t} + \Phi(t) - \frac{\ln I(t) - \ln I(0)}{t} \right], \end{aligned}$$

$$\text{where } m = \frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \left(\beta - \frac{\sigma^2(1-q)A}{\mu} \right).$$

Combining **Lemma 1** and conditions of **Theorem 3**, it is not difficult to derive that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \langle I(t) \rangle \\ & \leq \frac{\frac{\beta(1-q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{\sigma^2(1-q)^2 A^2}{2\mu^2}}{\frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \left(\beta - \frac{\sigma^2(1-q)A}{\mu} \right)} \\ & = \frac{(1-q)A[2\mu\beta - \sigma^2(1-q)A] - 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)}{2(\mu + \varepsilon_1 + \gamma + \delta)[\beta\mu - \sigma^2(1-q)A]} \\ & = \bar{I}^*. \end{aligned}$$

On the other hand, substituting (3.5) into the first equation of (3.1), one can see

$$\begin{aligned} & \frac{\ln I(t) - \ln I(0)}{t} \\ & = \beta \langle S(t) \rangle - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{1}{2} \sigma^2 \langle S^2(t) \rangle + \frac{M(t)}{t} \\ & = \frac{\beta(1-q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{1}{2} \sigma^2 \langle S^2(t) \rangle \\ & \quad - \frac{\beta(\mu + \varepsilon_1 + \gamma + \delta)}{\mu} \langle I(t) \rangle + \beta\varphi(t) + \frac{M(t)}{t} \\ & \geq \frac{\beta(1-q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{1}{2} \sigma^2 \left(\frac{A}{\mu} \right)^2 \\ & \quad - \frac{\beta(\mu + \varepsilon_1 + \gamma + \delta)}{\mu} \langle I(t) \rangle + \beta\varphi(t) + \frac{M(t)}{t}. \end{aligned}$$

Rearrange the above inequality and we get

$$\begin{aligned} \langle I(t) \rangle & \geq \frac{\mu}{\beta(\mu + \varepsilon_1 + \gamma + \delta)} \left[\frac{\beta(1-q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) \right. \\ & \quad \left. - \frac{\sigma^2 A^2}{2\mu^2} + \frac{M(t)}{t} + \beta\varphi(t) - \frac{\ln I(t) - \ln I(0)}{t} \right]. \end{aligned}$$

It can be concluded that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \langle I(t) \rangle \\ & \geq \frac{\frac{\beta(1-q)A}{\mu} - (\mu + \varepsilon_1 + \gamma + \delta) - \frac{\sigma^2 A^2}{2\mu^2}}{\frac{\mu + \varepsilon_1 + \gamma + \delta}{\mu} \beta} \\ & = \frac{2\mu\beta(1-q)A - \sigma^2 A - 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)}{2\beta\mu(\mu + \varepsilon_1 + \gamma + \delta)} \\ & = \bar{I}_*. \end{aligned}$$

From the third equation of (3.4), we can see

$$\langle Q(t) \rangle = \frac{\delta}{\mu + \varepsilon_2 + \lambda} \langle I(t) \rangle - \frac{Q(t) - Q(0)}{(\mu + \varepsilon_2 + \lambda)t}.$$

It follows that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \langle Q(t) \rangle & = \frac{\delta}{\mu + \varepsilon_2 + \lambda} \liminf_{t \rightarrow \infty} \langle I(t) \rangle \\ & \geq \frac{\delta \bar{I}_*}{\mu + \varepsilon_2 + \lambda} \\ & = \bar{Q}_*, \end{aligned}$$

and

$$\limsup_{t \rightarrow \infty} \langle Q(t) \rangle = \frac{\delta}{\mu + \varepsilon_2 + \lambda} \limsup_{t \rightarrow \infty} \langle I(t) \rangle$$

$$\begin{aligned} &\leq \frac{\delta \bar{I}^*}{\mu + \varepsilon_2 + \lambda} \\ &= \bar{Q}^* . \end{aligned}$$

Similarly, applying the same method to the last equation of (3.4), we have

$$\langle R(t) \rangle = \frac{qA}{\mu} + \frac{\gamma}{\mu} \langle I(t) \rangle + \frac{\lambda}{\mu} \langle Q(t) \rangle - \frac{R(t) - R(0)}{\mu t} .$$

Then, we obtain from the above equation that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \langle R(t) \rangle &= \frac{qA}{\mu} + \frac{\gamma}{\mu} \liminf_{t \rightarrow \infty} \langle I(t) \rangle + \frac{\lambda}{\mu} \liminf_{t \rightarrow \infty} \langle Q(t) \rangle \\ &\geq \frac{qA}{\mu} + \frac{\gamma}{\mu} \bar{I}^* + \frac{\lambda}{\mu} \bar{Q}^* \\ &= \bar{R}^* , \end{aligned}$$

and

$$\begin{aligned} \limsup_{t \rightarrow \infty} \langle R(t) \rangle &= \frac{qA}{\mu} + \frac{\gamma}{\mu} \limsup_{t \rightarrow \infty} \langle I(t) \rangle + \frac{\lambda}{\mu} \limsup_{t \rightarrow \infty} \langle Q(t) \rangle \\ &\leq \frac{qA}{\mu} + \frac{\gamma}{\mu} \bar{I}^* + \frac{\lambda}{\mu} \bar{Q}^* \\ &= \bar{R}^* . \end{aligned}$$

Therefore we get the desired assertion.

Remark 3 **Theorem 2** and **Theorem 3** give the sufficient conditions for extinction and persistence of model (1.2) respectively. Let $R_1 = (1 - q)A[2\mu\beta - \sigma^2(1 - q)A]$ and

$R_2 = 2\mu\beta(1 - q)A - \sigma^2 A$. From the above result we can conclude that under the premise of satisfying condition, $\sigma^2 < \frac{\mu\beta}{(1 - q)A}$, $R_1 < 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)$ will make the

disease extinct, and $R_2 > 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta)$ will lead to the epidemic of the disease. It is easy to see $R_1 > R_2$.

V. NUMERICAL SIMULATION

In order to verify the theoretical analysis in Sect. 3 and Sect. 4, we will present some numerical results in this section. By using Milsteins method described in [29], we consider the following discretization system :

$$\begin{cases} S_{k+1} - S_k = [(1 - q)A - \beta S_k I_k - \mu S_k] \Delta t - \sigma S_k I_k \sqrt{\Delta t} \xi_k \\ \quad + \frac{1}{2} \sigma^2 S_k I_k (\xi_k^2 - 1) \Delta t, \\ I_{k+1} - I_k = [\beta S_k I_k - (\mu + \varepsilon_1 + \gamma + \delta) I_k] \Delta t + \sigma S_k I_k \sqrt{\Delta t} \xi_k \\ \quad + \frac{1}{2} \sigma^2 S_k I_k (\xi_k^2 - 1) \Delta t, \\ Q_{k+1} - Q_k = [\delta I_k - (\mu + \varepsilon_2 + \lambda) Q_k] \Delta t, \\ R_{k+1} - R_k = [qA + \gamma I_k + \lambda Q_k - \mu R_k] \Delta t, \end{cases}$$

where Δt is time increment and ξ_k is a Gaussian random variable which follows $N(0,1)$. Without loss of generality, we choose initial value $(S(0), I(0), Q(0), R(0)) = (0.8, 0.4, 0.4, 0.5)$. The parameters of system are selected as follows:

$$\begin{aligned} A &= 0.5, q = 0.5, \beta = 1.0, \mu = 0.1, \varepsilon_1 = 0.4, \varepsilon_2 = 0.1, \\ \gamma &= 0.4, \delta = 0.4, \lambda = 0.2. \end{aligned}$$

According to the above parameters, we have

$$\frac{\mu\beta}{(1 - q)A} = 0.4 ,$$

and

$$2\mu^2(\mu + \varepsilon_1 + \gamma + \delta) = 0.026 .$$

Case 1. First, we investigate the extinction of model (1.2).

Assume $\sigma = 0.62$. Obviously, $\sigma^2 < \frac{\mu\beta}{(1 - q)A}$. Through

simple calculations, we can get

$$\begin{aligned} (1 - q)A[2\mu\beta - \sigma^2(1 - q)A] &= 0.025975 \\ &< 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta) , \end{aligned}$$

which means that condition (b) of **Theorem 2** hold. Therefore, the disease will be extinct. Numerical simulation can be displayed in Figure 1. It can be seen from Figure 1 that under the interference of white noise, $S(t)$ and $I(t)$ are strongly oscillated in the initial stage. As time t increases, disease gradually disappears, and $S(t)$ tends to $\frac{(1 - q)A}{\mu}$.

Case 2. We set $\sigma = 0.15$ under the same parameter conditions. It is easy to see that

$$2\mu\beta(1 - q)A - \sigma^2 A > 2\mu^2(\mu + \varepsilon_1 + \gamma + \delta) ,$$

which shows that conditions of **Theorem3** are satisfied. Therefore, we can conclude that the system is persistent, which is illustrated in Figure 2. Next, we set $\sigma = 0$. That is, we get the deterministic model corresponding to the random model. Figure 3 implies that the corresponding deterministic system is asymptotically stable. From Figure 2 and Figure 3, it can be seen that under the effect of stochastic disturbance, although the model does not have an endemic equilibrium point, it has similar stability to the deterministic model.

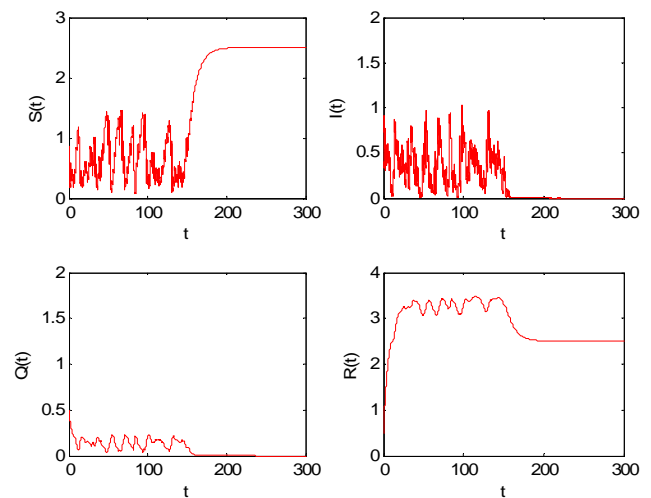


Fig.1. Numerical simulation of system (1.2) for $\sigma = 0.62$ with initial value $(0.8, 0.4, 0.4, 0.5)$.

Case 3. Finally, we consider the case where the intensity of white noise is relatively large. Let $\sigma = 1.0$ and we have

$$\sigma^2 > \frac{\beta^2}{2(\mu + \varepsilon_1 + \gamma + \delta)} .$$

This means that the condition (a) of **Theorem 2** is satisfied. Therefore, the disease will also be disappear. This property can be shown in Figure 4. Comparing Figure 1 with Figure 4, we can see that when the intensity of white noise increases, it will cause the disease to disappear quickly.

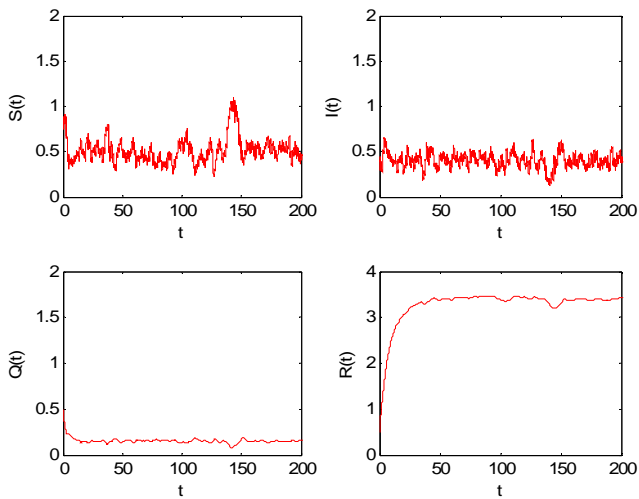


Fig.2. Numerical simulation of system (1.2) for $\sigma = 0.15$ with initial value (0.8, 0.4, 0.4, 0.5).

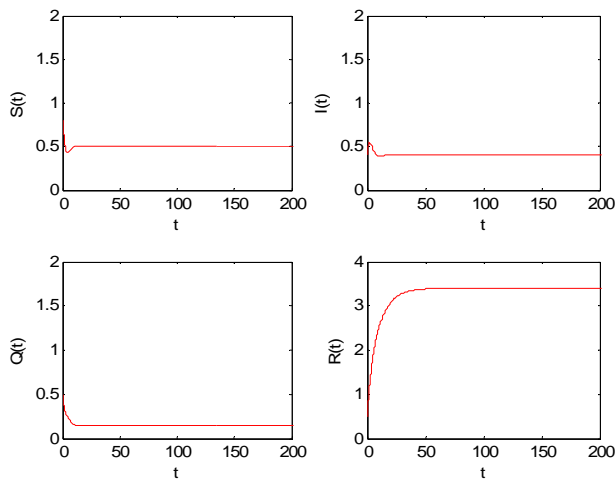


Fig.3. the corresponding deterministic system is asymptotically stable

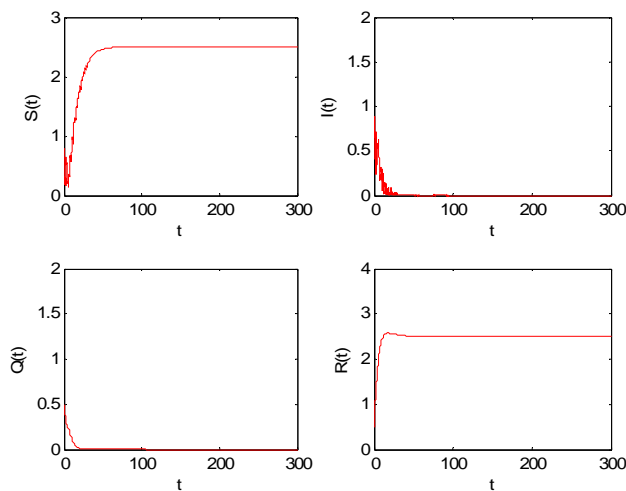


Fig.4. Numerical simulation of system (1.2) for $\sigma = 1.0$ with initial value (0.8, 0.4, 0.4, 0.5).

VI. CONCLUSION

In the present paper, a stochastic SIQR epidemic model with vaccination effect is studied. Compared with the corresponding deterministic epidemic system considered in [26], the stochastic system in this paper considers that the parameter β are affected by environmental perturbation. We incorporate a noise term into the system and investigate the effects of the white noise on the dynamics of the system. By taking the tricks in random calculations, some results on the existence, uniqueness and boundedness of the positive solution are obtained. It has been shown certain sufficient conditions that may drive infective and quarantined individuals to extinction. Further, the properties of system persistence are determined. Finally, we present some numerical results. From the simulations we can conclude that white noise can not only make the disease disappear, but also keep the system in a permanent state.

The effects of environmental disturbances in nature are everywhere, and ergodic property is an important element to characterize the dynamics of biological systems. we regret to point out that we only consider the extinction and persistence of the system in the present paper. In future studies, we will try to discuss other properties of the model, such as ergodic property and the existence of an invariant distribution. Also we would build some models with age structure and time delay, which are interesting for their biological significance.

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