Numerical Solution for Unsteady Diffusion Convection Problems of Anisotropic Trigonometrically Graded Materials with Incompressible Flow

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***Abstract*—The diffusion-convection equation with variable coefficients for anisotropic inhomogeneous media is discussed in this paper. Numerical solutions to problems that are governed by this equation are sought using a combined Laplace transform and boundary element method. The variable coefficients equation is transformed to a constant coefficients equation. The constant coefficients equation is then Laplace-transformed so that the time variable vanishes. The Laplace-transformed equation can then be written in a purely boundary integral equation that involves a time-free fundamental solution. The boundary integral equation is therefore employed to find numerical solutions using a standard boundary element method. Finally, the results obtained are inversely transformed numerically using the Stehfest formula to get solutions in the time variable. Some unsteady problems of anisotropic trigonometrically graded media governed by the diffusion convection equation are considered. The results show that the combined Laplace transform and boundary element method is easy to implement and accurate.**

***Index Terms*—variable coefficients, anisotropic trigonometrically graded materials, unsteady diffusion convection equation, Laplace transform, boundary element method**

I. Introduction

Nowadays, functionally graded materials (FGMs) have become an important topic, and numerous studies on them for a variety of applications have been reported. Authors commonly define an FGM as an inhomogeneous material having a specific property such as thermal conductivity, hardness, toughness, ductility, corrosion resistance, etc. that changes spatially in a continuous fashion. In this study, the unsteady anisotropic diffusion convection equation of incompressible flow and space variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[d_{ij}\left(\mathbf{x}\right)\frac{\partial c\left(\mathbf{x},t\right)}{\partial x_j}\right]-v_i\left(\mathbf{x}\right)\frac{\partial c\left(\mathbf{x},t\right)}{\partial x_i}=\alpha\left(\mathbf{x}\right)\frac{\partial c\left(\mathbf{x},t\right)}{\partial t} \tag{1}$$

will be considered. Equation (1) is usually used to model the physical phenomena such as pollutant transport and heat transfer.

Referred to the Cartesian frame Ox_1x_2 we will consider initial boundary value problems governed by (1) where $\mathbf{x} = (x_1, x_2)$. The coefficient $[d_{ij}]$ $(i, j = 1, 2)$ is a real positive definite symmetrical matrix. Also, in (1) the summation convention for repeated indices holds, so that explicitly (1) can be respectively written as

$$\frac{\partial}{\partial x_1}\left(d_{11}\frac{\partial c}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(d_{12}\frac{\partial c}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(d_{12}\frac{\partial c}{\partial x_1}\right)$$
$$+\frac{\partial}{\partial x_2}\left(d_{22}\frac{\partial c}{\partial x_2}\right)-v_1\frac{\partial c}{\partial x_1}-v_2\frac{\partial c}{\partial x_2}=\alpha\frac{\partial c}{\partial t}$$

In the last decade investigations on the diffusion-convection equation had been done for finding its numerical solutions. The investigations can be classified according to the anisotropy and inhomogeneity of the media under consideration. For example, Wu et al. [1], Hernandez-Martinez et al. [2], Wang et al. [3] and Fendoğlu et al. [4] had been working on problems of *isotropic diffusion and homogeneous media*, Yoshida and Nagaoka [5], Meenal and Eldho [6], Azis [7] (for Helmholtz type governing equation) studied problems of *anisotropic diffusion but homogeneous media*. Rap et al. [8], Ravnik and Škerget [9], [10], Li et al. [11] and Pettres and Lacerda [12] considered the case of *isotropic diffusion and variable coefficients (inhomogeneous media)*. Recently Azis and co-workers had been working on *steady state* problems of *anisotropic inhomogeneous* media for several types of governing equations, for examples [13], [14] for the modified Helmholtz equation, [15]–[21] for the diffusion convection reaction equation, [22]–[25] for the Laplace type equation, [26]–[30] for the Helmholtz equation. Azis et al. also had been working on *unsteady state* problems of *anisotropic inhomogeneous* media for some types of governing equations (see [31]–[34]).

This paper is intended to extend the recently published works on anisotropic diffusion convection equation with variable coefficients [35]–[39] from the steady state to unsteady state equation. Equation (1) applies for unsteady problems of *anisotropic and inhomogeneous* therefore provides a wider class of problems. It covers problems of isotropic and homogeneous media as special cases which occur respectively when $d_{11} = d_{22}, d_{12} = 0$ and the coefficients d_{ij}, v_i and α are constant.

II. The initial boundary value problem

Given the coefficients $d_{ij}\left(\mathbf{x}\right), v_i\left(\mathbf{x}\right), \alpha\left(\mathbf{x}\right)$, solutions $c\left(\mathbf{x},t\right)$ and its derivatives to (1) are sought which are valid for time interval $t \geq 0$ and in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth

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curves. On $\partial\Omega_1$ the dependent variable $c(\mathbf{x},t)$ is specified, and

$$P(\mathbf{x},t) = d_{ij}(\mathbf{x})\frac{\partial c(\mathbf{x},t)}{\partial x_i}n_j \tag{2}$$

is specified on $\partial\Omega_2$ where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$c(\mathbf{x},0) = 0 \tag{3}$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation, and then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable s. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution c and its derivatives for all $(\mathbf{x},t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula. The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $d_{11} = d_{22}$ and $d_{12} = 0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

III. The boundary integral equation

We restrict the coefficients d_{ij}, v_i, α to be of the form

$$d_{ij}(\mathbf{x}) = \hat{d}_{ij}\, g(\mathbf{x}) \tag{4}$$
$$v_i(\mathbf{x}) = \hat{v}_i\, g(\mathbf{x}) \tag{5}$$
$$\alpha(\mathbf{x}) = \hat{\alpha}\, g(\mathbf{x}) \tag{6}$$

where $\hat{d}_{ij}, \hat{v}_i, \hat{\alpha}$ are constants. Further we assume that the coefficients $d_{ij}(\mathbf{x})$, $v_i(\mathbf{x})$ and $\alpha(\mathbf{x})$ are trigonometrically graded by taking $g(\mathbf{x})$ as an trigonometric function

$$g(\mathbf{x}) = \left[\cos(\beta_0 + \beta_i x_i) + \sin(\beta_0 + \beta_i x_i)\right]^2 \tag{7}$$

where β_0 and β_i are constants. Therefore if

$$\hat{d}_{ij}\beta_i\beta_j + \lambda = 0 \tag{8}$$

then (7) satisfies

$$\hat{d}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \lambda g^{1/2} = 0 \tag{9}$$

Substitution of (4)-(6) into (1) gives

$$\hat{d}_{ij}\frac{\partial}{\partial x_i}\left(g\frac{\partial c}{\partial x_j}\right) - \hat{v}_i g\frac{\partial c}{\partial x_i} = \hat{\alpha} g\frac{\partial c}{\partial t} \tag{10}$$

Assume

$$c(\mathbf{x},t) = g^{-1/2}(\mathbf{x})\,\psi(\mathbf{x},t) \tag{11}$$

therefore substitution of (4) and (11) into (2) gives

$$P(\mathbf{x},t) = -P_g(\mathbf{x})\,\psi(\mathbf{x},t) + g^{1/2}(\mathbf{x})\,P_\psi(\mathbf{x},t) \tag{12}$$

where

$$P_g(\mathbf{x},t) = \hat{d}_{ij}\frac{\partial g^{1/2}(\mathbf{x})}{\partial x_j}n_i \quad P_\psi(\mathbf{x},t) = \hat{d}_{ij}\frac{\partial \psi(\mathbf{x},t)}{\partial x_j}n_i$$

Equation (10) can be written as

$$\hat{d}_{ij}\frac{\partial}{\partial x_i}\left[g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right] - \hat{v}_i g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_i} = \hat{\alpha} g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial t}$$

which can be simplified to

$$\hat{d}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j} + g\psi\frac{\partial g^{-1/2}}{\partial x_j}\right)$$
$$-\hat{v}_i\left(g^{1/2}\frac{\partial\psi}{\partial x_i} + g\psi\frac{\partial g^{-1/2}}{\partial x_i}\right) = \hat{\alpha} g^{1/2}\frac{\partial\psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i} = -g^{-1}\frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\hat{d}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j} - \psi\frac{\partial g^{1/2}}{\partial x_j}\right)$$
$$-\hat{v}_i\left(g^{1/2}\frac{\partial\psi}{\partial x_i} - \psi\frac{\partial g^{1/2}}{\partial x_i}\right)$$
$$= \hat{\alpha} g^{1/2}\frac{\partial\psi}{\partial t}$$

Rearranging and neglecting the zero terms yield

$$g^{1/2}\left(\hat{d}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j} - \hat{v}_i\frac{\partial\psi}{\partial x_j}\right)$$
$$-\psi\left(\hat{d}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j} - \hat{v}_i\frac{\partial g^{1/2}}{\partial x_i}\right)$$
$$+\left(\hat{d}_{ij}\frac{\partial\psi}{\partial x_j}\frac{\partial g^{1/2}}{\partial x_i} - \hat{d}_{ij}\frac{\partial\psi}{\partial x_j}\frac{\partial g^{1/2}}{\partial x_i}\right)$$
$$= \hat{\alpha} g^{1/2}\frac{\partial\psi}{\partial t} \tag{13}$$

For incompressible flow

$$\frac{\partial v_i(\mathbf{x})}{\partial x_i} = 2g^{1/2}(\mathbf{x})\hat{v}_i\frac{\partial g^{1/2}(\mathbf{x})}{\partial x_i} = 0$$

that is

$$\hat{v}_i\frac{\partial g^{1/2}(\mathbf{x})}{\partial x_i} = 0$$

Thus (13) becomes

$$g^{1/2}\left(\hat{d}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j} - \hat{v}_i\frac{\partial\psi}{\partial x_i}\right) - \psi\hat{d}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j} = \hat{\alpha} g^{1/2}\frac{\partial\psi}{\partial t}$$

Equation (9) then implies

$$\hat{d}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j} - \hat{v}_i\frac{\partial\psi}{\partial x_i} - \lambda\psi = \hat{\alpha}\frac{\partial\psi}{\partial t} \tag{14}$$

Taking a Laplace transform of (11), (12), (14) and applying the initial condition (3) we obtain

$$\psi^*(\mathbf{x},s) = g^{1/2}(\mathbf{x})\,c^*(\mathbf{x},s) \tag{15}$$

$$P_{\psi^*}(\mathbf{x},s) = \left[P^*(\mathbf{x},s) + P_g(\mathbf{x})\,\psi^*(\mathbf{x},s)\right]g^{-1/2}(\mathbf{x}) \tag{16}$$

$$\hat{d}_{ij}\frac{\partial^2\psi^*}{\partial x_i\partial x_j} - \hat{v}_i\frac{\partial\psi^*}{\partial x_i} - (\lambda + s\hat{\alpha})\,\psi^* = 0 \tag{17}$$

where s is the variable of the Laplace-transformed domain.

By using Gauss divergence theorem, equation (17) can be transformed into a boundary integral equation

$$\eta(\boldsymbol{\xi})\;\psi^*(\boldsymbol{\xi},s) = \int_{\partial\Omega}\left\{P_{\psi^*}(\mathbf{x},s)\;\Phi(\mathbf{x},\boldsymbol{\xi})\right.$$
$$\left. - \left[P_v(\mathbf{x})\;\Phi(\mathbf{x},\boldsymbol{\xi}) + \Gamma(\mathbf{x},\boldsymbol{\xi})\right]\psi^*(\mathbf{x},s)\right\}dS(\mathbf{x}) \tag{18}$$

where

$$P_v(\mathbf{x}) = \hat{v}_i\, n_i(\mathbf{x})$$

For 2-D problems the fundamental solutions $\Phi(\mathbf{x},\boldsymbol{\xi})$ and $\Gamma(\mathbf{x},\boldsymbol{\xi})$ for are given as

$$\begin{aligned}\Phi(\mathbf{x},\boldsymbol{\xi}) &= \frac{\rho_i}{2\pi D}\exp\left(-\frac{\dot{\mathbf{v}}.\dot{\mathbf{R}}}{2D}\right)K_0\left(\dot{\mu}\dot{R}\right)\\ \Gamma(\mathbf{x},\boldsymbol{\xi}) &= \hat{d}_{ij}\frac{\partial\Phi(\mathbf{x},\boldsymbol{\xi})}{\partial x_j}n_i\end{aligned}$$

where

$$\dot{\mu} = \sqrt{(\dot{v}/2D)^2 + [(\lambda + s\hat{\alpha})/D]}$$

$$D = \left[\hat{d}_{11} + 2\hat{d}_{12}\rho_r + \hat{d}_{22}\left(\rho_r^2 + \rho_i^2\right)\right]/2$$

$$\dot{\mathbf{R}} = \dot{\mathbf{x}} - \dot{\boldsymbol{\xi}}$$

$$\dot{\mathbf{x}} = (x_1 + \rho_r x_2, \rho_i x_2)$$

$$\dot{\boldsymbol{\xi}} = (\xi_1 + \rho_r \xi_2, \rho_i \xi_2)$$

$$\dot{\mathbf{v}} = (\hat{v}_1 + \rho_r \hat{v}_2, \rho_i \hat{v}_2)$$

$$\dot{R} = \sqrt{(x_1 + \rho_r x_2 - \xi_1 - \rho_r \xi_2)^2 + (\rho_i x_2 - \rho_i \xi_2)^2}$$

$$\dot{v} = \sqrt{(\hat{v}_1 + \rho_r \hat{v}_2)^2 + (\rho_i \hat{v}_2)^2}$$

where ρ_r and ρ_i are respectively the real and the positive imaginary parts of the complex root ρ of the quadratic equation

$$\hat{d}_{11} + 2\hat{d}_{12}\rho + \hat{d}_{22}\rho^2 = 0$$

and K_0 is the modified Bessel function. Use of (15) and (16) in (18) yields

$$\begin{aligned}\eta g^{1/2}c^* = &\int_{\partial\Omega}\left\{\left(g^{-1/2}\Phi\right)P^*\right.\\ &\left.+\left[\left(P_g - P_v g^{1/2}\right)\Phi - g^{1/2}\Gamma\right]c^*\right\}dS \qquad (19)\end{aligned}$$

Equation (19) provides a boundary integral equation for determining the numerical solutions of c^* and its derivatives $\partial c^*/\partial x_1$ and $\partial c^*/\partial x_2$ at all points of Ω.

Knowing the solutions $c^*(\mathbf{x}, s)$ and its derivatives $\partial c^*/\partial x_1$ and $\partial c^*/\partial x_2$ which are obtained from (19), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $c(\mathbf{x}, t)$ and its derivatives $\partial c/\partial x_1$ and $\partial c/\partial x_2$. The Stehfest formula is

$$\begin{aligned}c(\mathbf{x},t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m c^*(\mathbf{x},s_m)\\ \frac{\partial c(\mathbf{x},t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial c^*(\mathbf{x},s_m)}{\partial x_1} \qquad (20)\\ \frac{\partial c(\mathbf{x},t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial c^*(\mathbf{x},s_m)}{\partial x_2}\end{aligned}$$

where

$$s_m = \frac{\ln 2}{t}m$$

$$V_m = (-1)^{\frac{N}{2}+m}\times \sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)}\frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!}$$

A simple script has been developed to calculate the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any number N. Table (I) shows the values of V_m for $N = 4, 6, 8, 10$.

TABLE I
VALUES OF V_m OF THE STEHFEST FORMULA FOR $N = 4, 6, 8, 10$

V_m	$N=4$	$N=6$	$N=8$	$N=10$
V_1	-2	1	$-1/3$	$1/12$
V_2	26	-49	$145/3$	$-385/12$
V_3	-48	366	-906	1279
V_4	24	-858	$16394/3$	$-46871/3$
V_5		810	$-43130/3$	$505465/6$
V_6		-270	18730	-236957.5
V_7			$-35840/3$	$1127735/3$
V_8			$8960/3$	$-1020215/3$
V_9				164062.5
V_{10}				-32812.5

IV. NUMERICAL RESULTS

In order to verify the analysis derived in the previous sections, we will consider several problems either as test examples of analytical solutions or problems without simple analytical solutions.

We assume each problem belongs to a system which is valid within a given spatial and time domain, governed by equation (1), satisfying the initial condition (3) and some boundary conditions as mentioned in Section II. The characteristics of the system which are represented by the coefficients $d_{ij}(\mathbf{x}), v_i(\mathbf{x}), \alpha(\mathbf{x})$ are assumed to be of the form (4), (5) and (6) in which $g(\mathbf{x})$ is a trigonometric function of the form (7). The coefficients $d_{ij}(\mathbf{x}), v_i(\mathbf{x}), \alpha(\mathbf{x})$ represents respectively the diffusivity or conductivity, the velocity of flow existing in the system and the change rate of the unknown $c(\mathbf{x}, t)$. The flow is assumed to be incompressible so that the velocity $v_i(\mathbf{x})$ satisfies the condition $\partial v_i(\mathbf{x})/\partial x_i = 0$.

Standard BEM with constant elements is employed to obtain numerical results. For a simplicity, a unit square (depicted in Figure 1) will be taken as the geometrical domain for all problems. A number of 320 elements of equal length, namely 80 elements on each side of the unit square, are used. A FORTRAN script is developed to compute the solutions.

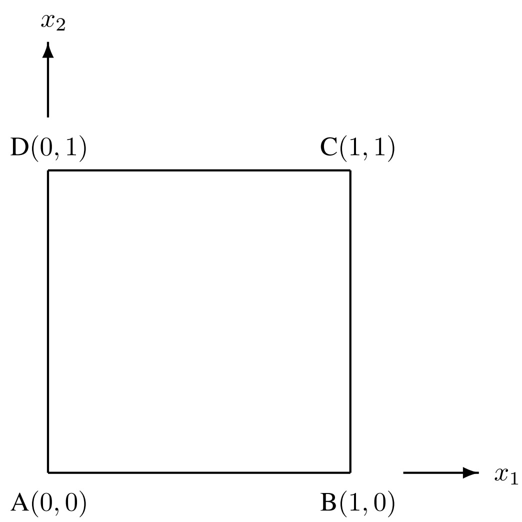

Fig. 1. The domain Ω

A. Test problems

Other aspects that will be verified are the accuracy and consistency (between the scattering and flow) of the numer-

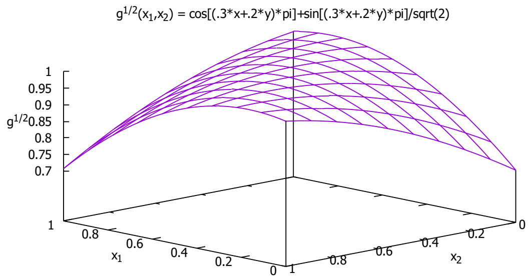

Fig. 2. Function $g(\mathbf{x})$

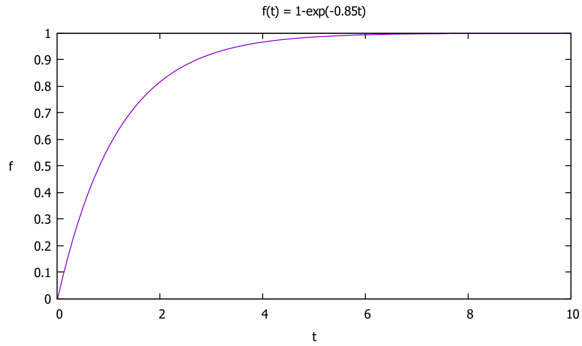

Fig. 3. Function $f(t)$ for Problem 1

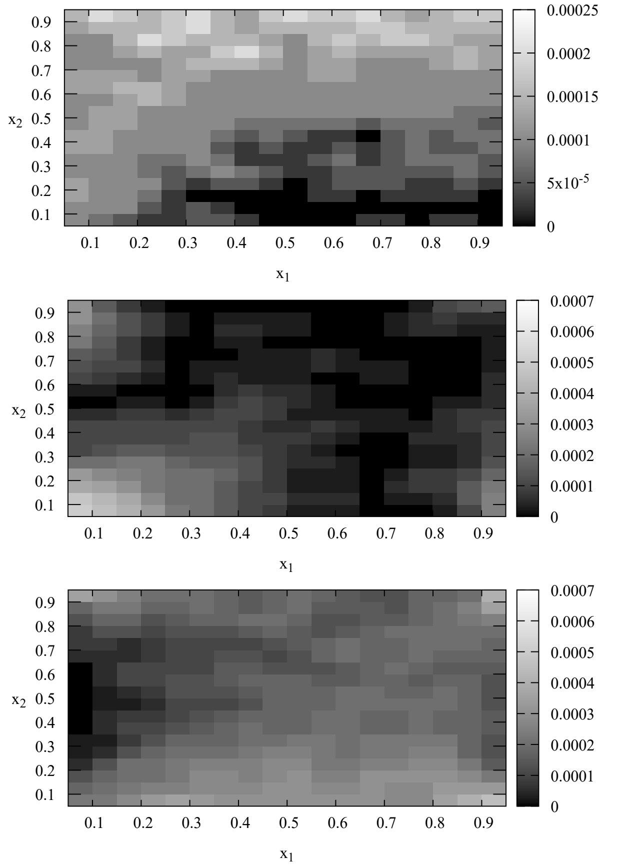

Fig. 4. The errors of solutions c (top), $\partial c/\partial x_1$ (center), $\partial c/\partial x_2$ (bottom) at $t = 5$ for Problem 1

ical solutions. The analytical solutions are assumed to take a separable variables form

$$c(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where

$$h(\mathbf{x}) = 2.5 \exp\left[-\left(1 + 0.35x_1 + 0.55x_2\right)\right]$$

The function $g^{1/2}(\mathbf{x})$ is

$$g^{1/2}(\mathbf{x}) = \left[\cos\left(0.3x_1 + 0.2x_2\right) + \sin\left(0.3x_1 + 0.2x_2\right)\right]/\sqrt{2}$$

and depicted in Figure 2. We will consider three forms of time variation functions $f(t)$ of time domain $t = [0:10]$ which are

$$\begin{aligned} f(t) &= 1 - \exp(-0.85t) \\ f(t) &= 0.1t \\ f(t) &= t(10 - t)/25 \end{aligned}$$

We take a mutual coefficient $\hat{d}_{ij}$ for the problems

$$\hat{d}_{ij} = \begin{bmatrix} 0.85 & 0.35 \\ 0.35 & 1 \end{bmatrix}$$

so that from (8) we have

$$\lambda = -0.1585$$

We choose

$$\hat{v}_i = (-0.2, 0.3), \hat{\alpha} = 0.794875/s$$

and a mutual set of boundary conditions (see Figure 1)

P is given on side AB, BC, CD
c is given on side AD

We try to change the value of N in the Stehfest formula (20) from $N = 6$ to $N = 12$ and find out that the optimized solutions (closest to the analytical solutions) are obtained when $N = 10$. Increasing N from $N = 10$ to $N = 12$ gives divergent solutions. According to Hassanzadeh and Pooladi-Darvish [40] these divergent solutions are induced by round-off errors. Hence, for all problems considered the value of N for the Stehfest formula in (20) is chosen to be $N = 10$.

Problem 1:: First, we suppose that the time variation function is

$$f(t) = 1 - \exp(-0.85t)$$

Function $f(t)$ is depicted in Figure 3. Figure 4 shows the accuracy of the BEM solutions. The errors occur in the fourth decimal place for the c and the derivatives $\partial c/\partial x_1$ and $\partial c/\partial x_2$ solutions. Figure 5 shows the consistency between the scattering and the flow solutions which verifies that the solutions for the derivatives had also been computed correctly. Figure 6 shows that the solution c changes with time t in a similar way the function $f(t) = 1 - \exp(-0.85t)$ does (see Figure 3) and tends to approach a steady state solution as the time goes to infinity, as expected.

Problem 2:: Next, we suppose that the time variation function is (see Figure 7)

$$f(t) = 0.1t$$

Figure 8 shows the accuracy of the BEM solutions. The errors occur in the fourth decimal place for the c and the derivatives $\partial c/\partial x_1$ and $\partial c/\partial x_2$ solutions. Figure 9 shows the consistency between the scattering and the flow solutions. Figure 10 shows that the solution c changes with time t in a manner which is almost similar to as the function $f(t) = 0.1t$ does (see Figure 7), as expected.

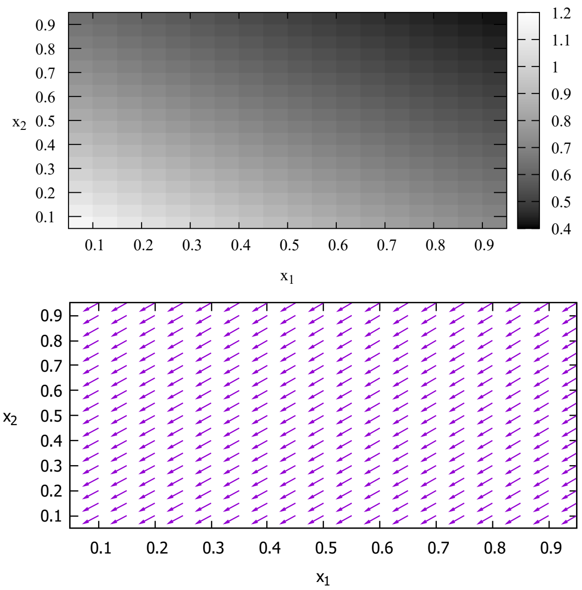

Fig. 5. Solutions c (top) and $(\partial c/\partial x_1, \partial c/\partial x_2)$ (bottom) at $t = 5$ for Problem 1

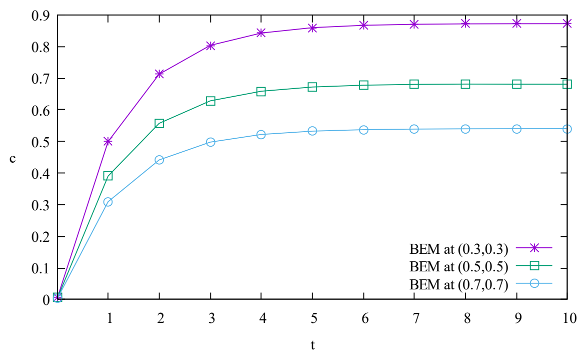

Fig. 6. Solutions c for Problem 1

Problem 3:: Now, we suppose that the time variation function is (see Figure 11)

$$f(t) = t(10-t)/25$$

Figure 12 shows the accuracy of the BEM solutions. The errors occur in the fourth decimal place for the c and the derivatives $\partial c/\partial x_1$ and $\partial c/\partial x_2$ solutions. Figure 13 shows the consistency between the scattering and the flow solutions which again verifies that the solutions for the derivatives had also been computed correctly. Figure 14 shows that the solution c changes with time t in a similar way the function

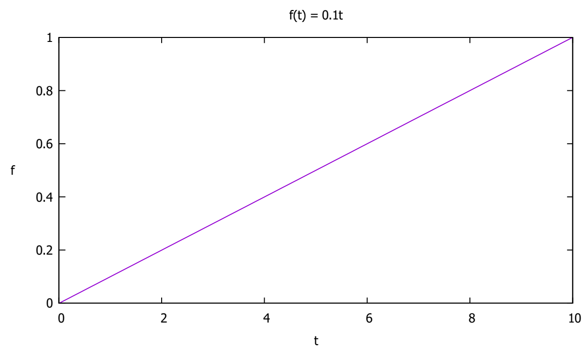

Fig. 7. Function $f(t)$ for Problem 2

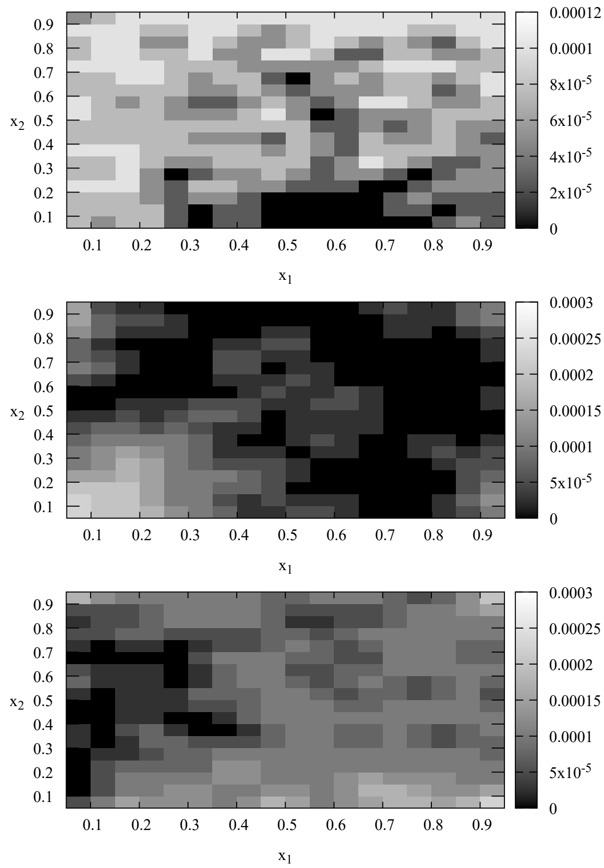

Fig. 8. The errors of solutions c (top), $\partial c/\partial x_1$ (center), $\partial c/\partial x_2$ (bottom) at $t = 5$ for Problem 2

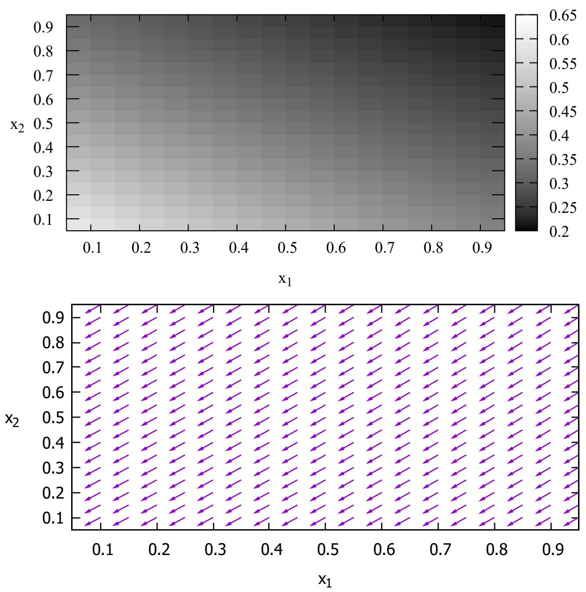

Fig. 9. Solutions c (top) and $(\partial c/\partial x_1, \partial c/\partial x_2)$ (bottom) at $t = 5$ for Problem 2

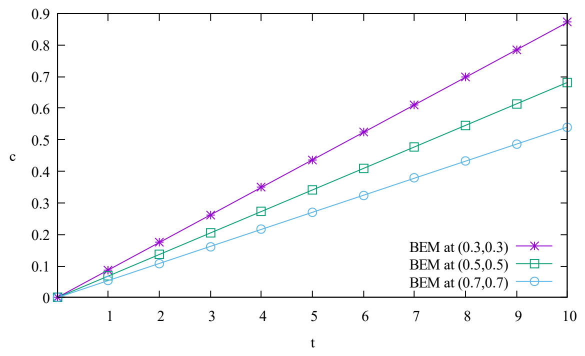

Fig. 10. Solutions c for Problem 2

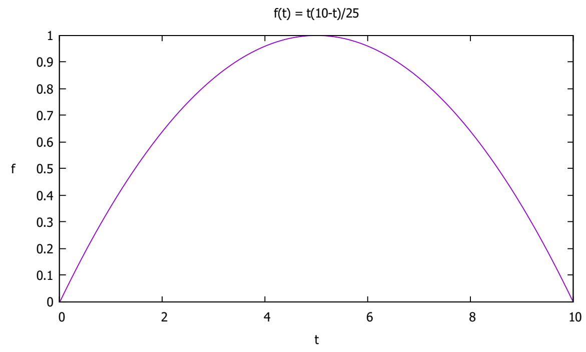

Fig. 11. Function $f(t)$ for Problem 3

$f(t) = t(10-t)/25$ does.

B. Examples without analytical solutions

Furthermore, we will show the impact of the anisotropy and the inhomogeneity of the material under consideration on the solutions. We choose

$$\hat{v}_i = (-0.2, 0.3) \qquad \hat{\alpha} = 1$$

Problem 4:: For this problem the medium is supposed to be inhomogeneous or homogeneous, anisotropic or isotropic with a gradation function $g(\mathbf{x})$, constant coefficients $\hat{d}_{ij}$ and corresponding λ satisfying (8) and (9) as respectively follows:

- inhomogeneous and anisotropic case

$$\begin{aligned} g^{1/2}(\mathbf{x}) &= [\cos(0.3x_1 + 0.2x_2) \\ &\quad + \sin(0.3x_1 + 0.2x_2)]/\sqrt{2} \\ \hat{d}_{ij} &= \begin{bmatrix} 0.85 & 0.35 \\ 0.35 & 1 \end{bmatrix} \\ \lambda &= -0.1585 \end{aligned}$$

- inhomogeneous and isotropic case

$$\begin{aligned} g^{1/2}(\mathbf{x}) &= [\cos(0.3x_1 + 0.2x_2) \\ &\quad + \sin(0.3x_1 + 0.2x_2)]/\sqrt{2} \\ \hat{d}_{ij} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \lambda &= -0.13 \end{aligned}$$

- homogeneous and isotropic case

$$\begin{aligned} g^{1/2}(\mathbf{x}) &= 1 \\ \hat{d}_{ij} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \lambda &= 0 \end{aligned}$$

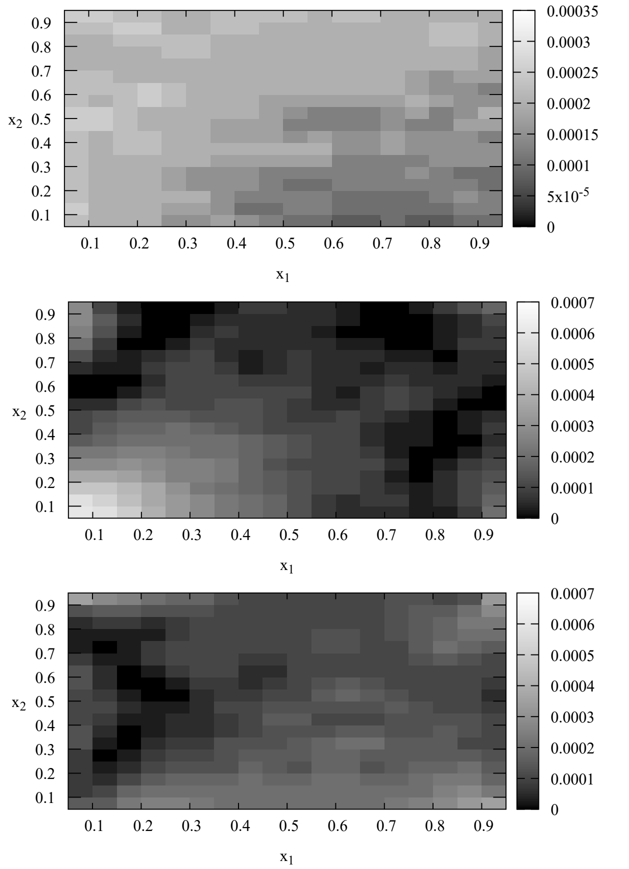

Fig. 12. The errors of solutions c (top), $\partial c/\partial x_1$ (center), $\partial c/\partial x_2$ (bottom) at $t = 5$ for Problem 3

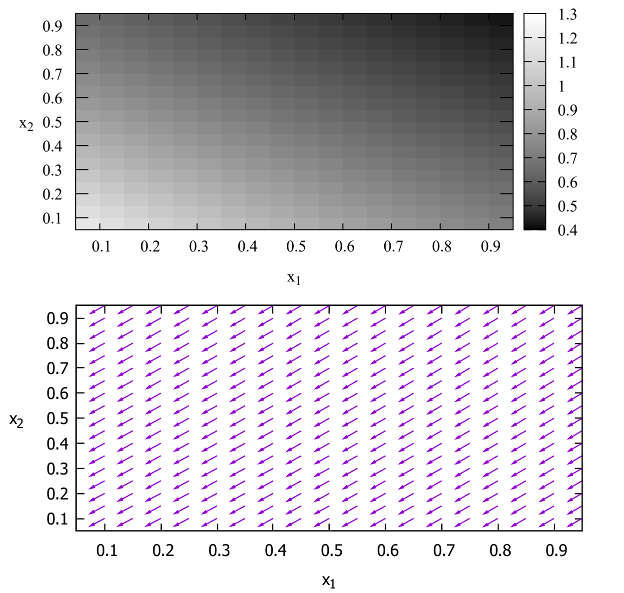

Fig. 13. Solutions c (top) and $(\partial c/\partial x_1, \partial c/\partial x_2)$ (bottom) at $t = 5$ for Problem 3

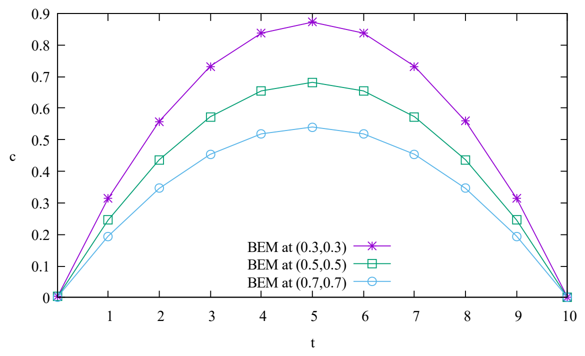

Fig. 14. Solutions c for Problem 3

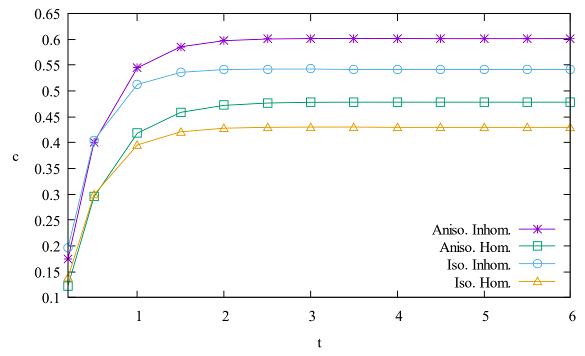

Fig. 15. Solutions c at $(x_1, x_2) = (0.5, 0.5)$ for Problem 4

TABLE II
SYMMETRY OF SOLUTIONS c ABOUT $x_2 = 0.5$ FOR PROBLEM 4

t	$(0.5, 0.3)$	$(0.5, 0.7)$	$(0.5, 0.3)$	$(0.5, 0.7)$
	Iso. Hom.		Aniso. Hom	
0.2	0.1389	0.1389	0.1032	0.1443
0.5	0.2993	0.2993	0.2551	0.3390
1.0	0.3955	0.3955	0.3659	0.4758
1.5	0.4210	0.4210	0.4023	0.5205
2.0	0.4279	0.4279	0.4144	0.5354
2.5	0.4297	0.4297	0.4184	0.5404
3.0	0.4301	0.4302	0.4197	0.5420
3.5	0.4302	0.4302	0.4201	0.5425
4.0	0.4301	0.4301	0.4202	0.5426
4.5	0.4301	0.4301	0.4201	0.5425
5.0	0.4300	0.4301	0.4201	0.5425
	Aniso. Inhom.		Iso. Inhom.	
0.2	0.1483	0.2023	0.2006	0.1931
0.5	0.3455	0.4602	0.4080	0.4014
1.0	0.4732	0.6226	0.5153	0.5098
1.5	0.5094	0.6683	0.5388	0.5336
2.0	0.5198	0.6815	0.5439	0.5388
2.5	0.5227	0.6853	0.5450	0.5399
3.0	0.5235	0.6862	0.5451	0.5400
3.5	0.5236	0.6864	0.5450	0.5399
4.0	0.5235	0.6863	0.5449	0.5398
4.5	0.5235	0.6862	0.5449	0.5398
5.0	0.5234	0.6862	0.5449	0.5398

- homogeneous and anisotropic case

$$
\begin{aligned}
g^{1/2}(\mathbf{x}) &= 1 \\
\hat{d}_{ij} &= \begin{bmatrix} 0.85 & 0.35 \\ 0.35 & 1 \end{bmatrix} \\
\lambda &= 0
\end{aligned}
$$

The boundary conditions are that (see Figure 1)

$$
\begin{gathered}
P = 0 \text{ on side AB} \\
c = 0 \text{ on side BC} \\
P = 0 \text{ on side CD} \\
P = 1 \text{ on side AD}
\end{gathered}
$$

There is no simple analytical solution for this problem. In fact the system is geometrically symmetric about the axis $x_2 = 0.5$. The results in Table II verify that anisotropy and inhomogeneity give impact to the values of solution c for being asymmetric about $x_2 = 0.5$. Solutions are symmetric only for homogeneous isotropic case, as expected. Moreover, for all cases the results in Figure 15 indicate that the system has a steady state solution. After all, the results suggest that it is important to take both aspects of inhomogeneity and anisotropy into account when doing an experimental study.

Problem 5:: We consider the inhomogeneous and anisotropic case of Problem 4 again. But we change the set

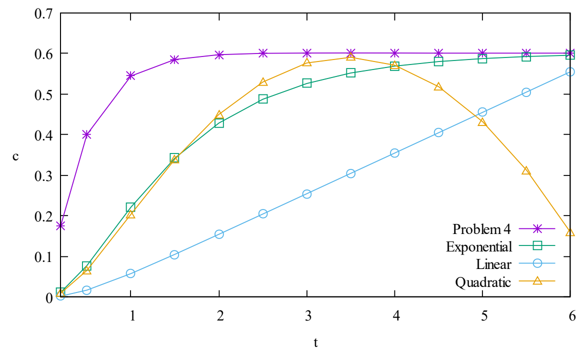

Fig. 16. Solutions c at $(x_1, x_2) = (0.5, 0.5)$ for Problem 5

of the boundary conditions of Problem 4 especially on the side AD. Now we use three cases of the boundary condition on the side AD, namely

$$
\begin{aligned}
&P = 1 - \exp(-0.85t) && \text{on side AD} \\
&P = t/6 && \text{on side AD} \\
&P = t(6-t)/9 && \text{on side AD}
\end{aligned}
$$

The results in Figure 16 are expected. The trends of the solutions c mimics the trends of the exponential function $1 - \exp(-0.85t)$, the linear function $t/6$ and the quadratic function $t(6-t)/9$ of the boundary condition on side AD. Specifically, for the exponential function $1 - \exp(-0.85t)$, as time t goes to infinity, values of this function go to 1. So for big value of t, Problem 5 is similar to Problem 4 of the anisotropic inhomogeneous case. And the two plots of solutions c for Problem 4 and Problem 5 in Figure 16 verifies this, they approach a same steady state solution as t gets bigger.

V. CONCLUSION

A mixed Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic trigonometrically graded materials which are governed by the diffusion-convection equation (1) of incompressible flow. The method is easy to implement and involves a time variable free fundamental solution therefore it gives more accurate solutions. It does not generate round-off error propagation as it solves the boundary integral equation (19) independently for each specific value of t at which the solution is computed. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution sometimes contain time singular points and also solution for the next time step is based on the solution of the previous time step so that the round-off error will propagate.

The numerical method has been applied to trigonometrically graded materials. As the coefficients $d_{ij}(\mathbf{x}), v_i(\mathbf{x}), \alpha(\mathbf{x})$ do depend on the spatial variable $\mathbf{x}$ only and on the same inhomogeneity or grading function $g(\mathbf{x})$, it will be of interest to extend the study in the future to the case when the coefficients depend on different gradation functions varying also with the time variable t.

In order to use the boundary integral equation (19), the values $c(\mathbf{x}, t)$ or $P(\mathbf{x}, t)$ of the boundary conditions as stated in Section (II) of the original system in time variable t have to be Laplace transformed first. This means that from the beginning when we set up a problem, we actually put a set of approximating boundary conditions. Therefore it is really important to find a very accurate technique of numerical Laplace transform inversion. Based on the results of the problems in Section IV-A, the Stehfest formula in (20) is quite accurate.

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