A Boundary Integral Equation Formulation for Unsteady Anisotropic Modified Helmholtz Problems of Spatial Variable Coefficients

Moh. Ivan Azis

Abstract—**In this paper a combined Laplace transform and boundary element method is used to solve numerically a class of variable coefficient unsteady modified Helmholtz equation. The variable coefficients equation is transformed to a constant coefficients equation. The constant coefficients equation after being Laplace transformed is then written in a boundary integral equation involving a time-free fundamental solution. The boundary-only integral equation is therefore employed to find the numerical solutions using a standard boundary element method. Finally, the results obtained are inversely transformed numerically using the Stehfest formula to get solutions in the time variable. Some problems of anisotropic exponentially graded media are considered. The results show that the combined Laplace transform and boundary element method is easy to implement and accurate.**

Index Terms—**unsteady modified Helmholtz problems, anisotropic functionally graded materials, Laplace transform, boundary element method**

I. Introduction

We will consider initial boundary value problems governed by a modified Helmholtz type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_j}\right]-\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=\alpha(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial t} \tag{1}$$

The coefficients $[\kappa_{ij}]$ $(i,j=1,2)$ is a real symmetric positive definite matrix. Also, in (1) the summation convention for repeated indices holds. Therefore equation (1) may be written explicitly as

$$\frac{\partial}{\partial x_1}\left(\kappa_{11}\frac{\partial \mu}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\kappa_{12}\frac{\partial \mu}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{12}\frac{\partial \mu}{\partial x_1}\right)$$
$$+\frac{\partial}{\partial x_2}\left(\kappa_{22}\frac{\partial \mu}{\partial x_2}\right)-\beta^2\mu=\alpha\frac{\partial c}{\partial t}$$

Equation (1) is usually used to model infiltration problems (see for examples [1]–[3]).

During the last decade functionally graded materials (FGMs) have become an important topic, and numerous studies on them for a variety of applications have been reported. FGMs are materials possessing characteristics which vary (with time and position) according to a mathematical function. Therefore equation (1) is relevant for FGMs. FGMs are mainly artificial materials which are produced to meet a preset practical performance (see for example [4], [5]). This constitutes relevancy of solving equation (1).

Manuscript received April 26, 2021; revised August 27, 2021.
This work was supported by Universitas Hasanuddin and Kementerian Pendidikan, Kebudayaan, Riset, dan Teknologi Indonesia.
M. I. Azis is a lecturer at the Department of Mathematics, Hasanuddin University, Makassar, Indonesia. e-mail: ivan@unhas.ac.id

A number of studies on the modified Helmholtz equation had been done for finding its numerical solutions. However the studies mainly focus on the modified Helmholtz equation with constant coefficients (see for example [1]–[3]). The constant coefficients equation is relevant to problems of homogeneous materials and the boundary element method (BEM) and other methods had been successfully used to find the numerical solutions to such kind of problems.

However, this is not the case for inhomogeneous materials. Due to the unavailability of fundamental solutions for equations of variable coefficients which govern problems of inhomogeneous media, the BEM can not be used straight away to solve the problems. Some progress of solving problems for inhomogeneous media using various techniques has been done. Timpitak and Pochai [6] investigated finite difference solutions of unsteady diffusion-convection problems for heterogeneous media. Noda et al. [7] studied the analytical solutions to a transient heat conduction equation of variable coefficients with a source term for a functionally graded orthotropic strip (FGOS). In this study, the inhomogeneity of the FGOS is simplified to be functionally graded in the x variable only. In [8] Azis and Clements worked on finding numerical solutions to nonlinear transient heat conduction problems for anisotropic quadratically graded materials using a boundary domain element method. The quadratically varying coefficient in the governing equation considered by Azis and Clements [8] can certainly be represented as a sum of constant and variable coefficients. Some later studies on the class of constant-plus-variable coefficients equations had been done a number of authors. Samec and Škerget [9] considered a non-steady diffusive–convective transport equation with variable velocity which is represented as a sum of constant and variable terms. Ravnik and Škerget in [10] studied steady state diffusion-convection problems with inhomogeneous isotropic diffusivity, variable velocity and incompressible fluid using a domain boundary integral equation method (DBIEM). In this work both the diffusivity and the velocity take a constant-plus-variable form. Ravnik and Škerget in [11] considered an unsteady state diffusion-convection problems with sources, inhomogeneous isotropic conductivity, variable velocity and incompressible fluid using a DBIEM. In this study both the diffusivity and the velocity are again taken to be of constant-plus-variable form. AL-Bayati and Wrobel [12], [13] focused on convection–diffusion–reaction equation of incompressible flow with constant diffusivity and variable velocity taking the form of constant-plus-variable terms. Ravnik and Tibuat [14] also considered an unsteady diffusion-convection equation with variable diffusivity and velocity. The diffusivity is of the constant-plus-variable form. By taking the variable coefficients as a sum of constant and variable coefficients, the derived integral equation will then involve both boundary and domain integrals. The constant coefficient term will contribute boundary integrals as the fundamental solutions are available, and the variable coefficient term will give domain integrals.

Reduction to constant coefficients equation is another technique that can be used to transform a variable coefficients equation to a constant coefficients equation. Recently, Azis and co-workers had been working on the steady state problems of anisotropic inhomogeneous media for several types of governing equations, for examples [15]–[20] for Helmholtz equation, [21]–[24] for the modified Helmholtz equation, [25] for elasticity problems, [26]–[30] for the diffusion convection equation, [31]–[34] for the Laplace type equation, [35]–[41] for the diffusion convection reaction equation. Some other classes of inhomogeneity functions for FGMs that differ from the class of constant-plus-variable coefficients are reported from these papers. Azis et al. also had been working on *unsteady state* problems of *anisotropic inhomogeneous* media for some types of governing equations (see [42]–[46]).

This paper is intended to extend the recently published works in [21]–[24] for steady anisotropic modified Helmholtz type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial x_j}\right]-\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=0$$

to unsteady anisotropic modified Helmholtz type equation with spatially variable coefficients of the form (1).

Equation (1) will be transformed to a constant coefficient equation from which a boundary integral equation will derived. It is necessary to place some constraint on the class of coefficients κ_{ij} and β^2 for which the solution obtained is valid. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (1).

II. The initial-boundary value problem

Referred to a Cartesian frame Ox_1x_2 solutions $\mu(\mathbf{x},t)$ and its derivatives to (1) are sought which are valid for time interval $t \geq 0$ and in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\mu(\mathbf{x},t)$ ($\mathbf{x}=(x_1,x_2)$) is specified and on $\partial\Omega_2$

$$P(\mathbf{x},t)=\kappa_{ij}(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial x_i}n_j \tag{2}$$

is specified where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\mathbf{n}=(n_1,n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$\mu(\mathbf{x},0)=0 \tag{3}$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation, and then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable s. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution μ and its derivatives for all $(\mathbf{x},t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula.

The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\kappa_{11}=\kappa_{22}$ and $\kappa_{12}=0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

III. The boundary integral equation

The coefficients $\kappa_{ij},\beta^2,\alpha$ are required to take the form

$$\kappa_{ij}(\mathbf{x}) = \overline{\kappa}_{ij}g(\mathbf{x}) \tag{4}$$

$$\beta^2(\mathbf{x}) = \overline{\beta}^2 g(\mathbf{x}) \tag{5}$$

$$\alpha(\mathbf{x}) = \overline{\alpha}g(\mathbf{x}) \tag{6}$$

where the $\overline{\kappa}_{ij},\overline{\beta}^2,\overline{\alpha}$ are constants and g is a differentiable function of $\mathbf{x}$. Further we assume that the coefficients $\kappa_{ij}(\mathbf{x})$, $\beta^2(\mathbf{x})$ and $\alpha(\mathbf{x})$ are exponentially graded by taking $g(\mathbf{x})$ as an exponential function

$$g(\mathbf{x})=\left[\exp(c_0+c_ix_i)\right]^2 \tag{7}$$

where c_0 and c_i are constants. Therefore if

$$\overline{\kappa}_{ij}c_ic_j-\lambda=0 \tag{8}$$

then (7) satisfies

$$\overline{\kappa}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j}-\lambda g^{1/2}=0 \tag{9}$$

Use of (4)-(6) in (1) yields

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g\frac{\partial\mu}{\partial x_j}\right)-\overline{\beta}^2 g\mu=\overline{\alpha}g\frac{\partial\mu}{\partial t} \tag{10}$$

Let

$$\mu(\mathbf{x},t)=g^{-1/2}(\mathbf{x})\,\psi(\mathbf{x},t) \tag{11}$$

therefore substitution of (4) and (11) into (2) gives

$$P(\mathbf{x},t)=-P_g(\mathbf{x})\,\psi(\mathbf{x},t)+g^{1/2}(\mathbf{x})\,P_\psi(\mathbf{x},t) \tag{12}$$

where

$$P_g(\mathbf{x})=\overline{\kappa}_{ij}\frac{\partial g^{1/2}}{\partial x_j}n_i \qquad P_\psi(\mathbf{x})=\overline{\kappa}_{ij}\frac{\partial\psi}{\partial x_j}n_i$$

Also, (10) may be written in the form

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left[g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right]-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial t}$$

which can be simplified

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}+g\psi\frac{\partial g^{-1/2}}{\partial x_j}\right)-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i}=-g^{-1}\frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}-\psi\frac{\partial g^{1/2}}{\partial x_j}\right)-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Rearranging and neglecting the zero terms yield

$$g^{1/2}\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\psi\overline{\kappa}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j}-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Equation (9) then implies

$$\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\left(\overline{\beta}^2+\lambda\right)\psi=\overline{\alpha}\frac{\partial\psi}{\partial t} \tag{13}$$

Taking the Laplace transform of (11), (12), (13) and applying the initial condition (3) we obtain

$$\psi^*(\mathbf{x},s)=g^{1/2}(\mathbf{x})\,\mu^*(\mathbf{x},s) \tag{14}$$

$$P_{\psi^*}(\mathbf{x},s)=\left[P^*(\mathbf{x},s)+P_g(\mathbf{x})\,\psi^*(\mathbf{x},s)\right]g^{-1/2}(\mathbf{x}) \tag{15}$$

$$\overline{\kappa}_{ij}\frac{\partial^2\psi^*}{\partial x_i\partial x_j}-\left(\overline{\beta}^2+\lambda+s\overline{\alpha}\right)\psi^*=0 \tag{16}$$

where s is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (16) is given in the form

$$\begin{aligned}\eta(\mathbf{x}_0)\,\psi^*(\mathbf{x}_0,s) &= \int_{\partial\Omega}\left[\Gamma(\mathbf{x},\mathbf{x}_0)\,\psi^*(\mathbf{x},s)\right.\\ &\quad \left.-\Phi(\mathbf{x},\mathbf{x}_0)\,P_{\psi^*}(\mathbf{x},s)\right]dS(\mathbf{x})\end{aligned} \tag{17}$$

where $\mathbf{x}_0=(a,b)$, $\eta=0$ if $(a,b)\notin\Omega\cup\partial\Omega$, $\eta=1$ if $(a,b)\in\Omega$, $\eta=\frac{1}{2}$ if $(a,b)\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a,b). The so called fundamental solution Φ in (17) is any solution of the equation

$$\overline{\kappa}_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j}-\left(\overline{\beta}^2+\lambda+s\overline{\alpha}\right)\Phi=\delta(\mathbf{x}-\mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x},\mathbf{x}_0)=\overline{\kappa}_{ij}\frac{\partial\Phi(\mathbf{x},\mathbf{x}_0)}{\partial x_j}n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\begin{aligned}\Phi(\mathbf{x},\mathbf{x}_0) &= \begin{cases}\frac{K}{2\pi}\ln R & \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}=0\\ \frac{\imath K}{4}H_0^{(2)}(\omega R) & \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}<0\\ \frac{-K}{2\pi}K_0(\omega R) & \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}>0\end{cases}\\ \Gamma(\mathbf{x},\mathbf{x}_0) &= \begin{cases}\frac{K}{2\pi}\frac{1}{R}\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{-\imath K\omega}{4}H_1^{(2)}(\omega R)\,\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{K\omega}{2\pi}K_1(\omega R)\,\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\end{cases}\\ &\quad\begin{cases}\text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}=0\\ \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}<0\\ \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}>0\end{cases}\end{aligned} \tag{18}$$

where

$$\begin{aligned}K &= \ddot{\tau}/D\\ \omega &= \sqrt{|\overline{\beta}^2+\lambda+s\overline{\alpha}|/D}\\ D &= \left[\overline{\kappa}_{11}+2\overline{\kappa}_{12}\dot{\tau}+\overline{\kappa}_{22}\left(\dot{\tau}^2+\ddot{\tau}^2\right)\right]/2\\ R &= \sqrt{(\dot{x}_1-\dot{a})^2+(\dot{x}_2-\dot{b})^2}\\ \dot{x}_1 &= x_1+\dot{\tau}x_2\\ \dot{a} &= a+\dot{\tau}b\\ \dot{x}_2 &= \ddot{\tau}x_2\\ \dot{b} &= \ddot{\tau}b\end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\kappa}_{11}+2\overline{\kappa}_{12}\tau+\overline{\kappa}_{22}\tau^2=0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (18) are given by

$$\begin{aligned}\frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1-\dot{a})\\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}(\dot{x}_1-\dot{a})\right]+\ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2-\dot{b}\right)\right]\end{aligned}$$

Use of (14) and (15) in (17) yields

$$\eta g^{1/2}\mu^*=\int_{\partial\Omega}\left[\left(g^{1/2}\Gamma-P_g\Phi\right)\mu^*-\left(g^{-1/2}\Phi\right)P^*\right]dS \tag{19}$$

This equation provides a boundary integral equation for determining the solutions μ^* and its derivatives at all points inside the domain Ω.

Knowing the solutions $\mu^*(\mathbf{x},s)$ and its derivatives $\partial\mu^*/\partial x_1$ and $\partial\mu^*/\partial x_2$ which are obtained from (19), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $\mu(\mathbf{x},t)$ and its derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The Stehfest formula is

$$\begin{aligned}\mu(\mathbf{x},t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\mu^*(\mathbf{x},s_m)\\ \frac{\partial\mu(\mathbf{x},t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_1}\\ \frac{\partial\mu(\mathbf{x},t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_2}\end{aligned} \tag{20}$$

where

$$\begin{aligned}&s_m=\frac{\ln 2}{t}m\\ &V_m=(-1)^{\frac{N}{2}+m}\times\\ &\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)}\frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!}\end{aligned}$$

IV. Numerical examples

In order to verify the analysis used to derive the boundary integral equation (19), we will consider several problems either as test examples of analytical solutions or problems without simple analytical solutions.

We assume each problem belongs to a system which is valid in given spatial and time domains, governed by equation (1), satisfying the initial condition (3) and some boundary conditions as mentioned in Section II. The characteristics of the system which are represented by the coefficients $\kappa_{ij}(\mathbf{x}),\beta^2(\mathbf{x}),\alpha(\mathbf{x})$ in equation (1) are assumed to be of the form (4), (5) and (6) in which $g(\mathbf{x})$ is an exponential function of the form (7). The coefficients

TABLE I
VALUES OF V_m OF THE STEHFEST FORMULA

V_m	$N=6$	$N=8$	$N=10$	$N=12$
V_1	1	$-1/3$	$1/12$	$-1/60$
V_2	-49	$145/3$	$-385/12$	$961/60$
V_3	366	-906	1279	-1247
V_4	-858	$16394/3$	$-46871/3$	$82663/3$
V_5	810	$-43130/3$	$505465/6$	$-1579685/6$
V_6	-270	18730	-236957.5	1324138.7
V_7		$-35840/3$	$1127735/3$	$-58375583/15$
V_8		$8960/3$	$-1020215/3$	$21159859/3$
V_9			164062.5	-8005336.5
V_{10}			-32812.5	5552830.5
V_{11}				-2155507.2
V_{12}				359251.2

$\kappa_{ij}(\mathbf{x}), \beta^2(\mathbf{x}), \alpha(\mathbf{x})$ may represent the diffusivity or conductivity, the wave number and the change rate of the unknown $\mu(\mathbf{x}, t)$, respectively.

Standard BEM with constant elements is employed to obtain numerical results. For a simplicity, a unit square will be taken as the geometrical domain for all problems. A number of 320 elements of equal length, namely 80 elements on each side of the unit square, are used. The time interval is chosen to be $0 \le t \le 5$. A FORTRAN script is developed to compute the solutions and a specific FORTRAN command is imposed to calculate the elapsed CPU time for obtaining the results. A simple script is also embedded to calculate the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any even number N for the Stehfest formula (20). Table I shows the values of V_m for several values of N.

For all problems the inhomogeneity function is taken to be

$$g^{1/2}(\mathbf{x}) \quad = \quad \exp(-0.75 + 0.35x_1 + 0.4x_2)$$

and the constant anisotropic coefficient $\overline{\kappa}_{ij}$

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

so that 8 implies

$$\lambda = 0.3065$$

We set the constant coefficient $\overline{\beta}^2$

$$\overline{\beta}^2 = 1$$

A. Examples with analytical solutions

1) Problem 1:: Other aspects that will be justified are the convergence (as N increases) and time efficiency for obtaining the numerical solutions. The analytical solutions are assumed to take a separable variables form

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where $h(\mathbf{x}), f(t)$ are continuous functions. The boundary conditions are assumed to be (see Figure 1)

P is given on side AB
P is given on side BC
μ is given on side CD
P is given on side AD

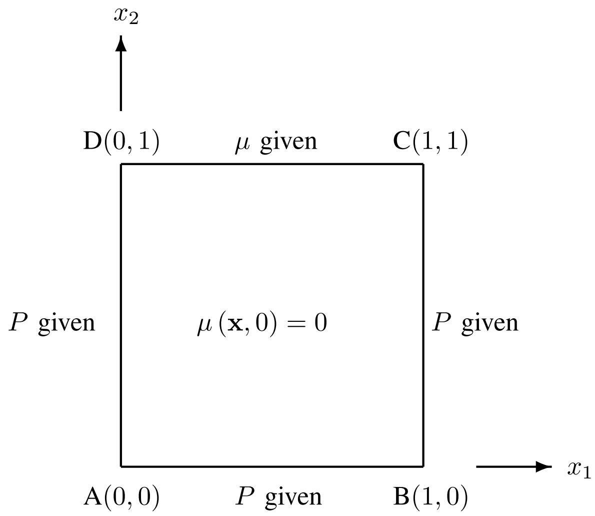

Fig. 1. The boundary conditions for the problems in Section IV-A

For each N, numerical solutions for μ and the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ at 19×19 points inside the space domain which are

$$\begin{aligned}(x_1, x_2) \quad &= \quad \{0.05, 0.1, 0.15, \ldots, 0.9, 0.95\} \\ &\qquad \times \{0.05, 0.1, 0.15, \ldots, 0.9, 0.95\}\end{aligned}$$

and 11 time-steps which are

$$t = 0.0005, 0.5, 1, 1.5, \ldots, 4, 4.5, 5$$

are computed. The aggregate relative error E is calculated using the norm

$$E = \left[\frac{\sum_t \sum_{i=1}^{19\times19} (\varsigma_{n,i} - \varsigma_{a,i})^2}{\sum_t \sum_{i=1}^{19\times19} \mu_{a,i}^2}\right]^{\frac{1}{2}}$$

where ς_n and ς_a represent respectively the numerical and analytical solutions μ or the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The elapsed CPU time τ (in seconds) is also computed and the time efficiency number ε for obtaining the numerical solutions of error E is defined as

$$\varepsilon = E\tau$$

This formula explains that the smaller time τ with smaller error E, the more efficient the procedure (smaller ε).

Case 1:: We take

$$\begin{aligned} h(\mathbf{x}) \quad &= \quad 1 - 0.35x_1 - 0.65x_2 \\ f(t) \quad &= \quad 1 - \exp(-1.75t) \end{aligned}$$

Thus for $h(\mathbf{x})$ to satisfy (16)

$$\overline{\alpha} = -1.3065/s$$

Figure 2 and Table II show the error E and efficiency number ε for solutions $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ as N increases from $N = 6$ to $N = 12$. For the solutions $\mu, \partial\mu/\partial x_2$ the error E and efficiency number ε get smaller as N moves up to $N = 10$ and for the solution $\partial\mu/\partial x_1$ the error E and efficiency number ε get smaller as N moves up to $N = 12$ and $N = 10$ respectively. According to Hassanzadeh and Pooladi-Darvish [47] increasing N will increase the accuracy up to a point, and then the accuracy will decline due to round-off errors.

As shown in Table III, the optimized value of N for solutions $\mu, \partial\mu/\partial x_2$ to achieve their smallest error E and

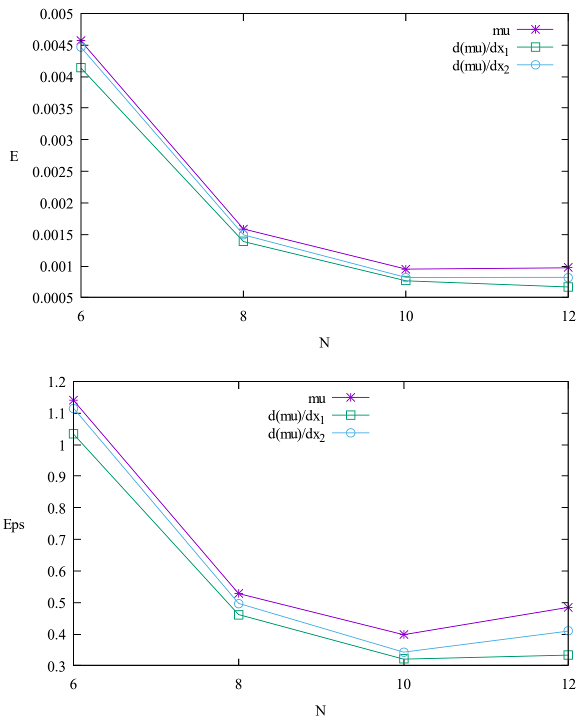

Fig. 2. The global average error E and efficiency number $\varepsilon = \tau E$ for Case 1.

TABLE II
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $\varepsilon = \tau E$ FOR CASE 1

N		6	8	10	12
τ		249.859	332.906	418.594	498.781
μ	E	0.00456413	0.00158719	0.00095128	0.00097223
	ε	1.140390	0.528384	0.398200	0.484928
$\frac{\partial \mu}{\partial x_1}$	E	0.00413618	0.00138737	0.00076626	0.00066876
	ε	1.033463	0.461865	0.320751	0.333567
$\frac{\partial \mu}{\partial x_2}$	E	0.00446496	0.00149382	0.00081896	0.00082135
	ε	1.115611	0.497304	0.342810	0.409674

time efficiency number ε is $N = 10$. For the solution $\partial \mu / \partial x_1$ the optimized value of N to reach its smallest error E and efficiency number ε is $N = 12$ and $N = 10$ respectively.

In addition Figure 3 shows the numerical and analytical solutions μ, $\partial \mu / \partial x_1$ and $\partial \mu / \partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$.

Case 2:: For the analytical solution we take

$$\begin{aligned} h(\mathbf{x}) &= \cos(1 - 0.35x_1 - 0.65x_2) \\ f(t) &= t/5 \end{aligned}$$

So that in order for $h(\mathbf{x})$ to satisfy (16)

$$\overline{\alpha} = -1.858/s$$

Figure 4 and Tables IV and V show that for solution μ the smallest error E and efficiency number ε are achieved when $N = 12$ and $N = 8$ respectively. Whereas for the solutions

TABLE III
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $\mu, \partial \mu / \partial x_1, \partial \mu / \partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER ε FOR CASE 1

	μ	$\frac{\partial \mu}{\partial x_1}$	$\frac{\partial \mu}{\partial x_2}$
E	$N = 10$	$N = 12$	$N = 10$
ε	$N = 10$	$N = 10$	$N = 10$

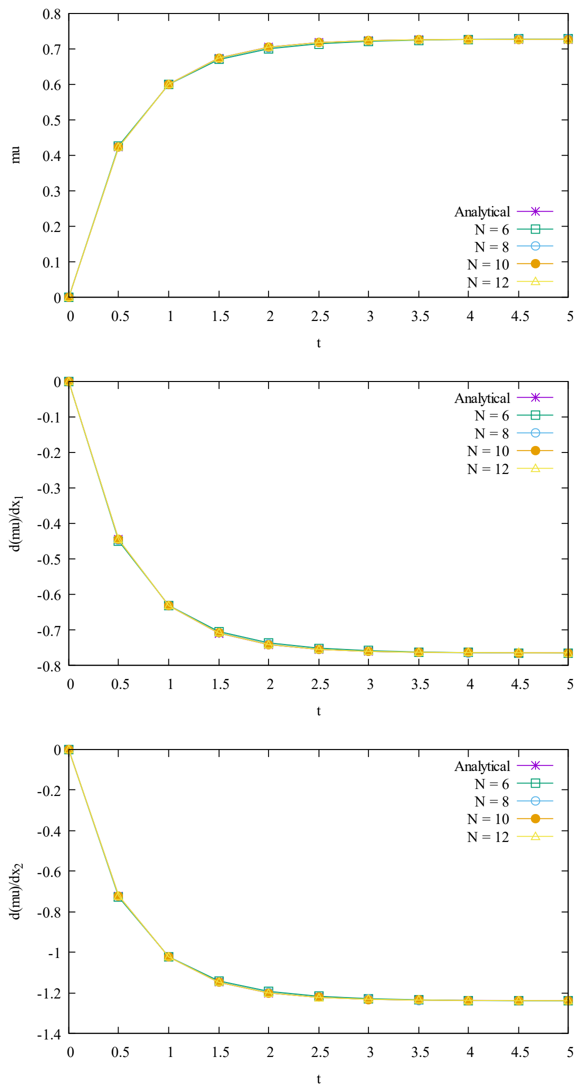

Fig. 3. The solutions μ, $\partial \mu / \partial x_1$ and $\partial \mu / \partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$ for Case 1.

TABLE IV
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $\varepsilon = \tau E$ FOR CASE 2

N		6	8	10	12
τ		399.906	530.906	653.391	765.234
μ	E	0.00168221	0.00071267	0.00060798	0.00057896
	ε	0.672728	0.378362	0.397250	0.443040
$\frac{\partial \mu}{\partial x_1}$	E	0.00228414	0.00040042	0.00040509	0.00041696
	ε	0.913440	0.212583	0.264681	0.319074
$\frac{\partial \mu}{\partial x_2}$	E	0.00335038	0.00114237	0.00125029	0.00128123
	ε	1.339838	0.606494	0.816928	0.980439

$\partial \mu / \partial x_1, \partial \mu / \partial x_2$ they are reached when $N = 8$. Figure 5 shows the numerical and analytical solutions μ, $\partial \mu / \partial x_1$ and $\partial \mu / \partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$.

Case 3:: We take

$$\begin{aligned} h(\mathbf{x}) &= \exp(-1 + 0.25x_1 + 0.65x_2) \\ f(t) &= 0.16t\,(5 - t) \end{aligned}$$

Therefore (16) gives

$$\overline{\alpha} = -0.841/s$$

Figure 6 and Tables VI and VII show that for solutions $\mu, \partial \mu / \partial x_1$ the smallest error E and efficiency number ε

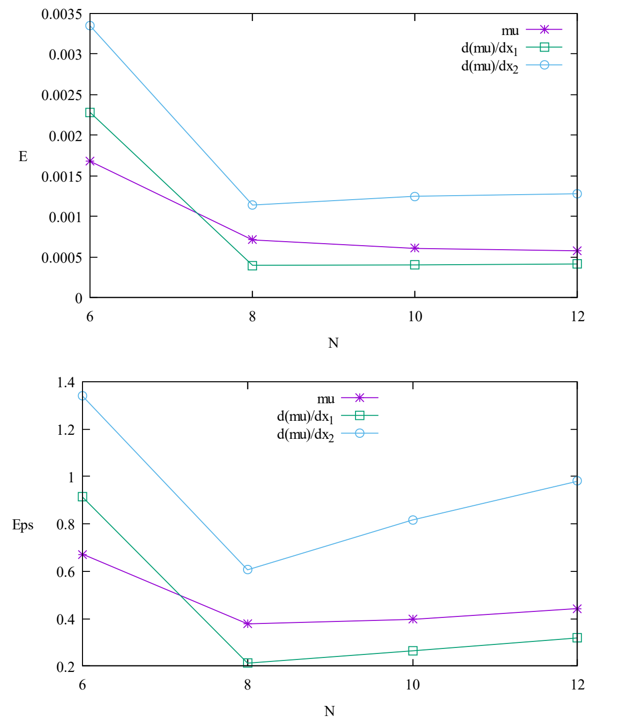

Fig. 4. The global average error E and efficiency number $\varepsilon = \tau E$ for Case 2

TABLE V
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER ε FOR CASE 2

	μ	$\frac{\partial\mu}{\partial x_1}$	$\frac{\partial\mu}{\partial x_2}$
E	$N = 12$	$N = 8$	$N = 8$
ε	$N = 8$	$N = 8$	$N = 8$

are achieved when $N = 12$ and $N = 10$ respectively, whereas for the solutions $\partial\mu/\partial x_2$ they are reached when $N = 10$ respectively. Meanwhile, Figure 7 shows the numerical and analytical solutions μ, $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$.

TABLE VI
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $\varepsilon = \tau E$ FOR CASE 3

N		6	8	10	12
τ		219.906	291.781	361.453	435.750
μ	E	0.16845606	0.01074386	0.00066556	0.00055494
	ε	37.044541	3.134857	0.240570	0.241814
$\frac{\partial\mu}{\partial x_1}$	E	0.16858928	0.01092946	0.00059684	0.00056974
	ε	37.073837	3.189010	0.215730	0.248265
$\frac{\partial\mu}{\partial x_2}$	E	0.16871683	0.01109693	0.00029777	0.00030753
	ε	37.101886	3.237877	0.107630	0.134007

TABLE VII
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER ε FOR CASE 3

	μ	$\frac{\partial\mu}{\partial x_1}$	$\frac{\partial\mu}{\partial x_2}$
E	$N = 12$	$N = 12$	$N = 10$
ε	$N = 10$	$N = 10$	$N = 10$

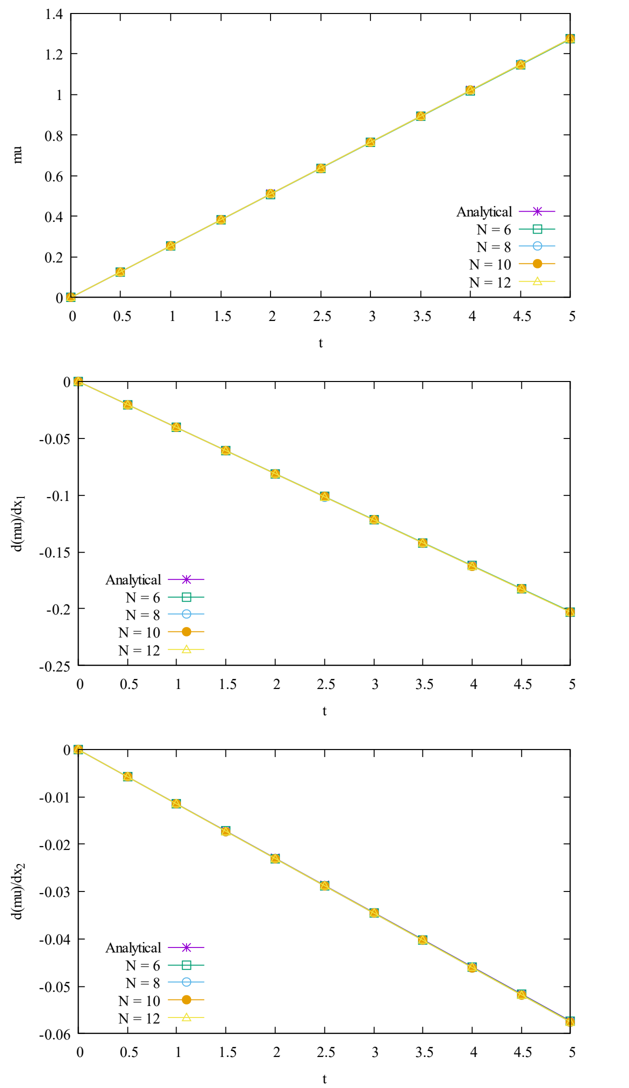

Fig. 5. The solutions μ, $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$ for Case 2

B. Examples without analytical solutions

The aim is to show the effect of inhomogeneity and anisotropy of the considered material on the solution μ.

1) Problem 2:: The material is supposed to be either inhomogeneous or homogeneous and either anisotropic or isotropic. If the material is homogeneous then

$$g(\mathbf{x}) = 1$$

and if it is isotropic then

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So that there are four cases regarding the material, namely anisotropic inhomogeneous, anisotropic homogeneous, isotropic inhomogeneous and isotropic homogeneous material. We set $\overline{\alpha} = 1$ and the boundary conditions are (see Figure 8)

$$\begin{aligned} P &= P(t) \text{ on side AB} \\ P &= 0 \text{ on side BC} \\ \mu &= 0 \text{ on side CD} \\ P &= 0 \text{ on side AD} \end{aligned}$$

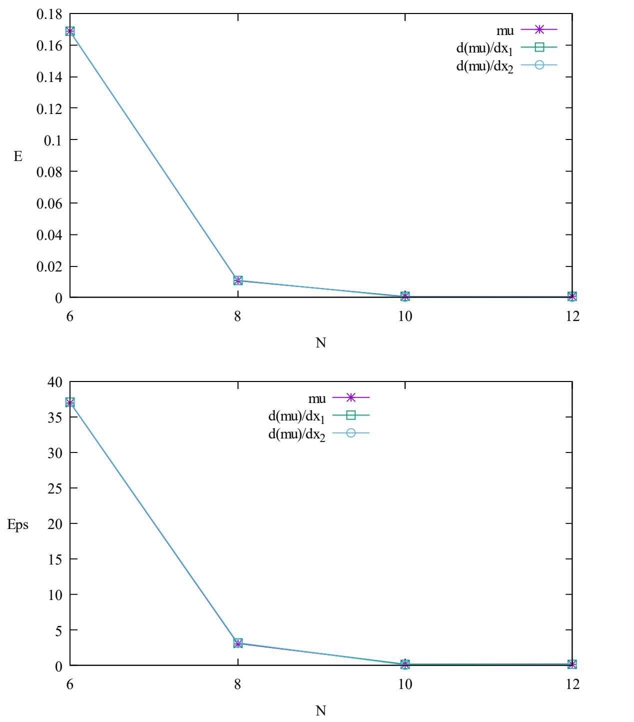

Fig. 6. The global average error E and efficiency number $\varepsilon = \tau E$ for Case 3

where $P(t)$ takes four forms

$$
\begin{aligned}
P(t) = P_1(t) &= 1 \\
P(t) = P_2(t) &= 1 - \exp(-1.75t) \\
P(t) = P_3(t) &= t/5 \\
P(t) = P_4(t) &= 0.16t(5-t)
\end{aligned}
$$

Therefore the system is geometrically symmetric about $x_1 = 0.5$. We use $N = 12$ for all cases of this problem.

The results are shown in Figures 9, 10 and 11. Figure 9 depicts solution μ at points $(0.2, 0.5), (0.8, 0.5)$ when the material under consideration is an isotropic homogeneous material. It can be seen that the values of μ at point $(0.2, 0.5)$ coincide with those at point $(0.8, 0.5)$. This is to be expected as the system is symmetrical about $x_1 = 0.5$ when the material is isotropic homogeneous. However, if the material is anisotropic homogeneous the values of μ at point $(0.2, 0.5)$ do not coincide with those at point $(0.8, 0.5)$. See Figure 10. This means anisotropy gives effect on the values of μ. Similarly, if the material is isotropic inhomogeneous (see Figure 11) the values of μ at point $(0.2, 0.5)$ differ from those at point $(0.8, 0.5)$. This indicates that inhomogeneity also gives effect on the values of μ.

In addition, Figures 9, 10 and 11 show that the trends of μ values (as the time t changes) follow the time variation of $P(t)$ except for the form of $P(t) = 1$. This is to be expected as $P(t)$ which acts as the boundary condition on side AB, is the only time-dependent quantity for the system. Moreover, as shown in Figures 9 and 11, it is also expected that the values of μ for the cases of $P_1(t) = 1$ and $P_2(t) = 1 - \exp(-1.75t)$ tend to approach the same steady state solution as t increases since both functions $P_1(t) = 1$ and $P_2(t) = 1 - \exp(-1.75t)$ will converge to 1 as t gets bigger.

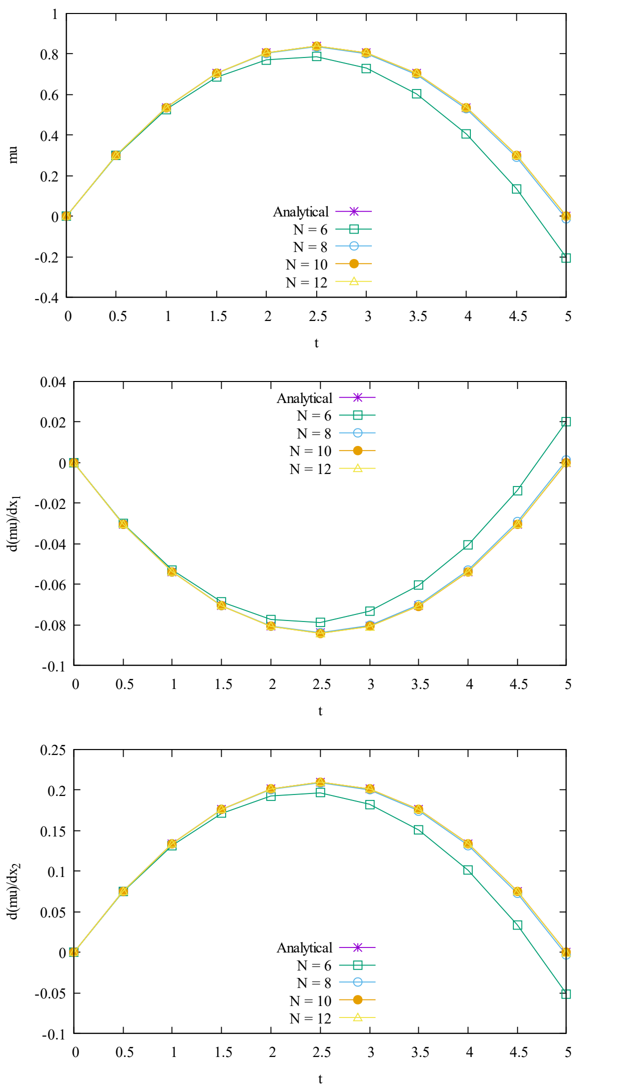

Fig. 7. The solutions μ, $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ at $(x_1, x_2) = (0.5, 0.5)$ for Case 3

V. Conclusion

A combined Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic exponentially graded materials which are governed by the modified Helmholtz type equation (1). It is easy to implement and involves a time variable free fundamental solution therefore it is quite accurate. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution usually has singular time points and the procedure may involve round-off error propagation.

In order to use the boundary integral equation (19), the values $\mu(\mathbf{x}, t)$ or $P(\mathbf{x}, t)$ of the boundary conditions as stated in Section II of the original system in time variable t have to be Laplace transformed first. This means that from the beginning when we set up a problem, we actually put a set of approached boundary conditions. Therefore it is really important to find a very accurate technique of numerical Laplace transform inversion. Based on the results of problems in Section IV-A, the Stehfest formula is quite accurate.

The combined Laplace transform and standard BEM has

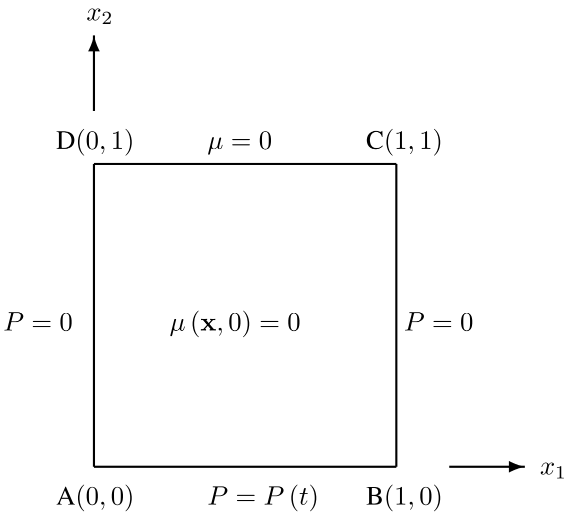

Fig. 8. The boundary conditions for Problem 2.

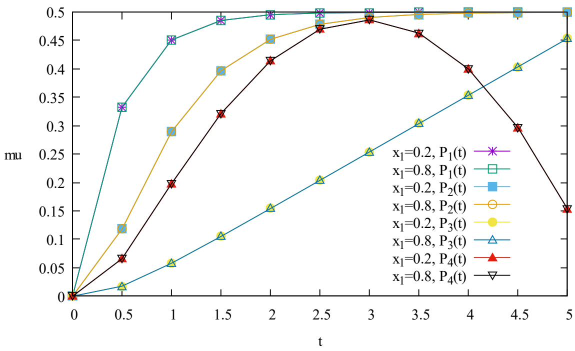

Fig. 9. Solution μ at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of isotropic homogeneous material.

been applied to a class of exponentially graded materials where the coefficients $\kappa_{ij}(\mathbf{x}), \beta(\mathbf{x}), \alpha(\mathbf{x})$ do depend only on the spatial variable $\mathbf{x}$ with the same inhomogeneity or gradation function $g(\mathbf{x})$. It will be interesting to extend the study in the future to the case when the coefficients depend on different gradation functions varying also with the time variable t.

References

[1] D. L. Clements, M. Lobo, "A BEM for time dependent infiltration from an irrigation channel," *Engineering Analysis with Boundary Elements*, vol. 34, pp. 1100, 2010.

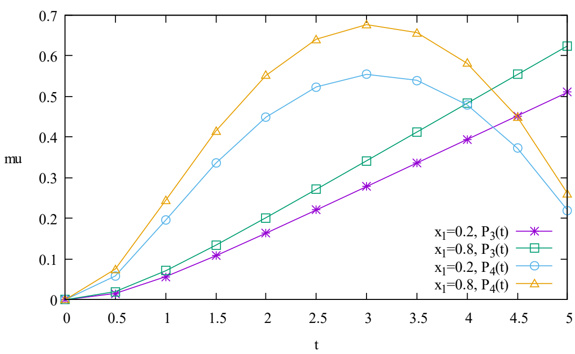

Fig. 10. Solution μ at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of anisotropic homogeneous material.

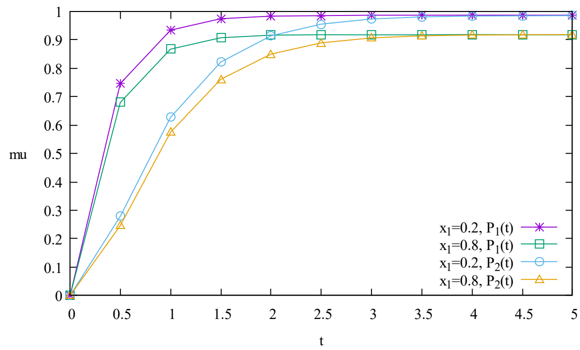

Fig. 11. Solution μ at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of isotropic inhomogeneous material.

[2] I. Solekhudin, K-C. Ang, "A DRBEM with a predictor-corrector scheme for steady infiltration from periodic channels with root-water uptake," *Engineering Analysis with Boundary Elements*, vol. 36, pp. 1199, 2012.

[3] I. Solekhudin, "Boundary interface water infiltration into layered soils using dual reciprocity methods," *Engineering Analysis with Boundary Elements*, vol. 119, pp. 280–292, 2020.

[4] S. Abotula, A. Kidane, V. B. Chalivendra, A. Shukla, "Dynamic curving cracks in functionally graded materials under thermo-mechanical loading," *Int. J. Solids Struct.*, vol. 49, pp. 1637, 2012.

[5] H. Abadikhah, P. D. Folkow, "Dynamic equations for solid isotropic radially functionally graded circular cylinders," *Compos. Struct.*, vol. 195, pp. 147, 2018.

[6] W. Timpitak, N. Pochai, "Numerical Simulations to a One-dimensional Groundwater Pollution Measurement Model Through Heterogeneous Soil," *IAENG International Journal of Applied Mathematics*, vol. 50, no.3, pp. 558–565, 2020.

[7] N. Noda, N. Sumi, M. Ohmichi, "Analysis of transient plane thermal stresses in functionally graded orthotropic strip," *Journal of Thermal Stresses*, vol. 41, pp. 1225–1243, 2018.

[8] M. I. Azis, D. L. Clements, "Nonlinear transient heat conduction problems for a class of inhomogeneous anisotropic materials by BEM," *Engineering Analysis with Boundary Elements*, vol. 32, pp. 1054–1060, 2008.

[9] N. Samec, L. Škerget, "Integral formulation of a diffusive–convective transport equation for reacting flows," *Engineering Analysis with Boundary Elements*, vol. 28, pp. 1055–1060, 2004.

[10] J. Ravnik, L. Škerget, "A gradient free integral equation for diffusion-convection equation with variable coefficient and velocity," *Eng. Anal. Boundary Elem.*, vol. 37, pp. 683–690, 2013.

[11] J. Ravnik, L. Škerget, "Integral equation formulation of an unsteady diffusion–convection equation with variable coefficient and velocity," *Computers and Mathematics with Applications*, vol. 66, pp. 2477–2488, 2014.

[12] S. A. AL-Bayati, L. C. Wrobel, "A novel dual reciprocity boundary element formulation for two-dimensional transient convection–diffusion–reaction problems with variable velocity," *Engineering Analysis with Boundary Elements*, vol. 94, pp. 60–68, 2018.

[13] S. A. AL-Bayati, L. C. Wrobel, "The dual reciprocity boundary element formulation for convection-diffusion-reaction problems with variable velocity field using different radial basis functions," *International Journal of Mechanical Sciences*, vol. 145, pp. 367–377, 2018.

[14] J. Ravnik, J. Tibuat, "Fast boundary-domain integral method for unsteady convection-diffusion equation with variable diffusivity using the modified Helmholtz fundamental solution," *Numerical Algorithms*, vol. 82, pp. 1441–1466, 2019.

[15] M. I. Azis, "Numerical solutions for the Helmholtz boundary value problems of anisotropic homogeneous media," *Journal of Computational Physics*, vol. 381, pp. 42–51, 2019.

[16] M. I. Azis, "BEM solutions to exponentially variable coefficient Helmholtz equation of anisotropic media," *J. Phys. Conf. Ser.*, vol. 1277, pp. 012036, 2019.

[17] B. Nurwahyu, B. Abdullah, A. Massinai, M. I. Azis, "Numerical solutions for BVPs governed by a Helmholtz equation of anisotropic FGM," *IOP Conf. Ser.: Earth Environ. Sci.*, vol. 279, pp. 012008, 2019.

[18] S. Hamzah, M. I. Azis, A. Haddade, A. K. Amir, "Numerical solutions to anisotropic BVPs for quadratically graded media governed by a Helmholtz equation," *IOP Conf. Ser.: Mater. Sci. Eng.*, vol. 619, pp. 012060, 2019.

[19] Paharuddin, Sakka, P. Taba, S. Toaha, M. I. Azis, "Numerical solutions to Helmholtz equation of anisotropic functionally graded materials," *J. Phys. Conf. Ser.*, vol. 1341, pp. 082012, 2019.

[20] Khaeruddin, A. Galsan, M. I. Azis, N. Ilyas, Paharuddin, "Boundary value problems governed by Helmholtz equation for anisotropic trigonometrically graded materials: A boundary element method solution," *J. Phys. Conf. Ser.*, vol. 1341, pp. 062007, 2019.

[21] M. I. Azis, I. Solekhudin, M. H. Aswad, A. R. Jalil, "Numerical simulation of two-dimensional modified Helmholtz problems for anisotropic functionally graded materials," *J. King Saud Univ. Sci.*, vol. 32, no. 3, pp. 2096–2102, 2020.

[22] R. Syam, Fahruddin, M. I. Azis, A. Hayat, "Numerical solutions to anisotropic FGM BVPs governed by the modified Helmholtz type equation," *IOP Conf. Ser.: Mater. Sci. Eng.*, vol. 619, no. 1, pp. 012061, 2019.

[23] N. Lanafie, M. I. Azis, Fahruddin, "Numerical solutions to BVPs governed by the anisotropic modified Helmholtz equation for trigonometrically graded media," *IOP Conf. Ser.: Mater. Sci. Eng.*, vol. 619, pp. 012058, 2019.

[24] M. I. Azis, R. Syam, S. Hamzah, "BEM solutions to BVPs governed by the anisotropic modified Helmholtz equation for quadratically graded media," *IOP Conference Series: Earth and Environmental Science*, vol. 279, pp. 012010, 2019.

[25] S. Hamzah, M. I. Azis, A. Haddade, E. Syamsuddin, "On some examples of BEM solution to elasticity problems of isotropic functionally graded materials," *IOP Conf. Ser.: Mater. Sci. Eng.*, vol. 619, pp. 012018, 2019.

[26] S. Suryani, J. Kusuma, N. Ilyas, M. Bahri, M. I. Azis, "A boundary element method solution to spatially variable coefficients diffusion convection equation of anisotropic media," *J. Phys. Conf. Ser.*, vol. 1341, no. 6, pp. 062018, 2019.

[27] S. Baja, S. Arif, Fahruddin, N. Haedar, M. I. Azis, "Boundary element method solutions for steady anisotropic-diffusion convection problems of incompressible flow in quadratically graded media," *J. Phys. Conf. Ser.*, vol. 1341, no. 8, pp. 062019, 2019.

[28] A. Haddade, E. Syamsuddin, M. F. I. Massinai, M. I. Azis, A. I. Latunra, "Numerical solutions for anisotropic-diffusion convection problems of incompressible flow in exponentially graded media," *J. Phys. Conf. Ser.*, vol. 1341, no. 8, pp. 082015, 2019.

[29] Sakka, E. Syamsuddin, B. Abdullah, M. I. Azis, A. M. A. Siddik, "On the derivation of a boundary element method for steady anisotropic-diffusion convection problems of incompressible flow in trigonometrically graded media," *J. Phys. Conf. Ser.*, vol. 1341, no. 6, pp. 062020, 2019.

[30] M. A. H. Assagaf, A. Massinai, A. Ribal, S. Toaha, M. I. Azis, "Numerical simulation for steady anisotropic-diffusion convection problems of compressible flow in exponentially graded media," *J. Phys. Conf. Ser.*, vol. 1341, no. 8, pp. 082016, 2019.

[31] S. N. Jabir, M. I. Azis, Z. Djafar, B. Nurwahyu, "BEM solutions to a class of elliptic BVPs for anisotropic trigonometrically graded media," *IOP Conf. Ser.: Mater. Sci. Eng.*, vol. 619, pp. 012059, 2019.

[32] N. Lanafie, N. Ilyas, M. I. Azis, A. K. Amir, "A class of variable coefficient elliptic equations solved using BEM," *IOP Conf. Ser.: Mater. Sci. Eng.*, vol. 619, pp. 012025, 2019.

[33] N. Salam, A. Haddade, D. L. Clements, M. I. Azis, "A boundary element method for a class of elliptic boundary value problems of functionally graded media," *Eng. Anal. Boundary Elem.*, vol. 84, pp. 186–190, 2017.

[34] A. Haddade, M. I. Azis, Z. Djafar, S. N. Jabir, B. Nurwahyu, "Numerical solutions to a class of scalar elliptic BVPs for anisotropic," *IOP Conf. Ser.: Earth Environ. Sci.*, vol. 279, pp. 012007, 2019.

[35] N. Salam, D. A. Suriamihardja, D. Tahir, M. I. Azis, E. S. Rusdi, "A boundary element method for anisotropic-diffusion convection-reaction equation in quadratically graded media of incompressible flow," *J. Phys. Conf. Ser.*, vol. 1341, no. 8, pp. 082003, 2019.

[36] N. Lanafie, P. Taba, A. I. Latunra, Fahruddin, M. I. Azis, "On the derivation of a boundary element method for diffusion convection-reaction problems of compressible flow in exponentially inhomogeneous media," *J. Phys. Conf. Ser.*, vol. 1341, no. 6, pp. 062013, 2019.

[37] M. I. Azis, "Standard-BEM solutions to two types of anisotropic-diffusion convection reaction equations with variable coefficients," *Eng. Anal. Boundary Elem.*, vol. 105, pp. 87–93, 2019.

[38] A. R. Jalil, M. I. Azis, S. Amir, M. Bahri, S. Hamzah, "Numerical simulation for anisotropic-diffusion convection reaction problems of inhomogeneous media," *J. Phys. Conf. Ser.*, vol. 1341, no. 8, pp. 082013, 2019.

[39] N. Rauf, H. Halide, A. Haddade, D. A. Suriamihardja, M. I. Azis, "A numerical study on the effect of the material's anisotropy in diffusion convection reaction problems," *J. Phys. Conf. Ser.* , vol. 1341, no. 8, pp. 082014, 2019.

[40] I. Raya, Firdaus, M. I. Azis, Siswanto, A. R. Jalil, "Diffusion convection-reaction equation in exponentially graded media of incompressible flow: Boundary element method solutions," *J. Phys. Conf. Ser.* , vol. 1341, no. 8, pp. 082004, 2019.

[41] S. Hamzah, A. Haddade, A. Galsan, M. I. Azis, A. M. Abdal, "Numerical solution to diffusion convection-reaction equation with trigonometrically variable coefficients of incompressible flow," *J. Phys. Conf. Ser.* , vol. 1341, no. 8, pp. 082005, 2019.

[42] M. I. Azis, I. Magdalena, Widowati, A. Haddade, "Numerical Simulation for Unsteady Helmholtz Problems of Anisotropic Functionally Graded Materials," *Engineering Letters*, vol. 29, issue 2, no. 22, pp. 526–533, 2021.

[43] M. I. Azis, I. Solekhudin, M. H. Aswad, S. Hamzah, A. R. Jalil, "A Combined Laplace Transform and Boundary Element Method for Unsteady Laplace Problems of Several Classes of Anisotropic Functionally Graded Materials," *Engineering Letters*, vol. 29, issue 2, no. 23, pp. 534–542, 2021.

[44] M. I. Azis, "A Combined Laplace Transform and Boundary Element Method for a Class of Unsteady Laplace Type Problems of Anisotropic Exponentially Graded Materials," *Engineering Letters*, vol. 29, issue 3, pp. 894–900, 2021.

[45] M. I. Azis, "An Integral Equation Method for Unsteady Anisotropic Diffusion Convection Reaction Problems of Exponentially Graded Materials and Incompressible Flow," *IAENG International Journal of Applied Mathematics*, vol. 51, issue 3, pp. 500–507, 2021.

[46] M. I. Azis, "Numerical Solution for Unsteady Diffusion Convection Problems of Anisotropic Trigonometrically Graded Materials with Incompressible Flow," *IAENG International Journal of Applied Mathematics*, vol. 51, issue 3, pp. 811–819, 2021.

[47] H. Hassanzadeh, H. Pooladi-Darvish, "Comparison of different numerical Laplace inversion methods for engineering applications," *Appl. Math. Comput.*, vol. 189, pp. 1966–1981, 2007.