Error of Approximation of Functions, Conjugate to the Functions Belonging to Weighted Lipschitz Class Using Matrix Means

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***Abstract*—A new function class $W(L^p, \Psi(t), \beta)$ was initially introduced by Srivastava and Singh [Trigonometric approximation of periodic functions belonging to weighted Lipschitz class $W(L^p, \Psi(t), \beta)$, Contemporary Mathematics, American Mathematical Society, 645 (2015) 283-291], which was a weighted version of $Lip(\omega(t), p)$-class, with weight function $\sin^{\beta p}(x/2)$, and authors obtained the error of approximation of functions belonging to this class by matrix means of its Fourier series. In this paper, we determine the error (degree) of approximation of conjugate functions belonging to this class using matrix means of its conjugate Fourier series. We also discuss some corollaries from our results.**

***Index Terms*—$W(L^p, \Psi(t), \beta)$-class, Degree of approximation, Conjugate Fourier series, Matrix Means.**

I. Introduction

MANY researchers had done a sufficient amount of work in the area of approximation of $\widetilde{g}(x)$ (conjugate function of g) lies in various Lipschitz classes. The error of approximation of $\widetilde{g}(x)$ lies in $Lip\alpha$ and $Lip(\alpha, p)$ was studied in [1]–[8]. After these studies many researchers worked in analyzing the error of approximation of the functions lies in $Lip(\xi(t), p)$ and obtained the error of the order of $m^{1/p}\xi(1/m)$, for instance, one can see [9]–[13]. Similarly, in the area of the error of approximation of the conjugate of a function lies in $W(L^p, \xi(t))$, some essential results was introduced by Nigam and Sharma [14], Qureshi [15] and Dhakal [16]. Lal [17], Rhoades [18] and Mittal et. al. [19], [20] have obtained the error of order $m^{\beta+1/p}\xi(1/m)$. Mishra et. al. [21], [22], Nigam and Sharma [23], Deǵer [24] and Singh and Srivastava [25] have got error of approximation of functions, conjugate to the functions belonging to $W(L^p, \xi(t))$-class and gave exciting results using different types of summability. Singh and Srivastava [25] obtained error of order $(m+1)^{\beta}\xi(1/(m+1))$, by using the general summability method. The result of Singh and Srivastava [25] was sharper than the previous result and also free from p. Rathore and Singh [26] has obtained error of approximation of function conjugate to $W(L^p, \xi(t))$. They also relaxed the conditions imposed on $\xi(t)$. Recently, Singh [27] has obtained an error of approximation of $\widetilde{g}(x)$, conjugate to functions lies in $W(L^p, \beta, \xi)$ $p \geq 1$ using product summability $C^1.T$-mean under the weighted norm. Recently, Sharma [28] worked on the error of approximation of conjugate functions lies in $W(L^p, \xi(t))$ and provided better results than previous ones. Using the properties of approximation of functions, many engineers and scientists have designed digital filters. Psarakis and Moustakiedes [29], developed a new L^2-based method for designing Finite Impulse Response digital filters to acquire ideal approximation. Also, L^p-space, L^2-space and L^∞-space are of much importance in designing digital filters.

In 2015, Srivastava and Singh [30] introduced a new Lipschitz class $W(L^p, \Psi(t), \beta)$. It is a weighted version of $Lip(\omega(t), p)$, with weight function $\sin^{\beta p}(x/2)$, defined as: $W(L^p, \Psi(t), \beta) = \{g \in L^p[0, 2\pi] : \|(g(x+t) - g(x)).\sin^{\beta}(x/2)\|_p = O(\Psi(t)/t^{1/p})\}$, where $\beta \geq 0$, $t > 0$, $p \geq 1$ and $\Psi(t)$ is an increasing and positive function depending on β.

For $\Psi(t) = t^{1/p}\xi(t)$, then $W(L^p, \Psi(t), \beta)$ coincides with $W(L^p, \xi(t)) = \{g \in L^p[0, 2\pi] : \|(g(x+t) - g(x)).\sin^{\beta}(x/2)\|_p = O(\xi(t))\}$, where $p \geq 1$, $\beta \geq 0$, $t > 0$ and $\xi(t)$ is an increasing and positive function [31]–[33]. Also, if we take $\beta = 0$ and $\Psi(t) = \omega(t)$, then $W(L^p, \Psi(t), \beta)$ reduces to $Lip(\omega(t), p) = \{g \in L^p[0, 2\pi] : \|g(x+t) - g(x)\|_p = O(t^{-1/p}\omega(t))\}$, where $p \geq 1, t > 0$ and $\omega(t)$ is an increasing and positive function [34]. Khan and Ram [35] defined $Lip(\xi(t), p) = \{g \in L^p[0, 2\pi] : |g(x+t) - g(x)| = O(\xi(t)t^{-1/p})\}$, $p > 1$, $t > 0$. For $\Psi(t) = \xi(t)$ and $\beta = 0$, $Lip(\xi(t), p)$ is a subset of $W(L^p, \Psi(t), \beta)$, since $\|.\|_p = O(\|.\|_\infty)$. For $\beta = 0$ and $\Psi(t) = t^{\alpha+1/p}, 0 < \alpha \leq 1$, $W(L^p, \Psi(t), \beta)$ reduces to $Lip(\alpha, p)$.

The $L^p[0, 2\pi]$-space is defined as follows:

$$L^p[0, 2\pi] = \left\{g : [0, 2\pi] \to \mathbf{R} : \int_0^{2\pi} |g(x)|^p \, dx < \infty\right\}, p \geq 1.$$

Then for $g \in L^p[0, 2\pi]$, the p-norm can be defined as given below:

$$\|g\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi} |\, g(x) \,|^p \, dx\right\}^{1/p}, \ (1 \leq p < \infty),$$

$$\text{and } \|g\|_\infty := \text{ess} \sup_{0 \leq x \leq 2\pi} |\, g(x) \,| \, .$$

Let g be a 2π periodic function belonging to $L^p[0, 2\pi]$ with $p \geq 1$. Then the trigonometric Fourier series and conjugate Fourier series of g can be written as

$$g(x) \sim \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \cos rx + b_r \sin rx) = \sum_{r=0}^{\infty} A_r,$$

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and

$$\widetilde{g}(x) \sim \sum_{r=1}^{\infty}(b_r \cos rx - a_r \sin rx) = \sum_{r=0}^{\infty} B_r,$$

respectively.
Let

$$s_m(g;x) := \frac{a_0}{2} + \sum_{r=1}^{m}(a_r \cos rx + b_r \sin rx),\ m \in \mathbf{N}$$

$$\text{with} \quad s_0(g;x) = \frac{a_0}{2},$$

and

$$\widetilde{s_m}(g;x) := \sum_{r=1}^{m}(b_r \cos rx - a_r \sin rx),\ m \in \mathbf{N}$$

$$\text{with} \quad \widetilde{s_0}(g;x) = 0,$$

denote the $(m+1)^{th}$ partial sums of Fourier series and conjugate Fourier series with respect to g, respectively. Here a_r and b_r, appeared in the above expressions, are called Fourier coefficients and can be evaluated easily as per the requirement.

The error of approximation $E_m(g)$ of function $g \in L^p$-space by a trigonometric polynomial $T_m(x)$ of order m is given by

$$E_m(g) := \min_{T_m} \| g(x) - T_m(x) \|_p,$$

where $T_m(x)$ is approximant of $g(x)$ and this approach is known as trigonometric Fourier approximation.

Define

$$\widetilde{t}_m(g;x) := \sum_{r=0}^{m} a_{m,r}\widetilde{s}_r(g;x), \;\; m \in \mathbf{N}_0,$$

where $T \equiv (a_{m,r} \geq 0$ for every $m, r)$ is a lower triangular matrix such that $a_{m,-1} = 0$, $A_{m,r} = \sum_{v=r}^{m} a_{m,v}$ and $A_{m,0} = 1$, $m \in \mathbf{N}_0$. If $\widetilde{t_m}(g;x) \to s$ as $m \to \infty$, then the Fourier series of the function g is called T-summable to s.
If $a_{m,r} = \frac{p_{m-r}}{P_m}$, for $0 \leq r \leq m$, and $a_{m,r} = 0$, for $r > m$, where $P_m = p_0 + p_1 + ... + p_m \neq 0 \to \infty$ as $m \to \infty$, then matrix T reduce to Nörlund matrix and denoted by N_p.

We also write

$$J(m,t) = \frac{1}{2\pi}\sum_{r=0}^{m} a_{m,m-r}\frac{\cos\left(m - r + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)},$$

$$\eta := [t^{-1}], \text{the integer part of } \; t^{-1},$$

and

$$\widetilde{g}(x) = -\frac{1}{2\pi}\lim_{\epsilon \to 0}\int_{\epsilon}^{\pi} \Psi_x(t)\cot(t/2)\,dt,$$

where $\Psi_x(t) = g(x+t) - g(x-t)$.

The conjugate function of the function g is defined as above and is denoted by $\widetilde{g}$. For $g \in W(L^p, \Psi(t), \beta)$, the function $\Psi_x(t)$ also belongs to $W(L^p, \Psi(t), \beta)$ [36].

II. MAIN RESULT

Recently, Srivastava and Singh [30] defined $W(L^p, \Psi(t), \beta)$- class, which was a weighted version of $Lip(\omega(t), p)$-class, with weight function $\sin^{\beta p}(x/2)$, and they obtained the error of approximation of functions belonging to this class by matrix means of its Fourier series. Here, we consider the same function class and determine the order of approximation of $\widetilde{g}$, conjugate to the function g belonging to this class using matrix means of its conjugate Fourier series.

Theorem. Let $T \equiv (a_{m,r})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to r, for $0 \leq r \leq m$) entries. Then the error of approximation of a 2π periodic function $\widetilde{g}$, conjugate of g lies in $W(L^p, \Psi(t), \beta)$ with $p \geq 1$ and $0 \leq \beta < 1/p$ by matrix means of its conjugate Fourier series is given by

$$\| \widetilde{t_m}(g;x) - \widetilde{g}(x) \|_p = O\left((m+1)^{\beta+1/p}\Psi\left(\frac{\pi}{m+1}\right)\right), \tag{1}$$

where $\Psi(t)$ is an increasing and positive function satisfying the following conditions:

$$\Psi(t)/t^{\beta+1/p} \text{ is an increasing function,} \tag{2}$$

$$\left(\frac{\Psi_x(t)\,\sin^{\beta}(t/2)}{\Psi(t)\,t^{-1/p}}\right) \text{is bounded function of } t, \tag{3}$$

$$\left(\int_{\pi/(m+1)}^{\pi}\left(\frac{\Psi(t)}{t^{1+1/p+\beta}}\right)^{p} dt\right)^{1/p} = O\left((m+1)^{\beta+1} \times \Psi\left(\frac{\pi}{m+1}\right)\right), \tag{4}$$

where condition (3) holds uniformly in x and $p^{-1}+q^{-1} = 1$.

III. LEMMAS

Here few lemmas are given, which are useful to prove our theorems:

Lemma 1. Let $T \equiv (a_{m,r})$ be a matrix defined in the Theorem then,

$$|J(m,t)| = O(t^{-1}), \;\; \text{for } t \in \left(0, \frac{\pi}{m+1}\right].$$

Proof. Applying $|\cos(t)| \leq 1$ and $t/\pi \leq \sin\left(\frac{t}{2}\right)$ for $t \in \left(0, \frac{\pi}{m+1}\right]$,

$$\begin{aligned}
|J(m,t)| &= \left|(2\pi)^{-1}\sum_{r=0}^{m}\frac{a_{m,m-r}\,\cos\left(m-r+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right| \\
&\leq (2\pi)^{-1}\sum_{r=0}^{m} a_{m,m-r}\left|\frac{\cos\left(m-r+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right| \\
&\leq (2\pi)^{-1}\sum_{r=0}^{m} a_{m,m-r}\frac{\pi}{t} \\
&\leq t^{-1}\sum_{r=0}^{m} a_{m,m-r} \\
|J(m,t)| &= O\left(t^{-1}\right).
\end{aligned}$$

Lemma 2. Let $T \equiv (a_{m,r})$ be a matrix defined in Theorem then,

$$|J(m,t)| = O\left(\frac{A_{m,m-\eta}}{t}\right), \text{ for } t \in \left(\frac{\pi}{m+1}, \pi\right].$$

Proof. Applying $t/\pi \le \sin\left(\frac{t}{2}\right)$ for $t \in \left(\frac{\pi}{m+1}, \pi\right]$, we have

$$\begin{aligned} |J(m,t)| &= \frac{1}{2\pi}\left|\sum_{r=0}^{m} \frac{a_{m,m-r}\cos\left(m-r+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right| \\ &\le O(t^{-1})\left|Re\sum_{r=0}^{m} a_{m,m-r}\, e^{i(m-r+1/2)t}\right| \\ &\le O(t^{-1})\left|\sum_{r=0}^{m} a_{m,m-r}\, e^{i(m-r)t}\right|. \end{aligned} \tag{5}$$

According to McFadden[[37], pp.8, Lemma 5.11], we have

$$\begin{aligned} \left|\sum_{r=0}^{m} a_{m,m-r}e^{i(m-r)t}\right| &= \left|e^{imt}\sum_{r=0}^{m} a_{m,m-r}e^{-irt}\right| \\ &\le \left|\sum_{r=0}^{\eta-1} a_{m,m-r}e^{-irt}\right| + \left|\sum_{r=\eta}^{m} a_{m,m-r}e^{-irt}\right| \\ &\le \sum_{r=0}^{\eta-1} a_{m,m-r} + 2a_{m,m-\eta}\sup_{\eta\le r\le m}\left|\frac{e^{-i(r+1)t}-1}{e^{-it}-1}\right| \\ &\le A_{m,m-\eta+1} + 2a_{m,m-\eta}\left(\frac{1}{\sin(t/2)}\right) \\ &\le A_{m,m-\eta} + 2a_{m,m-\eta}\,(\eta+1) \\ &= O(A_{m,m-\eta}). \end{aligned} \tag{6}$$

According to $2\,a_{m,m-\eta}(\eta+1) = O(A_{m,m-\eta})$(from increasing nature of $a_{m,r}$).

Thus from (5) and (6),

$$|J(m,t)| = O\left(\frac{A_{m,m-\eta}}{t}\right)$$

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IV. PROOF OF THEOREM

Case 1. For $p > 1$, we have

$$\widetilde{s_m}(g;x) - \widetilde{g}(x) = (2\pi)^{-1}\int_0^{\pi} \frac{\Psi_x(t)\cos\left(m+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt,$$

and $\widetilde{t_m}(g;x) - \widetilde{g}(x) = \sum_{r=0}^{m} a_{m,r}\left[\widetilde{s_r}(g;x) - \widetilde{g}(x)\right]$

$$\begin{aligned} &= \int_0^{\pi} \Psi_x(t)\,\frac{1}{2\pi}\sum_{r=0}^{m} a_{m,r}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt \\ &= \int_0^{\pi} \Psi_x(t)\,\frac{1}{2\pi}\sum_{r=0}^{m} a_{m,m-r}\frac{\cos\left(m-r+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt \\ &= \int_0^{\pi} \Psi_x(t)\,J(m,t)\,dt \\ &= \int_0^{\pi/(m+1)} \Psi_x(t)\,J(m,t)\,dt + \int_{\pi/(m+1)}^{\pi} \Psi_x(t)\,J(m,t)\,dt \\ &= R_1 + R_2, \text{ say.} \end{aligned} \tag{7}$$

Applying $\Psi_x(t) \in W(L^p, \Psi(t), \beta)$, Hölder's inequality, Lemma 1, $t/\pi \le \sin\left(\frac{t}{2}\right)$ for $t \in (0,\pi]$, condition (2), (3), mean value theorem for integrals and $p^{-1}+q^{-1}=1$, we have

$$\begin{aligned} |R_1| &\le \int_0^{\pi/(m+1)} \left(\frac{\sin^{\beta}(t/2)\,|\Psi_x(t)|}{t^{-1/p}\,\Psi(t)}\frac{|J(m,t)|\,\Psi(t)}{t^{1/p}\,\sin^{\beta}(t/2)}\right)dt \\ &\le \left[\int_0^{\pi/(m+1)}\left(\frac{\sin^{\beta}(t/2).\,|\Psi_x(t)|}{t^{-1/p}\,\Psi(t)}\right)^p dt\right]^{1/p} \times \\ &\quad \left[\int_0^{\pi/(m+1)}\left(\frac{|J(m,t)|.\,\Psi(t)}{t^{1/p}\,\sin^{\beta}(t/2)}\right)^q dt\right]^{1/q} \\ &= O(m+1)^{-1/p}\,\Psi\left(\frac{\pi}{m+1}\right) \times \\ &\quad \left[\int_0^{\pi/(m+1)}\left(\frac{1}{t^{1/p+\beta+1}}\right)^q dt\right]^{1/q} \\ &= O\left((m+1)^{\beta+1/p}\Psi\left(\frac{\pi}{m+1}\right)\right). \end{aligned} \tag{8}$$

Applying Hölder's inequality, Lemma 2, $t/\pi \le \sin\left(\frac{t}{2}\right)$ for $t \in (0,\pi]$, $p^{-1}+q^{-1}=1$, condition (4) and boundedness of $\left(\frac{\Psi_x(t)\,\sin^{\beta}(t/2)}{\Psi(t)\,t^{-1/p}}\right)$, we have

$$\begin{aligned} |R_2| &\le \int_{\pi/(m+1)}^{\pi} \left(\frac{|\Psi_x(t)|.\sin^{\beta}(t/2)}{\Psi(t)}\frac{|J(m,t)|\,\Psi(t)}{\sin^{\beta}(t/2)}\right)dt \\ &= O\left[\int_{\pi/(m+1)}^{\pi}\left(t^{-1/p}\frac{\Psi(t)}{t}\frac{A_{m,m-\eta}}{t^{\beta}}\right)dt\right] \\ &= O\left[\int_{\pi/(m+1)}^{\pi}\left(\frac{\Psi(t)}{t^{1+\beta+1/p}}\right)^p dt\right]^{1/p} \times \\ &\quad \left[\int_{\pi/(m+1)}^{\pi}\{A_{m,m-\eta}\}^q\,dt\right]^{1/q} \\ &= O\left[\int_{\pi/(m+1)}^{\pi}\left(\frac{\Psi(t)}{t^{1+\beta+1/p}}\right)^p dt\right]^{1/p} \times \\ &\quad \left[\int_{\pi/(m+1)}^{\pi}\left\{\frac{\pi}{(m+1)t}\right\}^q dt\right]^{1/q} \\ &= O(m+1)^{\beta+1}\,\Psi\left(\frac{\pi}{m+1}\right)(m+1)^{-1/q} \\ &= O\left((m+1)^{\beta+1/p}\Psi\left(\frac{\pi}{m+1}\right)\right), \end{aligned} \tag{9}$$

for $A_{m,m-\eta} = O\left(\frac{\pi}{(m+1)t}\right)$ (using regularity condition of $a_{m,r}$).

Combining (7)-(9), we have

$$|\widetilde{t_m}(g;x) - \widetilde{g}(x)| = O\left((m+1)^{\beta+1/p}\Psi\left(\frac{\pi}{m+1}\right)\right).$$

Case 2. For $p = 1$, as explained in the above proof and applying Hölder's inequality $(p = 1)$, we have

$$|R_1| \leq \int_0^{\pi/(m+1)} \left(\frac{|\Psi_x(t)|.\sin^\beta(t/2)}{t^{-1}\,\Psi(t)} \frac{\Psi(t)\,|J(m,t)|}{t\,\sin^\beta(t/2)} \right) dt$$

$$\leq O(m+1)\text{ess} \sup_{0<t\leq \pi/(m+1)} \left(\left| \frac{\Psi_x(t)}{t^{-1}\,\Psi(t)} \frac{\Psi(t)}{t^{\beta+1}} \right| \right) \times \int_0^{\pi/(m+1)} 1\,dt$$

$$= O\left((m+1)^{\beta+1} \Psi\left(\frac{\pi}{m+1} \right) \right), \quad (10)$$

according to Lemma 1, $t/\pi \leq \sin\left(\frac{t}{2}\right)$ for $t \in (0, \pi]$ and condition (2), (3).

Applying Hölder's inequality $(p = 1)$, $t/\pi \leq \sin\left(\frac{t}{2}\right)$ for $t \in (0, \pi]$, Lemma 2, condition (4) and boundedness of $\left(\frac{\Psi_x(t)\,\sin^\beta(t/2)}{\Psi(t)\,t^{-1}} \right)$, we have

$$|R_2| \leq \int_{\pi/(m+1)}^{\pi} \left(\frac{|\Psi_x(t)|.\sin^\beta(t/2)}{\Psi(t)} \frac{|J(m,t)|\,\Psi(t)}{\sin^\beta(t/2)} \right) dt$$

$$\leq O\left[\int_{\pi/(m+1)}^{\pi} \left(t^{-1} \frac{\Psi(t)}{t} \frac{A_{m,m-\eta}}{t^\beta} \right) dt \right]$$

$$\leq O\left[\int_{\pi/(m+1)}^{\pi} \left(\frac{\Psi(t)}{t^{2+\beta}} \right) dt \right] \times ess \sup_{\pi/(m+1)\leq t\leq \pi} |A_{m,m-\eta}|$$

$$= O(m+1)^{\beta+1} \Psi\left(\frac{\pi}{m+1} \right) \times ess \sup_{\pi/(m+1)\leq t\leq \pi} \left| \frac{\pi}{(m+1)t} \right|$$

$$= O\left((m+1)^{\beta+1} \Psi\left(\frac{\pi}{m+1} \right) \right), \quad (11)$$

in view of $A_{m,m-\eta} = O\left(\frac{\pi}{(m+1)t} \right)$.

Combining $(7), (10)$ and (11), we get

$$|\widetilde{t_m}(g;x) - \widetilde{g}(x)| = O\left((m+1)^{\beta+1} \Psi\left(\frac{\pi}{m+1} \right) \right).$$

Hence, for $p \geq 1$, we have

$$\|\widetilde{t_m}(g;x) - \widetilde{g}(x)\|_p = O\left((m+1)^{\beta+1/p} \Psi\left(\frac{\pi}{m+1} \right) \right).$$

It completes the proof of Theorem.

V. Corollaries

Here few corollaries are given, which are derived from our results.

1. If $\psi(t) = t^{1/p}\xi(t)$, then for $\widetilde{g} \in W(L^p, \xi(t))$,

$$\|\widetilde{t_m}(g;x) - \widetilde{g}(x)\|_p = O\left((m+1)^\beta\, \xi\left(\frac{\pi}{m+1} \right) \right),$$

where $p \geq 1$ and $\xi(t)$ is an increasing and positive function such that:

$$\xi(t)/t^\beta \text{is an increasing function}, \quad (12)$$

$$\frac{\psi_x(t)\,\sin^\beta(t/2)}{\xi(t)} \text{ is a bounded function of } t, \quad (13)$$

$$\left(\int_{\pi/(m+1)}^{\pi} \left(\frac{\xi(t)}{t^{1+\beta}} \right)^p dt \right)^{1/p} = O\left((m+1)^{\beta+1/q} \times \xi\left(\frac{\pi}{m+1} \right) \right). \quad (14)$$

Here $p^{-1} + q^{-1} = 1$ and condition (13) holds uniformly in x.

2. If $\beta = 0$ and $\Psi(t) = t^{1/p}\,\xi(t)$, then for $\widetilde{g} \in Lip(\xi(t), p)$,

$$\|\widetilde{t_m}(g;x) - \widetilde{g}(x)\|_p = O\left(\xi\left(\frac{\pi}{m+1} \right) \right),$$

where $p \geq 1$ and $\xi(t)$ is an increasing and positive function such that:

$$\frac{\Psi_x(t)}{\xi(t)} \text{is a bounded function of}\, t,$$

$$\left(\int_{\pi/(m+1)}^{\pi} \left(\frac{\xi(t)}{t} \right)^p dt \right)^{1/p} = O\left((m+1)^{1/q} \times \xi\left(\frac{\pi}{m+1} \right) \right).$$

3. If $\beta = 0$ and $\Psi(t) = t^{\alpha+1/p}$, $\alpha \in (0, 1/q)$, then for $\widetilde{g} \in Lip(\alpha, p)$,

$$\|\widetilde{t_m}(g;x) - \widetilde{g}(x)\|_p = O((m+1)^{-\alpha}),\ \alpha < 1/q, p > 1.$$

4. If $p \to \infty$ in Corollary 3, then for $\widetilde{g} \in Lip\alpha$ $(0 < \alpha \leq 1)$,

$$\|\widetilde{t_m}(g;x) - \widetilde{g}(x)\|_\infty = \begin{cases} O((m+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\frac{\log(m+1)}{m+1} \right), & \alpha = 1. \end{cases}$$

References

[1] K. Qureshi, "On the Degree of Approximation of a periodic function g by almost Riesz means of its conjugate series," *Indian Journal of Pure & Applied Mathematics*, vol.13, no.10, pp. 1136–1139, 1982.

[2] K. Qureshi, "On the Degree of Approximation of functions belonging to the Lipschitz class by means of a conjugate series," *Indian Journal of Pure & Applied Mathematics*, vol.12, no. 9, pp. 1120–1123, 1981.

[3] S. Lal and J. K. Kushwaha,"Approximation of conjugate of functions belonging to the generalized Lipschitz class by lower triangular matrix means," *International Journal of Mathematical Analysis*, vol. 21, pp. 1031–1041, 2009.

[4] H. K. Nigam and K. Sharma, "Approximation of conjugate of functions belonging to $Lip\alpha$ class and weighted class $W(L^p, \xi(t))$ by product means of conjugate series of Fourier series," *European Journal of Pure and Applied Mathematics*, vol. 4, no. 3, pp.276–286, 2011.

[5] B. P. Padhy, S. K. Buxi, M. Misra and U. K. Misra," On degree of approximation by product means of conjugate series of Fourier series," *International Journal of Mathematical Sciences and Applications*, vol. 6, pp. 363–370, 2012.

[6] S. Lal and H. P. Singh,"The degree of approximation of conjugates of almost Lipschitz functions by $(N_p, q)(E, 1)$ means," *International Mathematical Forum*, vol. 5, no. 34, pp. 1663–1671, 2010.

[7] K. Qureshi,"On the degree of approximation of functions belonging to the class $Lip(\alpha, p)$ by means of a conjugate series," *Indian Journal of Pure & Applied Mathematics*, vol. 13, no. 5, pp. 560–563, 1982.

[8] S. Lal and P. N. Singh,"Degree of approximation of conjugate of $Lip(\alpha, p)$ function by $(C, 1)(E, 1)$ means of conjugate of a Fourier series," *Tamkang Journal of Mathematics*, vol. 33, no. 3, pp. 269–274, 2002.

[9] H. K. Nigam and A. Sharma, "On Approximation of conjugate of a function belonging to $Lip(\xi(t), p)$-class by product summability means of conjugate series of Fourier series," *International Journal of Contemporary Mathematical Sciences*, vol. 05, no. 54, pp. 2673–2683, 2010.

[10] S. Lal and H. K. Nigam , "Degree of Approximation of conjugate of a function belonging to $Lip(\xi(t), p)$-class by matrix summability means of conjugate Fourier series," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 9, pp. 555–563, 2001.

[11] M. L. Mittal, U. Singh, V. N. Mishra, S. Priti and S. S. Mittal, "Approximation of functions (signals) belonging to $Lip(\xi(t), p)$, $p \geq$ 1-class by means of conjugate Fourier series using linear operators," *Indian Journal of Mathematics*, vol. 47, pp. 217–229, 2005.

[12] S. Lal and J. K.Kushwaha,"Approximation of conjugate of functions belonging to the generalized Lipschitz class by lower triangular matrix means," *International Journal of Mathematics Analysis*, vol. 3, no. 21, pp. 1031–1041, 2009.

[13] S. Lal and A. Srivastava,"Approximation of conjugate of $Lip(\xi(t), p)$ function by almost triangular matrix summability method," *Bulletin of the Calcutta Mathematical Society*, vol. 98, pp. 39–48, 2006.

[14] H. K. Nigam and A. Sharma,"On approximation of conjugate of functions belonging to different classes by product means," *International Journal of Pure & Applied Mathematics*, vol. 76, no. 2, pp. 303–316, 2012.

[15] K. Qureshi, "On the degree of approximation of a function belonging to weighted $W(L^p, \xi(t))$-class," *Indian Journal of Pure & Applied Mathematics*, vol. 13, no. 4, pp. 471–475, 1982.

[16] B. P. Dhakal, "Approximation of the conjugate of a function belonging to the $W(L^p, \xi(t))$ class by $(N, p_n)(E, 1)$ means of the conjugate series of the Fourier series," *Kathmandu University Journal of Science, Engineering and Technology*, vol. 5, pp. 30–36, 2009.

[17] S. Lal, "On the degree of approximation of conjugate of a function belonging to weighted $W(L^p, \xi(t))$ class by matrix summability means of conjugate series of a Fourier series," *Tamkang Journal of Mathematics*, vol. 31, no. 4, pp. 279–288, 2000.

[18] B. E. Rhoades, "On the degree of approximation of the conjugate of a function belonging to the weighted $W(L^p, \xi(t))$-class by matrix means of the conjugate series of a Fourier series," *Tamkang Journal of Mathematics*, vol. 33, no. 4, pp. 365–370, 2002.

[19] M. L. Mittal, B .E. Rhoades and V. N. Mishra, "Approximation of signals (functions) belonging to the weighted $W(L^p, \xi(t))$-class by linear operators," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, pp. 1–10, 2006.

[20] M. L. Mittal, U. Singh and V. N. Mishra, "Approximation of functions (signals) belonging to $W(L^p, \xi(t))$-class by means of conjugate Fourier series using Nörlund operators," *Varahmihir Journal of Mathematical Sciences*, vol. 6, no. 1, pp. 383–392, 2006.

[21] V. N. Mishra, H. H. Khan and K. Khatri, "Degree of approximation of conjugate of signals (functions) by lower triangular matrix operator," *Applied Mathematics*, vol. 2, no. 12, pp. 1448–1452, 2011.

[22] V. N. Mishra, H. H. Khan, K. Khatri and L. N. Mishra, "On approximation of conjugate of signals (functions) belonging to the generalized weighted $W(L^p, \xi(t)), p \geq 1$- class by product summability means of conjugate series of Fourier series," *International Journal of Mathematical Analysis*, vol. 6, no. 35, pp. 1703–1715, 2012.

[23] H. K. Nigam and A. Sharma, "On approximation of conjugate of a function belonging to weighted $W(L^p, \xi(t))$-class by product means," *International Journal of Pure and Applied Mathematics*, vol. 70, no. 03, pp. 317–328, 2011.

[24] U. Deger, "On approximation to functions in the $W(L^p, \xi(t))$ class by a new matrix mean," *Novi Sad Journal of Mathematics*, vol. 46, no. 1, pp. 1–14, 2016.

[25] U. Singh and S. K. Srivastava, "Fourier approximation of functions conjugate to the functions belonging to weighted Lipschitz class," in *Lecture Notes in Engineering and Computer Science: Wolrd Congress on Engineering 2013*, pp. 236–240.

[26] A. Rathore and U. Singh, "On the degree of approximation of functions in a weighted Lipschitz class by almost matrix summability method," *The Journal of Analysis*, vol. 28, no. 1, pp. 21–33, 2020.

[27] U. Singh, "On the trigonometric approximation of functions in a weighted Lipschitz class," *The Journal of Analysis*, vol. 29, no. 01, pp. 325–335, 2021.

[28] K. Sharma, "Study of error of approximation of conjugate Fourier series in weighted class by almost Riesz means," *International Journal of Applied Mathematics*, vol. 33 no. 5, pp. 867–877, 2020.

[29] E. Z. Psarakis, G. V. Moustakides, "An L_2-based method for the design of $1-D$ zero phase FIR digital filters," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 44, no. 07, pp. 551–601, 1997.

[30] S. K. Srivastava and U. Singh, "Trigonometric approximation of periodic functions belonging to weighted Lipschitz class $W(L^p, \psi(t), \beta)$," *Contemporary Mathematics, American Mathematical Society*, vol. 645, pp. 283–231, 2015.

[31] U. Singh and S. K. Srivastava, "Approximation of conjugate of functions belonging to weighted Lipschitz class $W(L^p, \xi(t))$ by Hausdorff means of conjugate Fourier series," *Journal of Computational and Applied Mathematics*, vol. 259, pp. 633–640, 2014.

[32] U. Singh and S. Sonker, "Trigonometric Approximation of signals (functions) belonging to weighted $(L^p, \xi(t))$ class by Hausdorff means," *Applied Mathematics and Approximation Theory 2012*, pp. 1–8.

[33] U. Singh, M. L. Mittal and S. Sonkar, "Trigonometric approximation of signals (functions) belonging to $W(L^r, \xi(t))$-class by $(C^1.N_p)$ operator," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, pp. 1–11, 2012.

[34] S. K. Srivastava and U. Singh, "Trigonometric approximation of periodic functions belonging to $Lip(\omega(t), p)$-class," *Journal of computational and applied mathematics*, vol. 270, pp. 223–230, 2014.

[35] H. H. Khan and G. Ram, "On the degree of approximation," *Facta Universitatis Series: Mathematics and Informatics*, vol. 18, pp. 47–57, 2003.

[36] U. Singh and S. K. Srivastava, "Trigonometric approximation of functions belonging to certain Lipschitz classes by $C^1.T$ operator," *Asian-European Journal of Mathematics*, vol. 7, no. 4, pp. 1–13, 2014.

[37] L. McFadden, "Absolute Nörlund summability," *Duke Mathematical Journal*, vol. 9, no. 1, pp. 168–207, 1942.