Generalized Migrativity Properties of Uninorms over Overlap Functions

Jingru Wang, Kuanyun Zhu* and Yongwei Yang

***Abstract*—In this paper, we study the generalized α-migrativity of uninorms over any two fixed overlap functions. At first, we propose the concept of (α, O_1, O_2)-migrative uninorms over any two fixed overlap functions O_1 and O_2. And then, we investigate the related properties for the (α, O_1, O_2)-migrativity equation of uninorms. In addition, we also discuss (α, O_1, O_2)-migrative uninorm U when U belongs to one of usual classes, i.e., $U \in \mathcal{U}_{\min}, \mathcal{U}_{\max}, \mathcal{U}_{id}, \mathcal{U}_{rep}$ or $\mathcal{U}_{\cos}$.**

***Index Terms*—Generalized migrativity; Uninorms; Overlap functions**

I. INTRODUCTION

A. A brief review of overlap and grouping functions

AS a special cases of binary aggregation functions, overlap functions is introduced, respectively, by Bustince et al. [10] in 2009. The concept of overlap functions, originate from some problems in image processing [9], classification [1], [22], and also in decision making [52]. In the past few years, overlap functions have had a rapid development both in theory and applications.

In theory, there exist many discussions involving various aspects of overlap functions, for example, the work related to some important properties [3], [12], [46], [47], [50], [59], [62], [63], [66], [68], [69], [71], [70], the investigations of the corresponding implication [13], [14], [16], [17], [57], the study of the additive, multiplicative generators and interval functions [4], [18], [15], [48]. The research related to the concept extension [11], [43], [29], [51], [58].

In applications, overlap functions play an important role in many aspects of real problems, for instance, in image processing [8], [32], classification [23], [24], [33], [34], [35], [36], [37], [38], [45], decision making [22] and fuzzy community detection problems [30].

B. A short introduction of migrativity

The α-migrativity [19] as an interesting property of two place functions on [0,1] has been studied in many works in the cases of t-norms in [26], [27], [28], [44], for t-subnorms in [60], for semicopulas, quasi-copulas and copulas in [5], [20], [21], [25], [42], for uninorms in [56], for nullnorms in [73] and for aggregation functions in general in [7], [6].

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In addition, the generalization of the migrative for t-norms has been studied in [27], [28]. In [40], Mas et al. gave a similar definition for t-conorms. Moreover, this study has been extended to uninorms with the same neutral elements in [41]. In [39], Mas et al. investigated the α-migrativity of uninorms and nullnorms over t-norms and t-conorms. As an addendum to [39], Zong et al. [72] studied the α-migrativity of t-norms and t-conorms over uninorms and nullnorms. Su et al. [54] investigated the α-migrativity properties for uninorms and semi t-operators. In addition, Su et al. [53], [55], [56] considered the α-migrativity equation for two uninorms with different neutral elements. In 2018, Qiao and Hu [49] studied the α-migrativity of uninorms and nullnorms over overlap and grouping functions. As an addendum to [49], Zhu and Hu [65] considered the α-migrativity of overlap and grouping functions over uninorms and nullnorms. Recently, Zhou and Yan [64] further investigated the α-migrativity properties of overlap functions over uninorms. In addition, they showed equivalent characterizations of the migrativity equation when the uninorms belong to one of the usual classes (e.g., $\mathcal{U}_{\min}$, $\mathcal{U}_{\max}$, $\mathcal{U}_{id}$, $\mathcal{U}_{rep}$ or $\mathcal{U}_{cos}$). In 2020, Zhu et al. [66] studied the α-migrativity of overlap functions over t-norms when t-norms are continuous and give their characterizations. In particular, they also pointed out that the relationship between the α-migrativity of overlap functions over uninorms and the α-migrativity of uninorms over overlap functions. In the same year, Zhu et al. [67] revisited the α-migrativity of uninorms and nullnorms over t-norms and t-conorms.

C. The motivation of our research

It has been pointed in Subsection 1.1 that the α-migrativity among some peculiar classes of binary aggregation functions, as a meaningful and hot research area in the topic of the α-migrativity of two operations, have been continuously studied in many recent literature.

Observe that there are no corresponding researches for the generalized α-migrativity of uninorms over overlap functions, although Qiao and Hu [46] have discussed the generalized α-migrativity for overlap functions. Therefore, as a supplement of this topic from the theoretical point of view, in this paper, we consider the generalized α-migrativity of uninorms over two fixed overlap functions. More precisely, for a given α in $[0,1]$ and any $x, y \in [0,1]$, we propose the following (α, O_1, O_2)-migrativity equations

$$U(O_1(\alpha, x), y) = U(x, O_2(\alpha, y))$$

where U is a uninorm, O_1 and O_2 are two given overlap functions.

The rest of this paper is organized as follows. In Section II, we present some notions and results on overlap functions and uninorms, which shall be used throughout this paper. In Section III, we discuss the generalized α-migrativity properties of a uninorm U over any two fixed overlap functions O_1 and O_2. In Section IV, we investigate (α, O_1, O_2)-migrative uninorm U when U belongs to one certain class, e.g., $\mathcal{U}_{\min}, \mathcal{U}_{\max}, \mathcal{U}_{id}, \mathcal{U}_{rep}$ or $\mathcal{U}_{\cos}$. In Section V, our researches are concluded.

II. Preliminaries

In this section, we recall some basic notions and results which shall be needed in the sequel. Firstly, we give the definition of overlap functions.

Definition 2.1: [10] A bivariate function $O : [0,1]^2 \longrightarrow [0,1]$ is said to be an overlap function if, for any $x, y \in [0,1]$, it satisfies the following conditions:

(O1) $O(x,y) = O(y,x)$;
(O2) $O(x,y) = 0$ iff $xy = 0$;
(O3) $O(x,y) = 1$ iff $xy = 1$;
(O4) O is incerasing;
(O5) O is continuous.

Moreover, an overlap function O is said to satisfy the property 1-section deflation [14] if

(O6) $O(1,y) \leq y$ for all $y \in [0,1]$

and the property 1-section inflation [14] if

(O7) $O(1,y) \geq y$ for all $y \in [0,1]$.

In what follows, we recall the notion and some basic conclusions related to uninorms.

Definition 2.2: ([61]) A bivariate function $U : [0,1]^2 \longrightarrow [0,1]$ is said to be a uninorm if, for any $x, y, z \in [0,1]$, it satisfies the following conditions:

(U1) $U(x,y) = U(y,x)$;
(U2) $U(U(x,y),z) = U(x,U(y,z))$;
(U3) U is non-decreasing in each place;
(U4) There has a neutral element $e \in [0,1]$, i.e., $U(x,e) = x$.

It follows from Definition 2.2 that a uninorm U becomes a t-norm T when $e = 1$ and a uninorm U becomes a t-conorm S when $e = 0$. A uninorm U is called conjunctive if $U(1,0) = 0$ and a uninorm U is called disjunctive if $U(1,0) = 1$.

In the following, we recall the definition of a uninorm locally internal on the boundary, which is firstly proposed in [39] for the discussion of migrative uninorms over t-norms and t-conorms.

Definition 2.3: ([39]) A conjunctive (resp. disjunctive) uninorm U is said to be locally internal on the boundary if it satisfies $U(1,x) \in \{1,x\}$ (resp. $U(0,x) \in \{0,x\}$) for all $x \in [0,1]$.

In what follows, we list a reminder of the structure of each usual class of uninorms, which shall be used in Section IV.

Lemma 2.4: ([27]) Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with neutral element $e \in]0,1[$. Then the sections $x \mapsto U(x,1)$ and $x \mapsto U(x,0)$ are continuous in each point, except perhaps for e, if and only if U is given by one of the following formulas.

(a) If $U(1,0) = 0$, then

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}), & \text{if } x,y \in [0,e], \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \text{if } x,y \in [e,1], \\ \min(x,y), & \text{otherwise.} \end{cases}$$

(b) If $U(1,0) = 1$, then

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}), & \text{if } x,y \in [0,e], \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \text{if } x,y \in [e,1], \\ \max(x,y), & \text{otherwise.} \end{cases}$$

In both case, T_U is a t-norm and S_U is a t-conorm.

In the sequel, we write $\mathcal{U}_{\min}$ as the class of uninorms in case (a) and $\mathcal{U}_{\max}$ as the class of uninorms in case (b).

A uninirm U is said to be idempotent if $U(x,x) = x$ for all $x \in [0,1]$ [2]. Moreover, the class of idempotent uninorms is denote by $\mathcal{U}_{id}$ and idempotent uninorms can be characterized as follows.

Lemma 2.5: ([39]) U is an idempotent uninorm with neutral element $e \in [0,1]$ if and only if there exists a non-increasing function $g : [0,1] \longrightarrow [0,1]$, symmetric with respect to the main diagonal with $g(e) = e$, such that

$$U(x,y) = \begin{cases} \min(x,y), & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g(g(x))), \\ \max(x,y), & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g(g(x))), \\ \min(x,y) \text{ or } \max(x,y), & \text{if } y = g(x) \text{ and } x = g(g(x)), \end{cases}$$

is commutative in the points (x,y) satisfies $y = g(x)$ with $x = g(g(x))$.

Definition 2.6: ([27], [39]) Consider $e \in]0,1[$. A binary function $U : [0,1]^2 \longrightarrow [0,1]$ is a representable uninorm if and only if there exists a continuous strictly increasing function $h : [0,1] \longrightarrow [-\infty, +\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for all $(x,y) \in [0,1]^2 \backslash \{(0,1),(1,0)\}$ and $U(0,1) = U(1,0) \in \{0,1\}$. The function h is usually called an additive generator of U.

The class of representable uninorms is denoted by $\mathcal{U}_{rep}$.

Now, we recall the structure of uninorms which are continous in the open unit square $]0,1[^2$.

Lemma 2.7: ([31]) Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm continuous in $]0,1[^2$ with neutral element $e \in]0,e[$. Then either one of the following cases is satisfied:

(a) There exist $u \in [0,e[$, $\lambda \in [0,u]$, two continuous t-norms T_1, T_2 and a representable uninorm R such that U can be represented as

$$U(x,y) = \begin{cases} \lambda T_1(\frac{x}{\lambda}, \frac{y}{\lambda}), & \text{if } x,y \in [0,\lambda], \\ \lambda + (u-\lambda)T_2(\frac{x-\lambda}{u-\lambda}, \frac{y-\lambda}{u-\lambda}), & \text{if } x,y \in [\lambda,u]), \\ u + (1-u)R(\frac{x-\lambda}{1-u}, \frac{y-u}{1-u}), & \text{if } x,y \in]u,1[, \\ 1, & \text{if } \min(x,y) \in]\lambda,1] \text{ and } \max(x,y) = 1, \\ \lambda \textit{ or } 1, & \text{if } (x,y) \in \{(\lambda,1),(1,\lambda)\}, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

(b) There exist $v \in]e,1]$, $\omega \in [v,1]$, two continuous t-conorms S_1, S_2, and a representable uninorm R such that U

can be represented as

$$U(x,y)=\begin{cases} \upsilon R(\frac{x}{\upsilon},\frac{y}{\upsilon}), & \text{if } x,y\in]0,\upsilon[,\\ \upsilon+(\omega-\upsilon)S_1(\frac{x-\upsilon}{\omega-\upsilon},\frac{y-\upsilon}{\omega-\upsilon}), & \text{if } x,y\in[\upsilon,\omega]),\\ \omega+(1-\omega)S_2(\frac{x-\omega}{1-\omega},\frac{y-\omega}{1-\omega}), & \text{if } x,y\in[\omega,1],\\ 0, & \text{if } \max(x,y)\in[0,\omega[\\ & \text{and } \min(x,y)=0,\\ \omega \ or\ 0, & \text{if } (x,y)\in\{(0,\omega),\\ & (\omega,0)\},\\ \max(x,y), & \text{otherwise.}\end{cases}$$

We write the class of all uninorms continuous in $]0,1[^2$ as $\mathcal{U}_{\cos}$. In particular, the class of all uninorms having the form in case (a) is denoted by $\mathcal{U}_{\cos,\min}$ while the class the class of all uninorms having the form in case (b) is denoted by $\mathcal{U}_{\cos,\max}$.

III. Generalized α-migrativity of uninorms over overlap functions

In this section, at first, we introduce the concept of the generalized α-migrativity of a uninorm U over two fixed overlap functions O_1 and O_2. And then, we study the related important properties on the (α, O_1, O_2)-migrativity equation of uninorms.

Definition 3.1: Consider $\alpha \in [0, 1]$. Let O_1 and O_2 be two fixed overlap functions. A uninorm $U : [0, 1]^2 \longrightarrow [0, 1]$ is said to be α-migrative with respect to O_1 and O_2 ((α, O_1, O_2)-migrative, for short) if

$$U(O_1(\alpha, x), y) = U(x, O_2(\alpha, y)) \tag{1}$$

for any $x, y \in [0, 1]$.

Now, we discuss the properties of Eq. (1). At first, it follows from Definition 3.1 that we have the following trivial conclusion.

Proposition 3.2: For any two given overlap functions O_1 and O_2, a uninorm U with neutral element $e \in [0, 1]$ is (α, O_1, O_2)-migrative if and only if U is (α, O_2, O_1)-migrative.

Proof. It is straightforward. □

For $\alpha = 0$, we obtain the following conclusion.

Proposition 3.3: Let O_1 and O_2 be two fixed overlap functions. A uninorm U with neutral element $e \in [0, 1]$ is $(0, O_1, O_2)$-migrative if and only if U is conjunctive.

Proof. Suppose that U is conjunctive. Then, one has that

$$U(O_1(0, x), y) = 0 = U(x, O_2(0, y))$$

for all $x, y \in [0, 1]$. Thus, U is $(0, O_1, O_2)$-migrative.

Conversely, assume that U is $(0, O_1, O_2)$-migrative. Then, one obtains that

$$U(0, 1) = U(O_1(0, e), 1) = U(e, O_2(0, 1)) = 0.$$

Thus, one gets that U is conjunctive. □

As a consequence of Proposition 3.3, in the following, we only consider $\alpha \in]0, 1]$. We begin with the situation for $\alpha = 1$.

Proposition 3.4: Suppose that O_1, O_2 are two fixed overlap functions amd U is a $(1, O_1, O_2)$-migrative uninorm with neutral element $e \in [0, 1]$. Then the following statements are equivalent:

(1) $O_1(1, e) = e$ and $O_2(1, e) = e$;
(2) $O_1(1, x) = x$ and $O_2(1, x) = x$ for any $x \in [0, 1]$.

Proof. (1) $\Rightarrow$ (2): Since U is a $(1, O_1, O_2)$-migrative uninorm, for any $x \in [0, 1]$, one has that

$$\begin{aligned} O_1(1,x) &= U(O_1(1,x),e)\\ &= U(x,O_2(1,e))\\ &= U(x,e)\\ &= x. \end{aligned}$$

In a similar way, one gets that $O_2(1, x) = x$.
(2) $\Rightarrow$ (1): It is straightforward. □

It follows from Proposition 3.4 that we get the following result.

Corollary 3.5: Let O_1 and O_2 be two fixed overlap functions, U be a uninorm with neutral element $e \in [0, 1]$, $O_1(1, e) = e$ and $O_2(1, e) = e$. Then the following statements are equivalent:
(1) U is $(1, O_1, O_2)$-migrative;
(2) $O_1(1, x) = x$ and $O_2(1, x) = x$ for any $x \in [0, 1]$.

Next, we discuss the situation for $\alpha \in]0, 1[$.

Proposition 3.6: Consider $\alpha \in]0, 1[$. Let O_1 and O_2 be two fixed overlap functions, U be a (α, O_1, O_2)-migrative uninorm with neutral element $e \in [0, 1]$. Then U is conjunctive.

Proof. Suppose that U is disjunctive. Take $x = 0$ and $y = 1$ in Eq. (1). Then, one has that

$$\begin{aligned} 1 &= U(0,1)\\ &= U(O_1(\alpha,0),1)\\ &= U(0,O_2(\alpha,1))\\ &\le U(e,O_2(\alpha,1))\\ &= O_2(\alpha,1), \end{aligned}$$

which implies that $O_2(\alpha, 1) = 1$. On the other hand, it follows from item (O3) of Definition 2.1 that $\alpha = 1$, which is a contradiction. Thus, U is conjunctive. □

Proposition 3.7: Consider $\alpha \in]0, 1[$. Let O_1 and O_2 be two fixed overlap functions, U be a uninorm with element $e \in [0, 1]$. If U is (α, O_1, O_2)-migrative, then $O_1(\alpha, x) = U(x, O_2(\alpha, e))$ for all $x \in [0, 1]$.

Proof. For all $x \in [0, 1]$, one obtains that $O_1(\alpha, x) = U(O_1(\alpha, x), e) = U(x, O_2(\alpha, e))$. □

In what follows, we discuss the (α, O_1, O_2)-migrativity of conjunctive uninorms locally internal on the boundary.

Proposition 3.8: Consider $\alpha \in]0, 1[$. Let U be a conjunctive uninorm locally internal on the boundary and O_1 and O_2 be two fixed overlap functions. If U is a (α, O_1, O_2)-migrative uninorm with neutral element $e \in [0, 1]$. Then the following statements hold.
(1) $O_1(\alpha, 1) = O_2(\alpha, e)$ and $e \in]0, 1]$. (2) $O_1(\alpha, 1) < e$.

Proof. (1) Take $x = 1$ and $y = e$ in Eq. (1). Then we have

$$\begin{aligned} O_1(\alpha,1) &= U(O_1(\alpha,1),e)\\ &= U(1,O_2(\alpha,e))\\ &\in \{1,O_2(\alpha,e)\}. \end{aligned}$$

Since $\alpha \in]0, 1[$, one has that $O_1(\alpha, 1) < 1$. Hence, we have $O_1(\alpha, 1) = O_2(\alpha, e)$. Further, if $e = 0$, then we have $O_1(\alpha, 1) = O_2(\alpha, e) = 0$. Thus, it follows from item (O2) of Definition 2.1 that $\alpha = 0$, which is a contradiction.

(2) If $O_1(\alpha, 1) \geq e$, then we have

$$\begin{aligned} O_1(\alpha, 1) &= U(O_1(\alpha, 1), e) \\ &= U(1, O_2(\alpha, e)) \\ &= U(1, O_1(\alpha, 1)) \\ &\geq U(1, e) \\ &= 1, \end{aligned}$$

i.e., $O_1(\alpha, 1) = 1$. Thus, it follows from item (O3) of Definition 2.1 that $\alpha = 1$, which is a contradiction. □

It follows from Proposition 3.8 that when U is a conjunctive uninorm locally internal on the boundary and $\alpha \in]0,1[$, we need only consider the case for $e \in]0,1]$. And, it follows from Proposition 3.7 and item (1) of Proposition 3.8 that we can obtain the following conclusion.

Proposition 3.9: Consider $\alpha \in]0,1[$. Let U be a conjunctive uninorm locally internal on the boundary with element $e \in]0,1]$ and O_1 and O_2 be two fixed overlap functions. If U is (α, O_1, O_2)-migrative, then $O_1(\alpha, x) = U(x, O_1(\alpha, 1))$ for all $x \in [0,1]$.

IV. Generalized α-migrativity for some usual classes of uninorms over overlap functions

In this section, we discuss the (α, O_1, O_2)-migrative uninorm U when U belongs to one of the usual classes, i.e., $U \in \mathcal{U}_{\min}, \mathcal{U}_{\max}, \mathcal{U}_{id}, \mathcal{U}_{rep}$ or $\mathcal{U}_{\cos}$. Since the cases for $e = 0$ and $e = 1$ that we have discussed, in the following, we only consider the case for $e \in]0,1[$.

A. The case for $U \in \mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$

In this subsection, we investigate the (α, O_1, O_2)-migrativity for $U \in \mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$. We start with $\alpha = 1$ and $U \in \mathcal{U}_{\min}$.

For the convenience of expression, in what follows, we denote $a = O_1(1, e)$ and $b = O_2(1, e)$. It follows from Corollary 3.5 that we only consider $a \neq e$ and $b \neq e$ for $(1, O_1, O_2)$-migrative uninorms.

Proposition 4.1: Let O_1 and O_2 be two fixed overlap functions, $U = \langle T_U, e, S_U \rangle_{\min}$ with neutral element $e \in]0,1[$. If $a \neq e$ and $b \neq e$, then U is not a $(1, O_1, O_2)$-migrative uninorm.

Proof. Case 1: $b > e$. It follows from Proposition 3.7 that for any $x \in [0, e[$, we have

$$\begin{aligned} O_1(1, x) &= U(b, x) \\ &= \min(b, x) \\ &= x. \end{aligned}$$

Further, it follows from the continuity of O_1 that $a = O_1(1, e) = e$, which is a contradiction with $a \neq e$.

Case 2: $b < e$. It follows from Proposition 3.7 that for any $x \in]e, 1]$, we have

$$\begin{aligned} O_1(1, x) &= U(b, x) \\ &= \min(b, x) \\ &= b. \end{aligned}$$

In particular, $b = O_1(1, 1) = 1$. Thus, $b = O_2(1, e) = 1$, it follows from the Definition 2.1 of O_2 that $e = 1$, which is a contradiction with $e < 1$. Therefore, U is not $(1, O_1, O_2)$-migrative. □

Proposition 4.2: Let O_1 and O_2 be two fixed overlap functions, $U_2 = \langle T_U, e, S_U \rangle_{\max}$ with neutral element $e \in]0,1[$. If $b \neq e$, then U is not $(1, O_1, O_2)$-migrative.

It follows from Propositions 4.1, 4.2 and 3.6, we only consider $\alpha \in]0,1[$ and $U \in \mathcal{U}_{\min}$.

Proposition 4.3: Consider $\alpha \in]0,1[$. Let O_1 and O_2 be two fixed overlap functions, $U = \langle T_U, e, S_U \rangle_{\min}$ with neutral element $e \in]0,1[$. If T_U is continuous, then the following statements hold.

(1) Let $O_1(\alpha, a) = a$. If U is (α, O_1, O_2)-migrative, then T_U is an ordinal sum of two continuous Archimedean t-norms T_1 and T_2, i.e., $T_U = (\langle 0, \frac{a}{e}, T_1 \rangle, \langle \frac{a}{e}, 1, T_2 \rangle)$ and $O_1(\alpha, x)$ has the following form

$$O_1(\alpha, x) = \begin{cases} x, & \text{if } x \in [0, a], \\ a, & \text{if } x \in]a, 1]. \end{cases}$$

(2) Let $O_1(\alpha, a) < a$. If U is (α, O_1, O_2)-migrative, then T_U is an ordinal sum of two the form $T_U = (..., \langle \rho_1, \rho_2, T^\omega \rangle, ...)$, where $\frac{a}{e} \in]\rho_1, \rho_2[$ and T^ω is a continuous Archimedean t-norms and $O_1(\alpha, x)$ has the following form

$$O_1(\alpha, x) = \begin{cases} x, \text{ if } x \in [0, e\rho_1[, \\ e(\rho_1 + (\rho_2 - \rho_1) T^\omega(\frac{a - e\rho_1}{e(\rho_2 - \rho_1)}, \frac{x - e\theta_1}{e(\rho_2 - \rho_1)})), \\ \text{if } x \in [e\rho_1, e\rho_2], \\ a, \text{ if } x \in]e\rho_2, 1]. \end{cases}$$

Proof. It follows from Proposition 3.8 that $a < e$. And, it follows from Proposition 3.9 that $O_1(\alpha, a) = U(a, a)$. Since $U(a, a) \leq a$, one gets that $O_1(\alpha, a) \leq a$. Now, we show that the conclusions as follows.

If $O_1(\alpha, a) = a$, then, one obtains that

$$a = U(a, a) = eT_U(\tfrac{a}{e}, \tfrac{a}{e}),$$

which implies that $\frac{a}{e}$ is an idempotent element of T_U. Since T_U is continuous, there exist two continuous Archimedean t-norms T_1 and T_2 such that $T_U = (\langle 0, \frac{a}{e}, T_1 \rangle, \langle \frac{a}{e}, 1, T_2 \rangle)$. In the following, we show that $O_1(\alpha, x)$ has the following form

$$O_1(\alpha, x) = \begin{cases} x, & \text{if } x \in [0, a], \\ a, & \text{if } x \in]a, 1]. \end{cases}$$

Case 1: If $x \in [0, a]$, then, one obtains that

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &= eT_U(\tfrac{a}{e}, \tfrac{x}{e}) \\ &= e(\tfrac{a}{e} T_1(1, \tfrac{x}{a})) \\ &= x. \end{aligned}$$

Case 2: If $x \in]a, e]$, then, one gets that

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &= eT_U(\tfrac{a}{e}, \tfrac{x}{e}) \\ &= e(\tfrac{a}{e} + (1 - \tfrac{a}{e}) T_2(0, \tfrac{\frac{x}{e} - \frac{a}{e}}{1 - \frac{a}{e}})) \\ &= a. \end{aligned}$$

Case 3: If $x \in]e, 1]$, then, one has that

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &= \min(a, x) \\ &= a. \end{aligned}$$

Let $O_1(\alpha, a) < a$. Then it follows from the proof of item (1) that

$$T_U(\frac{a}{e}, \frac{a}{e}) < \frac{a}{e}.$$

Since T_U is continuous, there exists a continuous Archimedean t-norm T^ω such that T_U is an ordinal sum of the form $T_U = (..., \langle \rho_1, \rho_2, T^\omega \rangle, ...)$, where $\frac{a}{e} \in]\rho_1, \rho_2[$. In the following, we verify that $O_1(\alpha, x)$ has the following form

$$O_1(\alpha, x) = \begin{cases} x, \text{ if } x \in [0, e\rho_1[, \\ e(\rho_1 + (\rho_2 - \rho_1)T^\omega(\frac{a-e\rho_1}{e(\rho_2-\rho_1)}, \frac{x-e\theta_1}{e(\rho_2-\rho_1)})), \\ \text{if } x \in [e\rho_1, e\rho_2], \\ a, \text{ if } x \in]e\rho_2, 1]. \end{cases}$$

Case 1: Let $x \in [0, e\rho_1]$. On the one hand,

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &= eT_U(\tfrac{a}{e}, \tfrac{x}{e}) \\ &\leq x. \end{aligned}$$

On the other hand, since $T_U(\rho_1, \rho_1) = \rho_1$ and T_U is continuous, one has that

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &= eT_U(\tfrac{a}{e}, \tfrac{x}{e}) \\ &\geq eT_U(\rho_1, \tfrac{x}{e}) \\ &= e\min(\theta_1, \tfrac{x}{e}) \\ &= x. \end{aligned}$$

Therefore, $O_1(\alpha, x) = x$.

Case 2: If $x \in [e\rho_1, e\rho_2]$, then, one gets that

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &= eT_U(\tfrac{a}{e}, \tfrac{x}{e}) \\ &= e(\rho_1 + (\rho_2 - \rho_1)T^\omega(\tfrac{\frac{a}{e}-\rho_1}{\rho_2-\rho_1}, \tfrac{\frac{x}{e}-\rho_1}{\rho_2-\rho_1}) \\ &= e(\rho_1 + (\rho_2 - \rho_1)T^\omega(\tfrac{a-e\rho_1}{e(\rho_2-\rho_1)}, \tfrac{x-e\rho_1}{e(\rho_2-\rho_1)})) \end{aligned}$$

Case 3: If $x \in]e\rho_2, e]$. On the one hand,

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &\leq U(a, e) \\ &= a. \end{aligned}$$

On the other hand, since $T_U(\rho_2, \rho_2) = \rho_2$ and T_U is continuous, one obtains that

$$\begin{aligned} O_1(\alpha, x) &= U(a, x) \\ &= eT_U(\tfrac{a}{e}, \tfrac{x}{e}) \\ &\geq eT_U(\tfrac{a}{e}, \rho_2) \\ &= e\min(\tfrac{a}{e}, \rho_2) \\ &= a. \end{aligned}$$

Therefore, $O_1(\alpha, x) = a$.

Case 4: If $x \in]e, 1]$. Then we have $O_1(\alpha, x) = U(a, x) = \min(a, x) = a$. □

B. The case for $U \in \mathcal{U}_{id}$

In this subsection, we investigate the (α, O_1, O_2)-migrativity for $U \in \mathcal{U}_{id}$. We start with $\alpha = 1$.

Proposition 4.4: Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle g, e \rangle_{id}$ with element $e \in]0, 1[$ and $g(b) = b$. Then the following statements are equivalent:
(1) U is $(1, O_1, O_2)$-migrative;
(2) $O_1(1, x) = x$ and $O_2(1, x) = x$ for all $x \in [0, 1]$.

Proof. It follows from Proposition 3.7. □

Proposition 4.5: Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle g, e \rangle_{id}$ be a uninorm with element $e \in]0, 1[$ and $g(b) \neq b$. Then U is not $(1, O_1, O_2)$-migrative.

Proof. Assume that U is $(1, O_1, O_2)$-migrative. Then, it follows from Proposition 3.7 that $O_1(1, x) = \min(b, x)$ for any $x \in [0, g(b)[$, and, for any $x \in]g(b), 1]$, $O_1(1, x) = \max(b, x)$. It follows from O_1 is continuous that

$$\begin{aligned} O_1(1, g(b)) &= O_1(b, \sup_{x \in [0, g(b)[} x) \\ &= \sup_{x \in [0, g(b)[} O_1(1, x) \\ &= \sup_{x \in [0, g(b)[} \min(b, x) \\ &= \min(b, \sup_{b \in [0, g(b)[} x) \\ &= \min(b, g(b)) \\ &\neq \max(b, g(b)) \\ &= \max(b, \inf_{x \in]g(b), 1]} x) \\ &= \inf_{x \in]g(b), 1]} \max(b, x) \\ &= \inf_{x \in]g(b), 1]} O_1(1, x) \\ &= O_1(1, \inf_{x \in]g(b), 1]} x) \\ &= O_1(1, g(b)), \end{aligned}$$

which is a contradiction. Therefore, U is not $(1, O_1, O_2)$-migrative. □

Now, we consider the (α, O_1, O_2)-migrativity for $U \in \mathcal{U}_{id}$ and $\alpha \in]0, 1[$.

Proposition 4.6: Let O_1 and O_2 be two fixed overlap functions, $U_2 \equiv \langle g, e \rangle_{id}$ be a conjunctive uninorm with element $e \in]0, 1[$. Then U is not (α, O_1, O_2)-migrative.

Proof. Assume that U is (α, O_1, O_2)-migrative, then we have the following two cases.

Case 1: If $g(b) = b$, then, from Proposition 3.9, for $x = 1$, one has that $O_1(\alpha, 1) = 1$, which implies that $\alpha = 1$, which is a contradiction.

Case 2: If $g(b) \neq b$, then it is similar to the proof of Proposition 4.5, one also can get a contradiction.

Therefore, U is not (α, O_1, O_2)-migrative. □

C. The case for $U \in \mathcal{U}_{rep}$

In this subsection, we investigate the (α, O)-migrativity for $U \in \mathcal{U}_{rep}$. We start with $\alpha = 1$.

Proposition 4.7: Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle g, e \rangle_{rep}$ with neutral element $e \in]0, 1[$, $b = O_1(1, b) = O_2(1, b)$. Then the following statements are equivalent
(1) U is $(1, O_1, O_2)$-migrative;
(2) $O_1(1, x) = x$ and $O_2(1, x) = x$ for all $x \in [0, 1]$.

Proof. (1) ⇒ (2): If U is $(1, O_1, O_2)$-migrative, then, it follows from Proposition 3.7 that

$$\begin{aligned} b &= O_1(1, b) \\ &= U(b, b) \\ &= h^{-1}(h(b) + h(b)) \\ &= h^{-1}(2h(b)). \end{aligned}$$

Hence, $h(b) = h(2b)$. Further, we obtain that $h(b) = -\infty$ or $h(b) = 0$ or $h(b) = +\infty$. It is worth noting that since $e \in]0, 1[$, it follows from (O2) and (O3) of Definition 2.1 that $b \neq 0$ and $b \neq 1$. Thus, $h(b) = 0$. Moreover, $b = h^{-1}(0) = e$. It follows from Proposition 3.4 that $O_1(1, x) = x$ and $O_2(1, x) = x$ for all $x \in [0, 1]$.

(2) $\Rightarrow$ (1): It is straightforward. $\square$

Proposition 4.8: Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle g, e\rangle_{rep}$ be $(1, O_1, O_2)$-migrative uninorm with neutral element $e \in]0, 1[$. If $b > O_1(1, b) = O_2(1, b)$, then O_1 satisfies $(O6)$.

Proof. If U is $(1, O_1, O_2)$-migrative. Then, it follows from the proof of Proposition 4.7 that $h(b) > h(2h(b))$, which implies $h(b) < 0$. Since $h(e) = 0$, we have $b < e$. Moreover, it follows from Proposition 3.7 that

$$\begin{aligned} O_1(1, x) &= U(b, x) \\ &\le U(e, x) \\ &= x \end{aligned}$$

for all $x \in [0, 1]$. $\square$

Proposition 4.9: Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle g, e\rangle_{rep}$ be $(1, O_1, O_2)$-migrative uninorms with neutral element $e \in]0, 1[$. If $b < O_1(1, b) = O_2(1, b)$, then O_1 satisfies $(O7)$.

Proof. The proof is similar to the one of Proposition 4.8. $\square$

In the following, we consider the case for $\alpha \in]0, 1[$.

Proposition 4.10: Let $\alpha \in]0, 1[$, O_1 and O_2 be two fixed overlap functions, $U \equiv \langle g, e\rangle_{rep}$ be a conjunctive uninorm with neutral element $e \in]0, 1[$. Then U is not (α, O_1, O_2)-migrative.

Proof. Suppose that U is (α, O_1, O_2)-migrative, then, it follows from Proposition 3.7 that $O_1(\alpha, x) = U(O_2(\alpha, e), x)$ for all $x \in [0, 1]$. On the other hand, since $e \in]0, 1[$ and $\alpha \in]0, 1[$, we have $O_2(\alpha, e) > 0$. Further, take $x = 1$ in Eq. (1), then, one has that

$$\begin{aligned} O_1(\alpha, 1) &= U(O_2(\alpha, e), 1) \\ &= h^{-1}(h(O_2(\alpha, e)) + h(1)) \\ &= h^{-1}(+\infty) \\ &= 1. \end{aligned}$$

Therefore, $\alpha = 1$, which is a contradiction. $\square$

D. The case for $U \in \mathcal{U}_{\cos}$

In this subsection, we investigate the (α, O_1, O_2)-migrativity for $U \in \mathcal{U}_{\cos}$. We start with $\alpha = 1$ and $U \in \mathcal{U}_{\cos,\max}$.

Proposition 4.11: Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle (R, e), \upsilon, S_1, \omega, S_2\rangle_{cos,max}$ such that $b \ge \upsilon$. Then U is not $(1, O_1, O_2)$-migrative.

Proof. Suppose that U is $(1, O_1, O_2)$-migrative, then, it follows from Proposition 3.7 that for any $x \in]0, \upsilon[$, one obtains that

$$\begin{aligned} O_1(1, x) &= U(b, x) \\ &= \max(b, x) \\ &= b. \end{aligned}$$

Moreover, it follows from the continuity of O_1, one gets that

$$\begin{aligned} 0 &= O_1(1, 0) \\ &= O_1(1, \inf_{x \in]0, \upsilon]} x) \\ &= \inf_{x \in]0, \upsilon]} O_1(1, x) \\ &= \inf_{x \in]0, \upsilon]} b \\ &= b, \end{aligned}$$

which is a contradiction. $\square$

Proposition 4.12: Suppose that O_1 and O_2 are two fixed overlap functions, $U \equiv \langle (R, e), \upsilon, S_1, \omega, S_2\rangle_{\cos,\max}$ such that $b < \upsilon$. If $b = O_1(1, b) = O_2(1, b)$, then the following statements are equivalent:

(1) U is $(1, O_1, O_2)$-migrative;

(2) $O_1(1, x) = x$ and $O_2(1, x) = x$ for any $x \in [0, 1]$.

Proof. (1) $\Rightarrow$ (2): Since $a = O_1(1, b) = O_2(1, b)$ and $e \in]0, 1[$, we have $b > 0$. In addition, since U is $(1, O_1, O_2)$-migrative, then, it follows from Proposition 3.7 that

$$\begin{aligned} b &= O_1(1, b) \\ &= U(b, b) \\ &= \upsilon R(\tfrac{b}{\upsilon}, \tfrac{b}{\upsilon}) \\ &= \upsilon h^{-1}(h(\tfrac{b}{\upsilon}) + h(\tfrac{b}{\upsilon})) \\ &= \upsilon h^{-1}(2h(\tfrac{b}{\upsilon})), \end{aligned}$$

where h is the additive generator of R. Hence, $h(\frac{b}{\upsilon}) = 2h(\frac{b}{\upsilon})$. one concludes that $h(\frac{b}{\upsilon}) = -\infty$ or $h(\frac{b}{\upsilon}) = 0$ or $h(\frac{b}{\upsilon}) = +\infty$. Since $x \in]0, \upsilon[$, we have $h(\frac{b}{\upsilon}) = 0$. Thus, $\frac{b}{\upsilon} = h^{-1}(0) = \frac{e}{\upsilon}$, which implies $b = e$. It follows from Proposition 3.3 that $O_1(1, x) = x$ and $O_2(1, x) = x$ for any $x \in [0, 1]$.

(2) $\Rightarrow$ (1): It is straightforward. $\square$

Proposition 4.13: Suppose that O_1 and O_2 are two a fixed overlap functions, $U \equiv \langle (R, e), \upsilon, S_1, \omega, S_2\rangle_{cos,max}$ is $(1, O_1, O_2)$-migrative such that $b < \upsilon$. Then the following statements hold.

(1) If $b > O_1(1, b) = O_2(1, b)$, then O_1 satisfies $(O6)$.

(2) If $b < O_1(1, b) = O_2(1, b)$, then O_1 satisfies $(O7)$.

Proof. (1) Since U is $(1, O_1, O_2)$-migrative and $b > 0$, it follows from Proposition 4.12 that $h(\frac{b}{\upsilon}) > 2h(\frac{b}{\upsilon})$. Further, one gets that $h(\frac{b}{\upsilon}) < 0$. Thus, we get that $\frac{b}{\upsilon} < \frac{e}{\upsilon}$. Moreover, it follows from Proposition 3.7 that

$$\begin{aligned} O_1(1, x) &= U(b, x) \\ &\le U(e, x) \\ &= x \end{aligned}$$

for any $x \in [0, 1]$.

(2) It can be proved in a similar way as (1). $\square$

Next, we consider the (α, O_1, O_2)-migrativity for $\alpha = 1$ and $U \in \mathcal{U}_{\cos,\min}$.

Proposition 4.14: Suppose that O_1 and O_2 are two fixed overlap functions, $U \equiv \langle T_1, \lambda, T_2, u, (R, e)\rangle_{\cos,\min}$ such that $b \le u$. Then U is not $(1, O_1, O_2)$-migrative.

Proof. Suppose that U is $(1, O_1, O_2)$-migrative, then, it follows from Proposition 3.7 that for any $x \in]u, 1[$, one has that

$$\begin{aligned} O_1(1, x) &= U(b, x) \\ &= \min(b, x) \\ &= b. \end{aligned}$$

Moreover, it follows from the continuity of O, one gets that

$$\begin{aligned} 1 &= O_1(1, 1) \\ &= O_1(1, \sup_{x \in]u, 1[} x) \\ &= \sup_{x \in]u, 1[} O_1(1, x) \\ &= \sup_{x \in]u, 1[} b \\ &= b, \end{aligned}$$

which is a contradiction. □

Proposition 4.15: Suppose that O_1 and O_2 are fixed overlap functions, $U \equiv \langle T_1, \lambda, T_2, u, (R,e)\rangle_{\cos,\min}$ such that $a > u$. If $a = O_1(1,a) = O_2(1,a)$, then the following statements are equivalent:
(1) U is $(1, O_1, O_2)$-migrative;
(2) $O_1(1,x) = x$ and $O_2(1,x) = x$ for any $x \in [0,1]$.

Proof. (1) $\Rightarrow$ (2): Since $a = O_1(1,a) = O_2(1,a)$ and $e \in]0,1[$, we have $a < 1$. In addition, since U is $(1, O_1, O_2)$-migrative, then, it follows from Proposition 3.7 that

$$\begin{aligned} a &= O_1(1,a) \\ &= U(a,a) \\ &= u + (1-u)R(\tfrac{a-u}{1-u}, \tfrac{a-u}{1-u}) \\ &= u + (1-u)h^{-1}(h(\tfrac{a-u}{1-u}) + h(\tfrac{a-u}{1-u})) \\ &= u + (1-u)h^{-1}(2h(\tfrac{a-u}{1-u})), \end{aligned}$$

where h is the additive generator of R. Hence, $h(\frac{a-u}{1-u}) = 2h(\frac{\beta-u}{1-u})$, one concludes that $h(\frac{a-u}{1-u}) = -\infty$ or $h(\frac{a-u}{1-u}) = 0$ or $h(\frac{a-u}{1-u}) = +\infty$. Since $x \in]u,1[$, we have $h(\frac{a-u}{1-u}) = 0$. Thus, $\frac{a-u}{1-u} = h^{-1}(0) = \frac{e-u}{1-u}$, which implies $a = e$. It follows from Proposition 3.3 that $O_1(1,x) = x$ and $O_2(1,x) = x$ for any $x \in [0,1]$.
(2) $\Rightarrow$ (1): It is straightforward. □

Proposition 4.16: Suppose that O_1 and O_2 are two fixed overlap functions, $U \equiv \langle T_1, \lambda, T_2, u, (R,e)\rangle_{\cos,\min}$ is $(1, O_1, O_2)$-migrative such that $O_1(1,a) = O_2(1,a) = a > u$. Then the following statements hold.
(1) If $a > O_1(1,a)$, then O_1 satisfies $(O6)$.
(2) If $a < O_1(1,a)$, then O_1 satisfies $(O7)$.

Proof. (1) Since U is $(1, O_1, O_2)$-migrative and $a < 1$, it follows from Proposition 4.15 that $h(\frac{a-u}{1-u}) > 2h(\frac{a-u}{1-u})$. Further, one gets that $h(\frac{a-u}{1-u}) < 0$. Thus, we obtain that $\frac{a-u}{1-u} < \frac{e-u}{1-u}$. Moreover, it follows from Proposition 3.7 that

$$\begin{aligned} O_1(1,x) &= U(a,x) \\ &\le U(e,x) \\ &= x \end{aligned}$$

for any $x \in [0,1]$.
(2) It can be proved in a similar way as (1). □

As a consequence of Proposition 4.16, we only need to consider $\alpha \in]0,1[$ and $U \in \mathcal{U}_{cos,min}$ or $U \equiv \langle (R,e), \upsilon, S_1, \omega, S_2\rangle_{cos,max}$ with $U(0,1) = U(1,0) = 0$. At first, we consider the situation for $U \equiv \langle (R,e), \upsilon, S_1, \omega, S_2\rangle_{cos,max}$ with $U(0,1) = U(1,0) = 0$.

Proposition 4.17: Consider $\alpha \in]0,1[$. Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle (R,e), \upsilon, S_1, \omega, S_2\rangle_{cos,max}$ with $U(0,1) = U(1,0) = 0$. Then U is not (α, O_1, O_2)-migrative.

Proof. $U(O_1(\alpha,1), e) = O_1(\alpha,1) < 1$. However, $U(O_2(\alpha,e),1) = 1$. Therefore, U is not (α, O_1, O_2)-migrative. □

Next, we discuss the situation for $U \in \mathcal{U}_{cos,min}$.

Proposition 4.18: Consider $\alpha \in]0,1[$. Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle T_1, \lambda, T_2, u, (R,e)\rangle_{cos,min}$ such that $a = O_1(\alpha,1) > \lambda$. Then U is not (α, O_1, O_2)-migrative.

Proof. $U(1, O_1(\alpha,1)) = U(1,a) = 1$. However, $O_1(\alpha,1) < 1$. It follows from Proposition 3.9 that U is not (α, O_1, O_2)-migrative. □

Proposition 4.19: Consider $\alpha \in]0,1[$. Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle T_1, \lambda, T_2, u, (R,e)\rangle_{\cos,\min}$ such that $a = O_1(\alpha,1) = \lambda$. If U_1 is (α, O_1, O_2)-migrative, then $U(\lambda,1) = \lambda$ and $O_1(\alpha,x) = \min(\lambda,x)$.

Proof. (1) $\Rightarrow$ (2): If U is (α, O_1, O_2)-migrative, then, one has that

$$\begin{aligned} U(\lambda,1) &= U(a,1) \\ &= a. \end{aligned}$$

Since $\alpha \in]0,1[$, we have $a < 1$. Thus, one gets that $U(\lambda,1) = \lambda$. Moreover, it follows from Proposition 3.9 that

$$\begin{aligned} O_1(\alpha,\lambda) &= U(a,\lambda) \\ &= U(\lambda,\lambda) \\ &= \lambda. \end{aligned}$$

And, for any $x \neq \lambda$, then, one gets that

$$\begin{aligned} O_1(\alpha,x) &= U(a,x) \\ &= U(\lambda,x) \\ &= \min(\lambda,x). \end{aligned}$$

□

Proposition 4.20: Consider $\alpha \in]0,1[$. Let O_1 and O_2 be two fixed overlap functions, $U \equiv \langle T_1, \lambda, T_2, u, (R,e)\rangle_{\cos,\min}$ such that $a = O_1(\alpha,1) < \lambda$. Then the following statements hold.
(1) Let $O_1(\alpha,a) = a$. If U is (α, O_1, O_2)-migrative, then T_1 is an ordinal sum of two continuous Archimedean t-norms $T^{\natural}$ and $T^{\sharp}$, i.e., $T_1 = (\langle 0, \frac{a}{\lambda}, T^{\natural}\rangle, \langle \frac{a}{\lambda}, 1, T^{\sharp}\rangle)$ and $O_1(\alpha,x)$ has the following form

$$O_1(\alpha,x) = \begin{cases} x, & \text{if } x \in [0,a], \\ a, & \text{if } x \in]a,1]. \end{cases}$$

(2) Let $O(\alpha,a) < a$. If U is (α, O_1, O_2)-migrative, then T_1 is an ordinal sum of the form $T_2 = (..., \langle \eta, \theta, T^{\S}\rangle, ...)$, where $T^{\S}$ is a continuous Archimedean t-norm and $\frac{a}{\lambda} \in]\eta,\theta[$. In addition, $O_1(\alpha,x)$ has the following form

$$O_1(\alpha,x) = \begin{cases} x, & \text{if } x \in [0,\lambda\eta[, \\ \lambda(\eta + (\theta-\eta))T^{\S}(\frac{a-\lambda\eta}{\lambda(\theta-\eta)}, \frac{x-\lambda\eta}{\lambda(\theta-\eta)}) & \text{if } x \in [\lambda\eta,\lambda\theta], \\ a, & \text{if } x \in]\lambda\theta,1]. \end{cases}$$

Proof. It can be proven in a similar way to Proposition 4.3. □

V. Conclusions

In this paper, we propose the concept of the (α, O_1, O_2)-migrativity of a uninorm U over any two fixed overlap functions O_1 and O_2. And then, we investigate some related properties for the (α, O_1, O_2)-migrativity equation of a uninorm. Meanwhile, we characterize the (α, O_1, O_2)-migrative uninorm U when U belongs to one of usual classes, i.e., $\mathcal{U}_{\min}, \mathcal{U}_{\max}, \mathcal{U}_{id}, \mathcal{U}_{rep}$ and $\mathcal{U}_{\cos}$, respectively.

As future works, we will consider the following topics:

(1) Investigating the generalized α-migrativity of overlap functions over uninorms;

(2) Studying the generalized α-migrativity of overlap functions over semi-uninorms;

(3) Discussing the generalized α-migrativity property between overlap (grouping) functions and semi-nullnorms.

(4) Considering the generalized α-migrativity between overlap functions and 2-uninorms.

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