Effects of Shear Stress on Magnetohydrodynamic (MHD) Powell Eyring Fluid over A Porous Plate: A Lift and Drainage Problem

Fawzia Mansour Elniel, Shaymaa Mustafa, Arifah Bahar, Zainal Abdul Aziz, and Faisal Salah

***Abstract*—To understand the isothermal flow of the non-Newtonian fluids through a porous media, it is necessary to know how shear stress affects fluid viscosity. The objective of this research is to investigate MHD non-Newtonian fluid flow that best fits the Powell-Eyring model in the presence of shear stress on an inclined porous plate for both lift and draining flow. The flow behaviour has been examined under the influence of a magnetic field and a shear distribution. In order to solve the non-dimensionalized governing equation, the Adomian Decomposition Method (ADM) is utilized. For both lifting and drainage flow, the influence of material parameters such as permeability, shear stress, and magnetic has been investigated and described. The physical interpretation of the essential factors which influenced the shear stress versus the velocity is presented. Finally, the analysis highlights the impact of permeability and magnetic parameters on the fluid flow graphically. The outcomes demonstrate that shear stress is an increasing function of fluid parameter in lifting case, but it is decreasing in drainage case. Moreover, the inclination is an important factor that affects the shear stress of both lifting and drainage flow.**

***Index Terms*—Adomian Decomposition Method (ADM), Darcy's law, Magnetohydrodynamic (MHD), Powell-Eyring fluid, Shear stress**

I. Introduction

THE dynamics of non-Newtonian fluids have an effect on fluid behaviour. Nonlinear constitutive relationships are found in a wide range of industrial products, including lubricants, paints, and foodstuffs. These constitutive relations for such fluids give rise to complex non-linear equations. The solution of non-linear mathematical problems is far more difficult than that of Newtonian fluids. The importance of finding a proper solution of theses equations comes from the need of understanding of rheological properties of the non-Newtonian fluids, which can help in describing the behaviour of the non-Newtonian fluid [1].

Powell-Eyring fluid model is a non-Newtonian fluid developed from Ree-Eyring fluid model by adding an infinite shear rate viscosity [2]. The Powell-Eyring fluid was introduced by Eyring and Powell [3]. This type of fluid is used commonly to describe viscoelastic fluids. This model is known for its accuracy and consistency in the calculation of experimental and analytical studies [4]. The main characteristic of this fluid model is that instead of scientific relationships, it is taken from the kinetic theory of liquids. Also, the liquid reduces Newtonian activity with a low and high shear rate [5], [6].

Many models have been developed in the last few years to describe the viscous flow of Powell-Eyring fluids through porous media. The incompressible Couette flow for the Eyring-Powell model on porous walls was proposed by Zaman [7]. In his study, he applied the Homotopy analysis method for uniform suction or injection. Similarly, in another study, Prasad et al. [8] investigated in a non-Darcy porous medium, the steady-state flow and heat transfer of non-Newtonian incompressible Eyring-Powell fluid from a vertical porous layer. Their study shows that the Darcy parameter enhances both velocity and temperature. Gupta, et al. [9] adopted the Homotopy perturbation technique solution for the heat transfer flow between vertical plates that moved in the reverse direction and packed with a porous medium partially.

Additionally, different researchers have analysed the impact of MHD on Powell-Eyring fluid. To analyse Powell-Eyring flow to a stretching sheet, Akbar et al [10] applied the boundary layer approach and similarity transformations. Their outcome showed that the rise in magnetic field strength produced resistance to the flow. Khan, et al [11] investigated the MHD flow of Powell–Eyring fluid over a rotating disk Due to the movement of Powell-Eyring fluid through a stretching cylinder with Newtonian heating. Hussain et al. [12] studied the influence of inclined magnetic field. From

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the literature review, it depicts that most of these studies have not considered the dynamics of the problem for the combined effects of porous media, steadiness and impact of Magnetohydrodynamic (MHD) on the flow system. These effects have not been discussed with the use of ADM.

It is worth to note that most of these problems have been solved numerically despite obtaining an approximate or analytical solution is most favourable than numerical solutions. In literature, various approximate theoretical approaches have been used to solve the governing equations of the fluid flow of the Powell Eyring, such as the variation iteration method [13], the method of Homotopy analysis [6], [11], Adomian Decomposition Method [14]–[16]. However, the exact solution regarding the thin film flow of non-Newtonian fluid is very challenging to obtain. Therefore, numerical methods take apart besides the approximate analytical methods.

Recently, different modelling studies have been carried out on thin-film flow for the Powell Eyring fluid by various researchers using numerical methods [17], [18]. In light of these efforts, Salah, et al. [19] have described the Powell Eyring fluid flow on the vertically moving belt using successive linearization method. Ellahi, Shivanian [20] also studied the numerical solutions for Couette flow of Powell-Eyring fluid under slip boundary condition and heat transfer. Akinshilo and Olaye [21] studied the Powell-Eyring pipe flow model with variable temperature, variable viscosity and internal heat generation using Perturbation method.

Many researchers have been modelling applied sciences and engineering problems through ADM. ADM is known due to its effectiveness and reliability for handling linear and non-linear differential equations [22], [23]. In the form of an infinite sequence, it provides the required analytical solution that quickly defines each term. The ADM requires no discretization, linearization, or spatial transformation, unlike conventional approaches. It offers some valuable benefits as well as numerical methods over other analytical techniques.

The object of this paper was to investigate the impact of shear stress and inclination when the Powell-Eyring fluid flow over porous medium. The material parameters of the shear stress for the lifting and drainage flow are taken into accounts, such as magnetic permeability, friction and heat transfer parameters. The governing equations representing the problem are converted into a non-linear ordinary differential equation. Using the Adomian decomposition process, the transformed ordinary differential equation was solved to get approximate solution. The results of relevant parameters on the velocity and stress of the fluid have been graphically displayed.

II. Problem Formulation

Consideration was provided to the one-dimensional flow of an incompressible Powell-Eyring fluid running over an inclined porous layer. The flow regime is supposed to be steady, laminar, uniform, and its direction is perpendicular to the magnetic field of strength B_0. A small magnetic Reynolds number is expected so that the induced magnetic field is neglected. Suddenly, the plate moves upward with constant velocity U_0, but due to gravity effects, the fluid

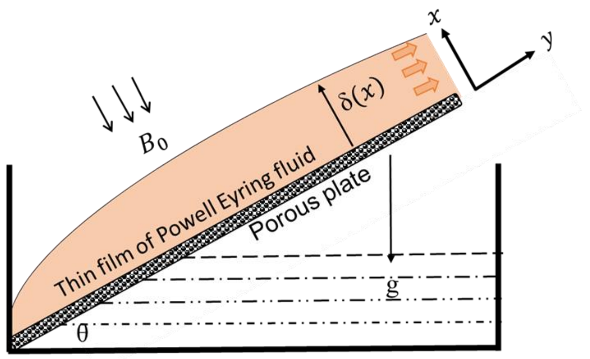

Fig. 1. A schematic diagram of lifting flow of Powell-Eyring over an inclined porous plate.

drains down again. The thickness of thin-film is assumed to equal to δ, and the external pressure is uniform everywhere. Figure 1 shows the x-axis perpendicular to the plate, and the y-axis is taken along the plate after rotation.

A. *Lift Flow of Powell-Eyring Fluid*

The flow is one dimensional in which the velocity varies only in one direction, and the flow is a function of only one co-ordinate axis. Then, the only velocity component and the extra stress tensor $\mathbf{S}$ in the y-direction is written in the following form

$$\mathbf{u} = (0, u(x), 0),\ \mathbf{S} = \mathbf{S}(x) \tag{1}$$

If we neglect the thermal effects, then the momentum and continuity equations of an incompressible fluid in lifting case are given by:

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \mathbf{T} - \sigma B_0^2 \mathbf{u} - \rho g \sin\theta \tag{3}$$

where ‘ρ’ is the constant density, ‘u’ is the velocity vector, the B_0 magnetic field of strength, and T is the stress tensor. Cauchy stress tensor $\mathbf{T}$ for an incompressible fluid is given by Yunus et al. [24].

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \tag{4}$$

where the extra stress tensor $\mathbf{S}$ for the Powell-Eyring fluid satisfies the relation [16]:

$$\mathbf{S} = \mu \nabla \mathbf{u} + \frac{1}{\beta} \sinh^{-1}\left(\frac{1}{C} \nabla \mathbf{u}\right) \tag{5}$$

where μ the coefficient of shear viscosity, and β, C are the material constants of the Powell-Eyring fluid. As stated in [17], the Taylor expansion of the inverse hyperbolic term in the above equation (5) is approximately given by the following equation (6):

$$\sinh^{-1}\left(\frac{1}{C}\nabla\mathbf{u}\right)\cong\frac{1}{C}\nabla\mathbf{u}-\frac{1}{6}\left(\frac{1}{C}\nabla\mathbf{u}\right)^{3},\quad \left|\frac{1}{C}\nabla\mathbf{u}\right|<<1 \qquad (6)$$

which leads to:

$$\mathbf{S}=\left(\mu+\frac{1}{\beta C}\right)\left(\frac{du}{dx}\right)-\frac{1}{6\beta C^{3}}\left(\frac{du}{dx}\right)^{3} \qquad (7)$$

Since the flow is steady then $\frac{du}{dt}=0$. By substituting equation (6) into equation (5) and with the help of equation (4), the non-zero components of the momentum equation (3) excluding pressure gradient in matrix form written as follows:

$$0=\mu\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\left(\frac{du}{dx}\right)+\frac{1}{\beta C}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\left(\frac{du}{dx}\right)-\frac{1}{6\beta C^{3}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\left(\frac{du}{dx}\right)^{3}-\sigma B_0^2\left(0,u(x),0\right)-\rho g\sin\theta-\mu\frac{\phi_i}{K}(0,u(x),0) \qquad (8)$$

From equation (8), the values of $S_{xx}=S_{yy}=S_{zz}=S_{xz}=S_{zx}=S_{yz}=S_{zy}=0$ and consequently $\mathbf{S}=S_{xy}$ in equation (7). Equation (8) is written in a simple form as:

$$0=\left(\mu+\frac{1}{\beta C}\right)\left(\frac{du}{dx}\right)-\frac{1}{6\beta C^{3}}\left(\frac{du}{dx}\right)^{3}-\sigma B_0^2 u-\rho g\sin\theta-\mu\frac{\phi_i}{K}u \qquad (9)$$

Subject to the boundary conditions

$$u(x)=U_0 \text{ at } x=0 \text{ (no-slip condition)} \qquad (10)$$
$$S_{xy}=0 \text{ at } x=\delta \text{ (free surface)} \qquad (11)$$

Introducing the non-dimensional quantities to simplify the equations (8)-(11), as follows:

$$u=u^{*}U_0,\ x=x^{*}\delta,\ \alpha=1+\frac{1}{\mu\beta C},\quad \xi=\frac{\delta^2}{K}\quad M=\frac{\sigma B_0^2\delta^2}{\mu},\quad St=\frac{\rho g\delta^2\sin\theta}{\mu U_0},\tilde{\beta}=\frac{U_0^2}{2\mu\beta\delta^2C^3} \qquad (12)$$

where α and $\tilde{\beta}$ are the dimensionless parameters of Eyring-Powell fluid, St is the Stokes number, M is the magnetic parameter, and ξ is the permeability parameters.

The dimensionless form of equations (9)-(11), omitting the (*) notation, can now be written as:

$$\alpha\left(\frac{d^2u}{dx^2}\right)-\tilde{\beta}\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dx}\right)^{2}-(M+\xi\phi_i)u-St\sin\theta=0 \qquad (13)$$

subject to the boundary conditions

$$u=1 \text{ at } x=0 \qquad (14)$$
$$\frac{du}{dx}=0 \text{ at } x=1 \qquad (15)$$

Equation (13) under the boundary conditions (14) and (15), is a non-linear differential equation of the second-order that challenges its exact solution. Therefore, we use ADM to get the solution.

B. Application of the ADM in lifting problem

Firstly, we define the linear operator $L_{xx}=\frac{d^2}{dx^2}$. Suppose that the inverse of this operator exists and defined by $L_{xx}^{-1}(*)=\iint(*)\,dxdx$, accordingly. Equation (13) can be written in the operator form as follows:

$$L_{xx}u(x)-\frac{\tilde{\beta}}{\alpha}\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dx}\right)^{2}-\frac{1}{\alpha}(M+\xi\phi_i)u=\frac{1}{\alpha}St\sin\theta \qquad (16)$$

By applying the inverse operator L_{xx}^{-1} on both sides of equation (16), we obtain

$$L_{xx}^{-1}L_{xx}u(x)=\frac{1}{\alpha}L_{xx}^{-1}St\sin\theta+\frac{\tilde{\beta}}{\alpha}L_{xx}^{-1}\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dx}\right)^{2}+\frac{1}{\alpha}(M+\xi\phi_i)L_{xx}^{-1}u \qquad (17)$$

Or

$$u(x)=\frac{1}{\alpha}\left(St\frac{x^2}{2}\sin\theta+c_0x+c_1\right)+\frac{\tilde{\beta}}{\alpha}L_{xx}^{-1}\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dx}\right)^2+\frac{1}{\alpha}(M+\xi\phi_i)L_{xx}^{-1}u \quad (18)$$

c_0 and c_1 are integration constants. As the ADM is a series method [9], it is possible to give the solution $u(x)$, and the non-linear terms can be given as,

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \quad (19)$$

$$\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dx}\right)^2=\sum A_n \quad (20)$$

where A_n is the Adomian polynomial which is calculated by using the following equation:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=1}^{n}\lambda^i u_i\right)\right]_{\lambda=0} \quad n=0,1,2,.. \quad (21)$$

To get:

$$A_0=\left(\frac{d^2u_0}{dx^2}\right)\left(\frac{du_0}{dx}\right)^2 \quad (22)$$

$$A_1=\left(\frac{d^2u_1}{dx^2}\right)\left(\frac{du_0}{dx}\right)^2+2\left(\frac{d^2u_0}{dx^2}\right)\left(\frac{du_0}{dx}\right)\left(\frac{du_1}{dx}\right) \quad (23)$$

$$A_2=\left(\frac{d^2u_2}{dx^2}\right)\left(\frac{du_0}{dx}\right)^2+2\left(\frac{d^2u_0}{dx^2}\right)\left(\frac{du_0}{dx}\right)\left(\frac{du_2}{dx}\right)+2\left(\frac{d^2u_1}{dx^2}\right)\left(\frac{du_0}{dx}\right)\left(\frac{du_1}{dx}\right)+\left(\frac{d^2u_1}{dx^2}\right)\left(\frac{d^2u_0}{dx^2}\right) \quad (24)$$

Substituting equations (19) and (20) in (18), we obtain

$$\sum_{n=0}^{\infty}u_n(x)=\frac{1}{\alpha}\left(St\frac{x^2}{2}\sin\theta+c_0x+c_1\right)+\frac{\tilde{\beta}}{\alpha}L_{xx}^{-1}\sum_{n=0}^{\infty}A_n+\frac{1}{\alpha}(M+\xi\phi_i)L_{xx}^{-1}\sum_{n=0}^{\infty}u_n \quad (25)$$

The zero component problem for the MHD Powell-Eyring fluid $u_0(x)$ is identified as follows:

$$u_0(x)=\frac{1}{\alpha}\left(St\frac{x^2}{2}\sin\theta+c_0x+c_1\right) \quad (26)$$

Subject to the boundary conditions

$$u_0(x)=1 \quad \text{at } x=0 \quad (27)$$

$$\frac{du_0}{dx}=0 \quad \text{at} \quad x=1 \quad (28)$$

Involving the boundary conditions (27) and (28) in equation (26), we obtain the zero component as:

$$u_0(x)=\frac{1}{\alpha}\left(St\sin\theta\frac{x^2}{2}-St\sin\theta x+1\right) \quad (29)$$

The remaining components can be written as:

$$u_{n+1}(x)=\frac{\tilde{\beta}}{\alpha}L_{xx}^{-1}\sum A_n+\frac{1}{\alpha}(M+\xi\phi_i)L_{xx}^{-1}\sum u_n \quad (30)$$

Subject to the boundary condition

$$u_{n+1}(x)=0 \quad \text{at } x=0 \quad (31)$$

$$\frac{du_{n+1}}{dx}=0 \quad \text{at} \quad x=1 \quad (31a)$$

Substitute $n=0,1,2,..$ in equation (29) and apply the inverse operator with the help of the boundary conditions (31) and (31a) to obtain the first several component solutions.

$$u_1(x)= \left(-\frac{1}{\alpha}(M+\xi\phi_i)\left(-\frac{1}{3}\frac{St\sin\theta}{\alpha}+1\right)-\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^4}\right)x +\left(\frac{1}{2}\frac{\beta St^3\sin\theta^3}{\alpha^4}+\frac{1}{2\alpha}(M+\xi\phi_i)\right)x^2 +\left(-\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^4}-\frac{1}{6}\frac{(M+\xi\phi_i)St\sin\theta}{\alpha^2}\right)x^3 +\left(\frac{1}{12}\frac{\beta St^3\sin\theta^3}{\alpha^4}+\frac{1}{24}\frac{(M+\xi\phi_i)St\sin\theta}{\alpha^2}\right)x^4 \quad (32)$$

$$u_2(x)= \frac{1}{\alpha}\left(\beta\left(\frac{1}{\alpha}\left(\frac{1}{30\alpha^2}\frac{\beta St^3\sin\theta^3}{\alpha^3}+\frac{1}{2}\frac{(M+\xi\phi_i)St\sin\theta}{\alpha}\right)St^2\sin\theta^2\right)\right) +\frac{1}{12}\frac{1}{\alpha^2}\left(\left(\frac{-2\beta St^3\sin\theta^3}{\alpha^3}-\frac{1}{\alpha}\frac{(M+\xi\phi_i)St\sin\theta}{\alpha}\right)St^2\sin\theta^2\right)+\ldots \quad (33)$$

The noticeable length and complexity of $u_2(x)$ terms make it difficult to display. The solution of the differential equation (13) takes the form:

$$u(x)=\sum_{n=0}^{\infty}u_n(x)=u_0(x)+u_1(x) +u_2(x)+u_3(x)+\ldots. \quad (34)$$

Or equivalently as

$$u_{lift}(x)= \frac{1}{\alpha}\left(St\sin\theta\frac{x^2}{2}-St\sin\theta x+1\right)+ \left(-\frac{1}{\alpha}(M+\xi\phi_i)\left(-\frac{1}{3}\frac{St\sin\theta}{\alpha}+1\right) -\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^4}\right)x +\left(\frac{1}{2}\frac{\beta St^3\sin\theta^3}{\alpha^4}+\frac{1}{2\alpha}(M+\xi\phi_i)\right)x^2 +\left(-\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^4}-\frac{1}{6}\frac{(M+\xi\phi_i)St\sin\theta}{\alpha^2}\right)x^3 +\left(\frac{1}{12}\frac{\beta St^3\sin\theta^3}{\alpha^4}+\frac{1}{24}\frac{(M+\xi\phi_i)St\sin\theta}{\alpha^2}\right)x^4 +\ldots. \quad (35)$$

where $u_{lift}(x)$ is the solution of lifting case. The accuracy of the ADM solution increases by increasing the complexity of the solution. A truncated number of terms has used for the solution. If $\theta=\frac{\pi}{2}$ and the effect of MHD and the porosity parameters are dominated (i.e., M and ξ are equals to zero) in equation (35), then:

$$u(x)=\frac{1}{2}\frac{Stx^2}{\alpha}-\frac{Stx}{\alpha}+1+\frac{1}{12}\frac{\beta St^3x(x^3-4x^2+6x-4)}{\alpha^4} +\frac{1}{18}\frac{\beta^2St^5x(x^5-6x^4+15x^3-20x^2+15x-6)}{\alpha^7}+\ldots. \quad (36)$$

Moreover, at $\beta=0$ and $\alpha=1$ in equation (36), the solution for the Newtonian fluid is recovered, and a similar result was derived in earlier research by [10].

C. Drainage Flow of Powell-Eyring Fluid

In this section, the MHD Powell-Eyring fluid steeps down on the stationary inclined plate due to the gravity. The same assumptions of lifting problem are used, except that the gravity which is the dominant driving force for the flow acts in the downward direction. Thus, the governing is converted to:

$$0=\left(\mu+\frac{1}{\beta C}\right)\left(\frac{du}{dx}\right)-\frac{1}{6\beta C^3}\left(\frac{du}{dx}\right)^3 -\sigma B_0^2u+\rho g\sin\theta-\mu\frac{\phi_i}{K}u \quad (37)$$

subject to the boundary conditions

$$u(x)=0 \text{ at } x=0 \text{ (at the plate surface)} \quad (38)$$

$$\frac{du}{dx}=0 \text{ at } x=\delta \text{ (shear stress at the free surface)} \quad (38a)$$

By using equation (11), the dimensionless form of equation (36) is

$$\alpha\left(\frac{d^2u}{dx^2}\right)-\tilde{\beta}\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dx}\right)^2 -(M+\xi\phi_i)u+St\sin\theta=0 \quad (39)$$

subject to the boundary conditions

$$u(x)=0 \text{ at } x=0 \quad (40)$$

$$\frac{du}{dx}=0 \text{ at } x=1 \quad (41)$$

To solve equation (39), we use the ADM by the same process. The components of the solution are:

$$u_0(x)=\frac{1}{\alpha}\left(-St\sin\theta\frac{x^2}{2}+St\sin\theta x\right) \qquad (42)$$

$$u_1(x)=\left(-\frac{1}{3}\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha^2}+\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^4}\right)x$$
$$-\frac{1}{2}\left(\frac{\beta St^3\sin\theta^3}{\alpha^4}\right)x^2$$
$$+\frac{1}{\alpha}\left(\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^3}+\frac{1}{6}\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha}\right)x^3 \qquad (43)$$
$$+\frac{1}{\alpha}\left(-\frac{1}{12}\frac{\beta St^3\sin\theta^3}{\alpha^3}-\frac{1}{24}\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha}\right)x^4$$

$$u_2(x)=$$
$$-\frac{1}{\alpha}\left(\beta\left(\frac{1}{\alpha}\left(\begin{matrix}\frac{1}{30\alpha^2}\frac{-\beta St^3\sin\theta^3}{\alpha^3}\\ -\frac{1}{2}\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha}\end{matrix}\right)St^2\sin\theta^2\right)\right) \qquad (44)$$
$$+\frac{1}{12}\frac{1}{\alpha^2}\left(\left(\begin{matrix}\frac{2\beta St^3\sin\theta^3}{\alpha^3}\\ +\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha}\end{matrix}\right)St^2\sin\theta^2\right)+\ldots$$

By substituting the values of equations (42) and (44) in equation (33), the solution of the non-linear differential equation (36) takes the form

$$u_{Drainage}(x)=\frac{1}{\alpha}\left(-St\sin\theta\frac{x^2}{2}+St\sin\theta x\right)$$
$$+\left(-\frac{1}{3}\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha^2}+\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^4}\right)x$$
$$-\frac{1}{2}\left(\frac{\beta St^3\sin\theta^3}{\alpha^4}\right)x^2$$
$$+\frac{1}{\alpha}\left(\frac{1}{3}\frac{\beta St^3\sin\theta^3}{\alpha^3}+\frac{1}{6}\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha}\right)x^3 \qquad (45)$$
$$+\frac{1}{\alpha}\left(-\frac{1}{12}\frac{\beta St^3\sin\theta^3}{\alpha^3}-\frac{1}{24}\frac{\left(M+\xi\phi_i\right)St\sin\theta}{\alpha}\right)x^4$$
$$+\ldots.$$

$$M\to 0,\xi\phi_i\to 0 \text{ and } \theta=\frac{\pi}{2}$$

In the case that the effects of porosity and MHD are neglected, which means equation (45) becomes:

$$u(x)=-\frac{1}{2}\frac{Stx^2}{\alpha}-\frac{Stx}{\alpha}$$
$$-\frac{1}{12}\frac{\beta St^3x(x^3-4x^2+6x-4)}{\alpha^4} \qquad (46)$$
$$-\frac{1}{18}\frac{\beta^2St^5x(x^5-6x^4+15x^3-20x^2+15x-6)}{\alpha^7}$$
$$+\ldots.$$

TABLE I
COMPARING THE NUMERICAL VALUE OF THE VELOCITY $u(x)$ TO THE PREVIOUS STUDY USING A SUCCESSIVE LINEARIZATION METHOD (SLM) FOR THE LIFTING CASE

x	Newtonian fluid St=1, M=0, $\tilde{\beta}$ =0		Non-Newtonian fluid St =1, M=0, $\tilde{\beta}$ =0.15, ξ =0	
	Present study $u(x)$ (ADM)	[19] $u(x)$ (SLM)	[19] $u(x)$ (SLM)	Present study $u(x)$ (ADM)
0	1	1	1	1
0.1	0.9050	0.9050	0.9015	0.9013
0.2	0.8200	0.8200	0.8145	0.8138
0.3	0.7450	0.7450	0.7386	0.7374
0.4	0.6800	0.6800	0.6736	0.6716
0.5	0.6250	0.6250	0.6191	0.6163
0.6	0.5800	0.5800	0.5749	0.5713
0.7	0.5450	0.5450	0.5411	0.5364
0.8	0.5200	0.5200	0.5173	0.5116
0.9	0.5050	0.5050	0.5036	0.4967
1.0	0.5000	0.5000	0.5000	0.4918

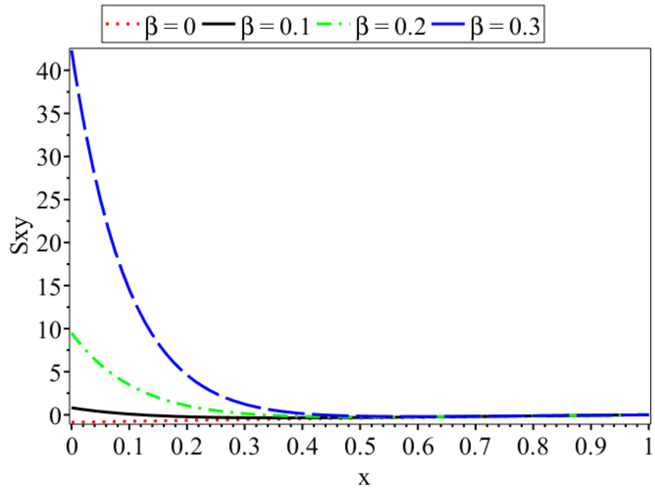

Fig. 2. Effect of the shear stress on the plate.

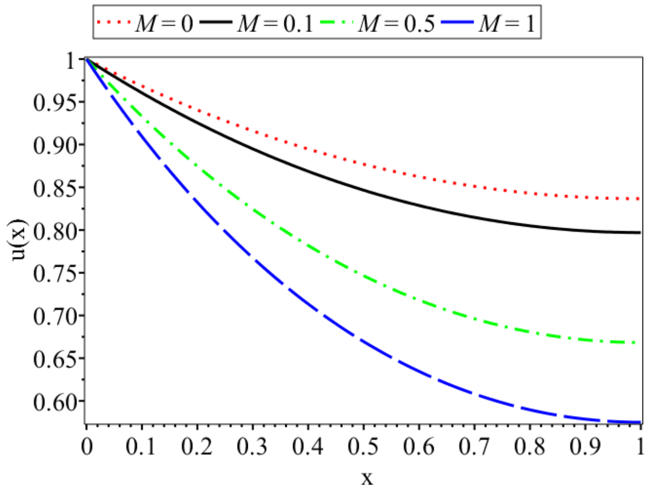

Fig. 4. Effect of Parameter M on the lift velocity $St = 0.5,\ \alpha = 1.1,\ \phi_i = 0,\ \beta = 1,\ \xi = 0,\ \theta = \frac{\pi}{4}$.

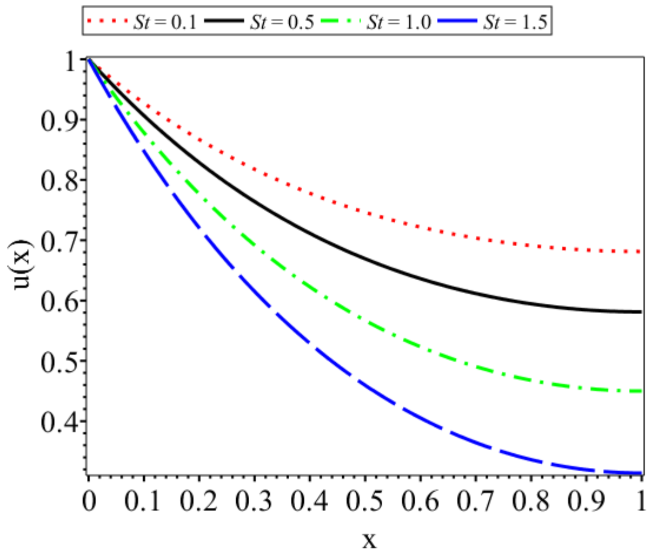

Fig. 3. Effect of the Stokes number on the lift velocity $M = 0.5,\ \alpha = 1.1,\ \phi_i = 1,\ \beta = 0.15,\ \xi = 1,\ \theta = \frac{\pi}{4}$.

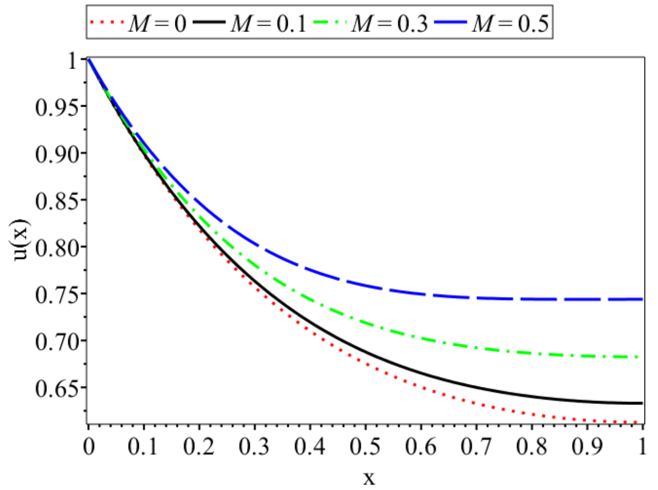

Fig. 5. Effect of the parameter on the lift velocity $St = 0.5,\ \alpha = 1.1,\ \phi_i = 1,\ \beta = 1,\ \xi = 2,\ \theta = \frac{\pi}{4}$.

In equation (46), it is noticeable that when $\beta = 0, \alpha = 1$ the solution for the Newtonian fluid is recovered.

III. Result and Discussion

The problem has been divided into two sections: lifting and a drainage fluid flow. Therefore, the results are also discussed individually in two separate sections.

A. *Lift Flow of Powell-Eyring Fluid*

In this section, the effect of the embedded flow parameters for the lifting problem of the Powell-Eyring fluid is discussed. Table I elucidates to compare $u_{lift}(x)$ for the Newtonian film. When we set the porosity and magnetic parameters equal to zero in equation (35), the Newtonian film solution is obtained. The results are tabulated in columns 1 and 2, and these results are similar to an earlier study conducted by Salah, et al. [19]. Moreover, the effect of the magnetic and the porosity parameter in the non-Newtonian film are tabulated in columns 3 and 4. The results reflect that the speed of the non-Newtonian film is strongly dependent on Sisko fluid parameter, and the Newtonian film lift faster than the non-Newtonian film.

The material parameters α *and* β are significant parameters that control friction and heat transfer. These parameters are also affecting the behaviour of the viscoelastic fluid, as illustrated in an earlier study [5]. Thus, the sensitivity of the solution due to the change of these two parameters is investigated. Substitute $x = 0$ in equation (8) to obtain the shear stress on the plate.

Figure 2 shows that the shear stress is susceptible to any small variations in the parameter β. The flow of Powell-Eyring fluid is too slow at high shear rate values [5]. The shear stress increased with the increment of β for $x < 0.5$, and thus the fluid flow is reduced. After $x > 0.5$ there is no noticeable effect of β on the shear stress, this result is in agreement with the results obtained by Van Rossum [11].

Figure 3-8 discuss the effect of the different parameters on velocity profile. Figure 3 illustrates the impact of Stoke's number St on the fluid velocity. It is found that the speed decreases by increasing St due to the rising of liquid thickness. Figure 4 and Figure 5 portray the effects of magnetic parameter. Generally, the velocity values reduce

TABLE II
EFFECT OF THE INCLINATION ON $u(x)$ FOR THE DRAINAGE CASE

x	Newtonian fluid when $St=1, M=0, \beta=0, \alpha=1$	Non- Newtonian fluid flow through a porous medium $u(x)$ under these conditions St =1, M=0, $\tilde{\beta}$ =0.15, ξ =0	
	$St=1, M=0, \beta=0, \alpha=1$	$St=1.5, M=0.1, \alpha=1$, $\beta=0.5, \xi=0.1, \theta=\frac{\pi}{2}$	$St=1.5, M=0.1, \alpha=1$, $\beta=0.9, \xi=0.1, \theta=\frac{\pi}{4}$
0	0	0	1
0.1	0.19	0.2187	0.1455
0.2	0.36	0.3904	0.2613
0.3	0.51	0.5258	0.3539
0.4	0.64	0.6329	0.4279
0.5	0.75	0.7172	0.4867
0.6	0.84	0.7827	0.5326
0.7	0.91	0.8317	0.5672
0.8	0.96	0.8658	0.5913
0.9	0.99	0.8860	0.6056
1.0	1.00	0.8927	0.6104

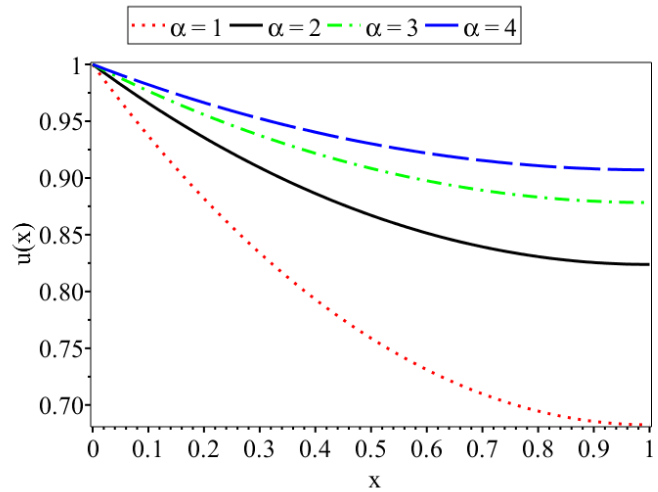

Fig. 6. Effect of the material parameter on the lift velocity $St=0.4,\ \phi_i=0,\ \beta=0.15,\ \xi=0, M=0.5,\ \theta=\frac{\pi}{4}$.

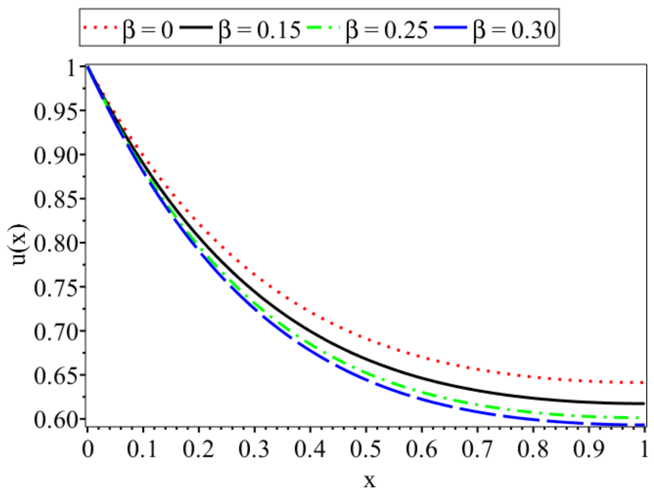

Fig. 7. Effect of non-Newtonian parameter on the lift velocity $St=1,\ \phi_i=1,\ \alpha=1.1, \xi=2, M=0.5,\ \theta=\frac{\pi}{4}$.

along the x axis. In the case of $\xi=\phi_i=0$, the fluid moves slower with the increment of M, as shown in Figure 4. Physically, the presence of a transverse magnetic field raises the Lorentz force, which acts as a retarding force on the velocity field. On the other hand, Figure 5 presents the relation between the velocity and M in the presence of porous medium. The generation of intermolecular porous force limits the effect of the magnetic force.

B. Drainage Flow of Powell-Eyring Fluid

The effects of different parameters on the drainage velocity $u(x)$ are investigated. Initially, the impact of inclination is summarised in Table II. In column 1, the drainage velocity of the Newtonian fluid is controlled by St value by affecting the fluid layer thickness. For non-Newtonian fluids in columns 2 and 3, it is clear that the liquid drained fast when the plate is in vertical position. Additionally, increasing either of St or M values reduces the rate of drainage while increasing β value raises the drain velocity because a large stokes number is indicating the dominance of inertial force. Figure 6 demonstrates the minimising of the material parameter α, which leads to the velocity decaying and at $\alpha=1$, the fluid converts to the Newtonian fluid. It is worthy to note the

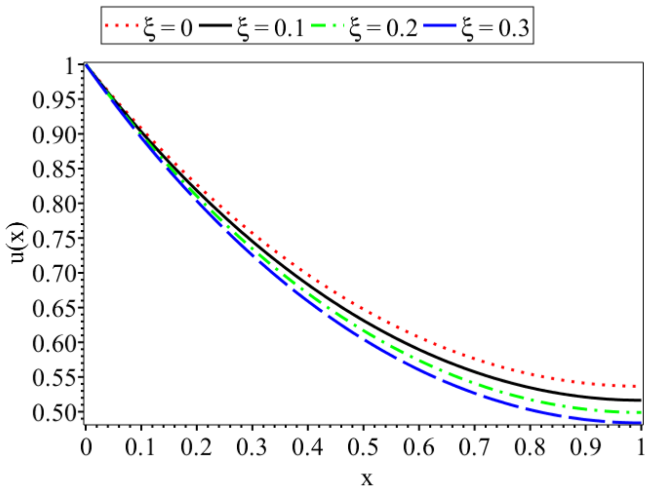

Fig. 8. Effect of the permeability on the lift velocity $St=1,\ \phi_i=1,\ \beta=0.15,\ \alpha=1.1, M=0.5,\ \theta=\frac{\pi}{4}$.

significant effect of β the flow velocity, as shown in Figure 7. In particular, the increment of β produces friction force which results in more fluid layer thickness, which means that

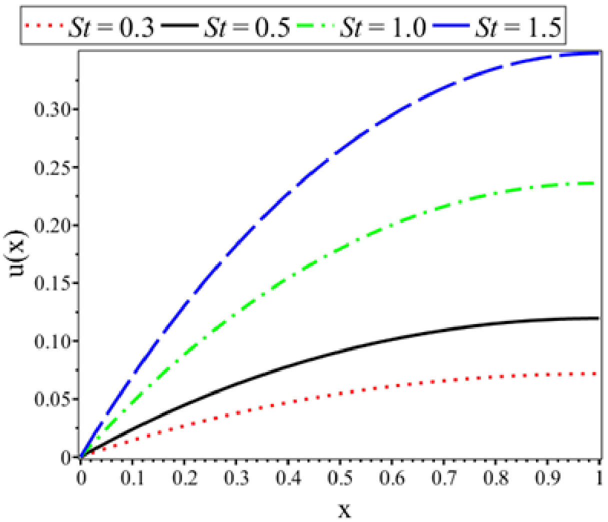

Fig. 9. Effect of the Stokes number on the velocity $M = 0.5,\ \phi_i = 1,\ \beta = 0.15,\ \alpha = 1.1,\ \xi = 1,\ \theta = \frac{\pi}{4}$.

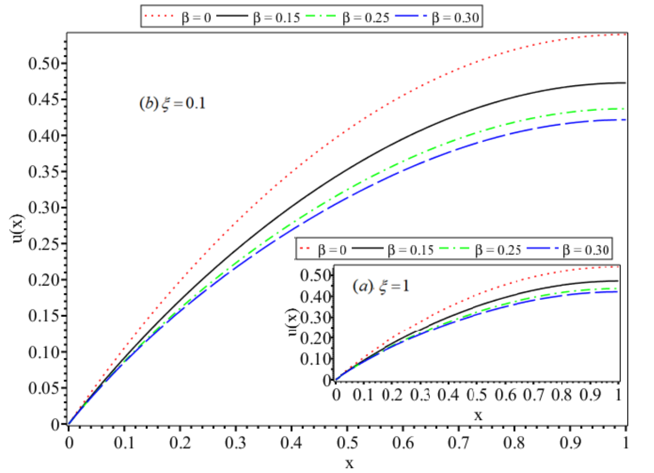

Fig. 11. Magnetic parameter's effect on the drainage velocity $\phi_i = 1,\ \alpha = 0.8, M = 0.5,\ \theta = \frac{\pi}{4}$.

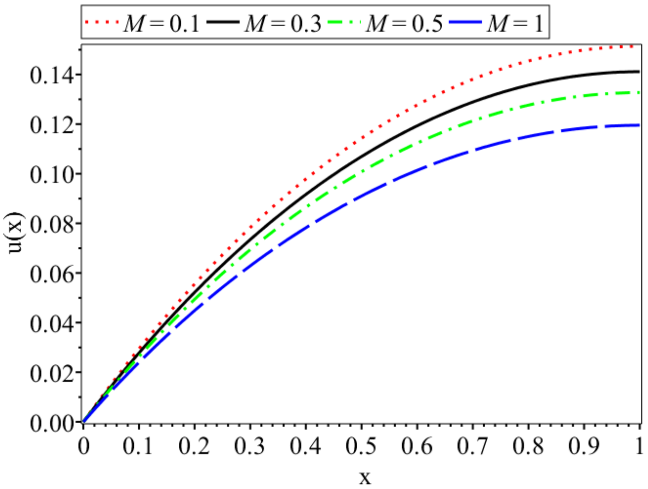

Fig. 10. Magnetic parameter's effect on the drainage velocity $St = 0.5,\ \phi_i = 1,\ \beta = 0.15,\ \alpha = 1.1,\ \xi = 0.1,\ \theta = \frac{\pi}{4}$.

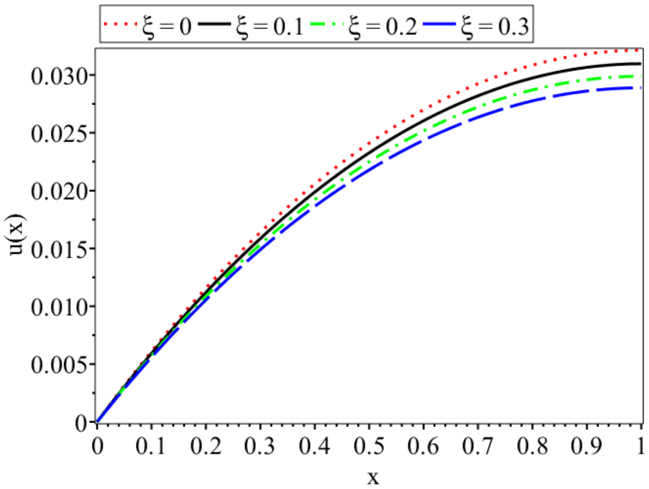

Fig. 12. Effect of the parameter on the drainage velocity $St = 1,\ \phi_i = 1,\ \beta = 0.15,\ \alpha = 1.1, M = 0.5,\ \theta = \frac{\pi}{4}$.

the fluid movement becomes slower. Figure 8 highlights the inverse relation between ξ and fluid velocity. The large values of ξ mean that a more considerable amount of fluid will be allowed to enter the medium with higher speed. Moreover, it is noticed that the flows of non-Newtonian fluid in a saturated porous medium are slower than the Newtonian fluid.

Figure 9 indicates that the drainage velocity increases at high values of the Stokes number St. This phenomenon has occurred as the friction is low near to the plate and higher at the surface, which results in flattening of the fluid layer.

Figure 10 shows that due to the high viscous force, which increases the adhesion of the liquid coatings, the velocity is a decreasing function due to the magnetic parameter effect.

Figure 10 displays the influence of non-Newtonian parameter β on drainage velocity. Generally, the fluid appears different behaviour by increasing β when assuming different values of the ξ. Utilising ξ that controlled the tone of the flow. Additionally, it is found that the fluid performs a shear-thinning when, $\beta = 0$ as shown in Figure 11. Figure 12 illustrates that the increment of ξ increases the fluid layer thickness near the plate and flattens it far away from the plate. The effect of the non-Newtonian parameter on the shear stress is displayed in Figure 13. The graph indicates that the shear stress increases as the non-Newtonian parameter increases.

IV. Conclusion

This study investigated the approximate analytical solution for the steady flow of magnetohydrodynamic (MHD) Powell-Eyring fluid over an inclined porous plate, for lifting and drainage problem. The non-linear differential equation arising from the mathematical modelling was analytically solved using ADM. The effect of the different parameters on the solutions was presented graphically. Certain major results are summarised as follows:

1) The effect of the magnetic parameter on velocity is opposite in lifting and drainage situations.
2) Shear stress is an increasing function of the fluid parameter in the lifting case and a decrease in the drainage case.
3) The adhesive force reduces the flow as ξ increases for both lifting and drainage flow.
4) Newtonian fluid results can be taken as a particular case

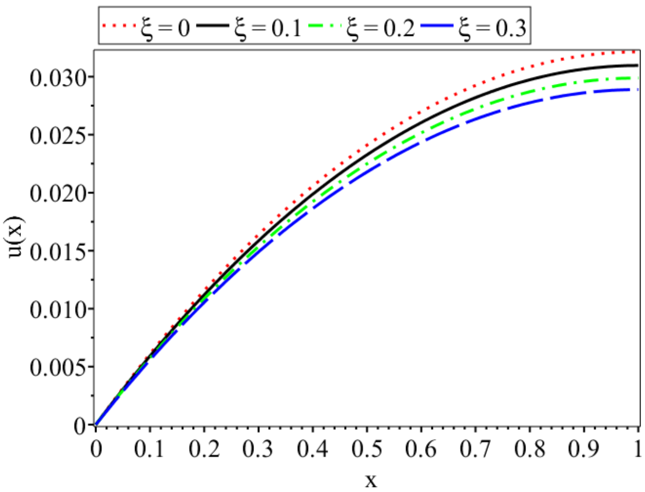

Fig. 13. Effect of the parameter on the shear stress $St = 1,\ \phi_i = 1,\ \beta = 0.15,\ \alpha = 1.1, M = 0.5,\ \theta = \frac{\pi}{4}$.

of the present problem $\beta = 0,$ higher velocities are observed with the increase of Stokes number when the fluid is drained down.

REFERENCES

[1] D. Malkus, J. Nohel, and B. Plohr, "Time-dependent shear flow of a non-Newtonian fluid," *J. Comput. Phys.*, vol. 87, pp. 464–487, 1990, doi: 10.1090/conm/100/08.

[2] F. Ree, T. Ree, and H. Eyring, "Relaxation theory of transport problems in condensed systems," *Ind. Eng. Chem.*, vol. 50, no. 7, pp. 1036–1040, 1958.

[3] R. E. Powell and H. Eyring, "Mechanisms for the relaxation theory of viscosity," *Nature*, vol. 154, no. 3909, pp. 427–428, 1944.

[4] A. M. Siddiqui, T. Haroon, and M. Zeb, "Analysis of eyring-powell fluid in helical screw rheometer," *Sci. World J.*, vol. 2014, 2014, doi: 10.1155/2014/143968.

[5] H. K. Yoon and A. J. Ghajar, "A note on the Powell-Eyring fluid model.," *Int. Commun. heat mass Transf.*, vol. 14, no. 4, pp. 381–390, 1987.

[6] T. Hayat, Z. Iqbal, M. Qasim, and S. Obaidat, "Steady flow of an Eyring Powell fluid over a moving surface with convective boundary conditions," *Int. J. Heat Mass Transf.*, vol. 55, no. 7–8, pp. 1817–1822, 2012, doi: 10.1016/j.ijheatmasstransfer.2011.10.046.

[7] H. Zaman, M. A. Shah, and M. Ibrahim, "Unsteady incompressible Couette flow problem for the Eyring-Powell model with porous walls," *Am. J. Comput. Math.*, vol. 2013, 2013.

[8] V. R. Prasad, S. A. Gaffar, E. K. Reddy, and O. A. Bég, "Computational study of non-newtonian thermal convection from a vertical porous plate in a non-darcy porous medium with biot number effects," *J. Porous Media*, vol. 17, no. 7, 2014, doi: 10.1615/JPorMedia.v17.i7.40.

[9] V. G. Gupta, A. Jain, and A. K. Jha, "Convective Effects on MHD Flow and Heat Transfer between Vertical Plates Moving in Opposite Direction and Partially Filled with a Porous Medium," *J. Appl. Math. Phys.*, vol. 4, no. 2, pp. 341–358, 2016, doi: 10.4236/jamp.2016.42041.

[10] N. Sher Akbar, A. Ebaid, and Z. H. Khan, "Numerical analysis of magnetic field effects on Eyring-Powell fluid flow towards a stretching sheet," *J. Magn. Magn. Mater.*, vol. 382, pp. 355–358, 2015, doi: 10.1016/j.jmmm.2015.01.088.

[11] N. A. Khan, S. Aziz, and N. A. Khan, "MHD flow of Powell-Eyring fluid over a rotating disk," *J. Taiwan Inst. Chem. Eng.*, vol. 45, no. 6, pp. 2859–2867, 2014, doi: 10.1016/j.jtice.2014.08.018.

[12] T. Hayat, Z. Hussain, M. Farooq, and A. Alsaedi, "Magnetohydrodynamic flow of powell-eyring fluid by a stretching cylinder with Newtonian heating," *Therm. Sci.*, vol. 22, no. 1B, pp. 371–382, 2018, doi: 10.2298/TSCI150717162H.

[13] Y. A. S. El-Masry, Y. Abd Elmaboud, and M. A. Abdel-Sattar, "The impacts of varying magnetic field and free convection heat transfer on an Eyring–Powell fluid flow with peristalsis: VIM solution," *J. Taibah Univ. Sci.*, vol. 14, no. 1, pp. 19–30, 2020, doi: 10.1080/16583655.2019.1698277.

[14] F. M. Elniel, Z. Abdul Aziz, F. Salah, and S. Mustafa, "Approximate analytical solution of the MHD Powell-Eyring fluid flow near accelerated plate," *Malaysian J. Fundam. Appl. Sci.*, vol. 13, pp. 416–420, 2017, doi: 10.11113/mjfas.v13n4-1.860.

[15] S. Adesanya, J. Falade, and R. Rach, "Effect of couple stresses on hydromagnetic Eyring-Powell fluid flow through a porous channel," *Theor. Appl. Mech. i Primenj. Meh.*, vol. 42, no. 2, pp. 135–150, 2015, doi: 10.2298/tam1502135a.

[16] A. M. Siddiqui, A. A. Farooq, and B. S. Babcock, "Two analytical methods applied to study thin film flow of an eyring-powell fluid on a vertically moving belt," *Appl. Math. Sci.*, vol. 7, no. 69–72, pp. 3469–3478, 2013, doi: 10.12988/ams.2013.34200.

[17] M. Jalil, S. Asghar, and S. M. Imran, "Selfsimilar solutions for the flow and heat transfer of Powell-Eyring fluid over a moving surface in a parallel free stream," *Int. J. Heat Mass Transf.*, vol. 65, pp. 73–79, 2013, doi: 10.1016/j.ijheatmasstransfer.2013.05.049.

[18] I. Khan, M. Y. Malik, T. Salahuddin, M. Khan, and K. U. Rehman, "Homogenous–heterogeneous reactions in MHD flow of Powell–Eyring fluid over a stretching sheet with Newtonian heating," *Neural Comput. Appl.*, vol. 30, no. 11, pp. 3581–3588, 2018, doi: 10.1007/s00521-017-2943-6.

[19] F. Salah, A. K. Alzahrani, A. O. Sidahmed, and K. K. Viswanathan, "A note on thin-film flow of Eyring-Powell fluid on the vertically moving belt using successive linearization method," *Int. J. Adv. Appl. Sci.*, vol. 6, no. 2, pp. 17–22, 2019, doi: 10.21833/ijaas.2019.02.004.

[20] R. Ellahi, E. Shivanian, S. Abbasbandy, and T. Hayat, "Numerical study of magnetohydrodynamics generalized Couette flow of Eyring-Powell fluid with heat transfer and slip condition," *Int. J. Numer. Methods Heat Fluid Flow*, vol. 26, no. 5, pp. 1433–1445, 2016, doi: 10.1108/HFF-04-2015-0131.

[21] A. T. Akinshilo and O. Olaye, "On the analysis of the Erying Powell model based fluid flow in a pipe with temperature dependent viscosity and internal heat generation," *J. King Saud Univ. - Eng. Sci.*, vol. 31, no. 3, pp. 271–279, 2019, doi: 10.1016/j.jksues.2017.09.001.

[22] G. Adomian and K. Malakian, "Operator-theoretic solution of stochastic systems," *J. Math. Anal. Appl.*, vol. 76, no. 1, pp. 183–201, 1980, doi: 10.1016/0022-247X(80)90071-2.

[23] G. Adomian, "A Review of the Decomposition Method and Some Recent Results for Nonlinear Equations," *Comput. Math. with Appl.*, vol. 21, no. 5, pp. 101–127, 1991.

[24] Y. A. Cengel and J. M. Cimbala, *Fluid Mechanics: Fundamentals and Applications*, 1st Editio. New York, NY, USA: McGraw-Hill, 2006.