Total Dominator Edge Chromatic Number of Graphs

Minhui Li, Shumin Zhang, Caiyun Wang and Chengfu Ye

***Abstract*—Let G be a simple graph with no isolated edge. A total dominator edge coloring (TDEC) of a graph G is a proper edge coloring such that each edge of G is adjacent to every edge of some (other) color class. The total dominator edge chromatic number (TDEC-number) of G is the minimum number of color classes among all total dominator edge colorings of G, denoted by $\chi_d'^t(G)$. The stability number of TDEC-number of G is the minimum number of vertices whose removal changes the value of $\chi_d'^t(G)$ and remaining graph is connected. The bondage number of TDEC-number of G is the minimum number of edges whose removal changes the value of $\chi_d'^t(G)$. In this paper, we establish the lower and upper bounds of the TDEC-number of a graph G, obtain the TDEC-number and the stability and bondage numbers of some graphs.**

***Index Terms*—total edge domination number, total dominator edge chromatic number, stability number, bondage number**

I. Introduction

ALL graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [1]. Let $G = (V, E)$ be a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The order and size of G are denoted by $n = |V|$ and $m = |E|$ respectively.

The open neighborhood $N(v)$ of a vertex v consists of the set of all vertices adjacent to v. The degree of a vertex v is the number of edges incident with v, denoted by $deg(v)$. Let $\delta(G)$ and $\Delta(G)$ be the minimum degree and the maximum degree of G, respectively. A vertex v is called a pendant vertex if $deg(v) = 1$. The open neighborhood $N(e)$ of an edge e consists of the set of all edges adjacent to e. The degree of an edge uv is defined as $deg(u) + deg(v) - 2$. An edge uv is called an isolated edge if $deg(uv) = 0$. An edge incident with a pendant vertex is called a pendant edge. For a subset $X \subseteq V(G)$, let $G - X$ denote a subgraph of G obtained by removing X. For a subset $Y \subseteq E(G)$, let $G - Y$ denote a subgraph of G obtained by removing Y. We denote a path, a cycle, a star and a complete graph of order n by P_n, C_n, $K_{1,n-1}$ and K_n, respectively. We say that a graph is connected if there is a path between every two vertices of the graph, and otherwise is called disconnected.

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A total dominating set of a graph G is a set $S \subseteq V(G)$ such that any vertex v of G satisfies $N(v) \cap S \neq \varnothing$, that is, every vertex of G is adjacent to at least one vertex in S. The total domination number of G is the cardinality of a minimum total dominating set, denoted by $\gamma_t(G)$. Some literatures on the subject on total domination of graphs has been surveyed in the book [5].

A proper coloring of a graph G is a function $f : V(G) \to \{1, \ldots, k\}(k \in N)$ such that $f(u) \neq f(v)$ if u and v are adjacent. The chromatic number of G is the minimum number of colors needed in a proper coloring of G, denoted by $\chi(G)$. Kazemi in [8] defined the new concept of total dominator coloring of graphs as following. A total dominator coloring of a graph G, briefly TDC, is a proper coloring such that each vertex of G is adjacent to every vertex of some (other) color class. The total dominator chromatic number of G, briefly TDC-number, is the minimum number of color classes among all total dominator colorings of G, denoted by $\chi_d^t(G)$.

The TDC-number of some graphs was determined and the computation of the TDC-number was NP-complete in [8]. Henning in [4] established the lower and upper bounds on the TDC-number of a graph G in terms of its $\gamma_t(G)$. And the properties of a TDC in trees were studied in [4, 8]. More details about the TDC-number of a graph can be found in [7, 9, 13].

In addition, the definitions related to the TDC-number of a graph were introduced by Ghanbari and Alikhani [3]. The total dominator chromatic stability (bondage) number of a graph G, is the minimum number of vertices (edges), whose removal changes the TDC-number.

Motivated by a TDC of a graph, Ghanbari and Alikhani [2] considered the proper edge coloring of G and introduced the total dominator edge coloring of G. A proper edge coloring of a graph G is a function $c : E(G) \to \{1, \ldots, k\}(k \in N)$ such that $c(e) \neq c(f)$ for any adjacent edges e, f. The edge chromatic number of G is the minimum number of colors needed in a proper edge coloring of G, denoted by $\chi'(G)$. A total dominator edge coloring of G, briefly TDEC, is a proper edge coloring such that each edge of G is adjacent to every edge of some (other) color class. The total dominator edge chromatic number of G, briefly TDEC-number, is the minimum number of color classes among all total dominator edge colorings of G, denoted by $\chi_d'^t(G)$. We say that a color is free if no edge is adjacent to all edges of this color class in a TDEC of G.

The TDEC-number of a graph is related to its total edge domination number. A set D of edges is a total edge dominating set of G, briefly TEDS, if every edge of G is adjacent to at least one edge in D. The total edge domination number of

G, briefly TED-number, is the minimum cardinality among all total edge dominating sets of G, denoted by $\gamma_t^{'}(G)$. A $\gamma_t^{'}(G)$-set is a TEDS of G with minimum cardinality. The total edge dominating of a graph was introduced and studied by Kulli and Patwari [6] in 1991. Readers may refer to [10–12, 14] for more information.

At present, in [2], Ghanbari and Alikhani obtained some properties of the TDEC-number and computed the values of the parameter for specific graphs and k-subdivision of graphs.

In this paper, we obtain some bounds of the TDEC-number of general graphs, and determine the TDEC-number, the TDEC-stability number and the TDEC-bondage number of some special graphs.

II. Preliminaries

Now we briefly introduce some related results.

Lemma 1. [11] *For any path P_n with $n \geq 2$, then*

$$\gamma_t^{'}(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \text{ or } 2(mod\ 4), \\ \frac{n-1}{2} & \text{if } n \equiv 1(mod\ 4), \\ \frac{n+1}{2} & \text{if } n \equiv 3(mod\ 4). \end{cases}$$

Lemma 2. [11] *For any cycle C_n with $n \geq 3$, then*

$$\gamma_t^{'}(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0(mod\ 4), \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{ or } 3(mod\ 4), \\ \frac{n+2}{2} & \text{if } n \equiv 2(mod\ 4). \end{cases}$$

Lemma 3. [11] *For any complete graph K_n with $n \geq 3$, then $\gamma_t^{'}(K_n) = \lfloor \frac{2n}{3} \rfloor$.*

III. Bounds of the TDEC-number of general graphs

In this section, we obtain some bounds of the TDEC-number of general graphs.

Observation 1. *Let G be a connected graph of size m with the maximum degree Δ. Then $\Delta \leq \chi_d^{'t}(G) \leq m$. Also the bounds are sharp.*

Theorem 1. *Let G be a graph with no isolated edge. Then*

$$max\{\chi^{'}(G), \gamma_t^{'}(G)\} \leq \chi_d^{'t}(G) \leq \gamma_t^{'}(G) + \chi^{'}(G).$$

Also the bounds are sharp.

Proof: First we prove the lower bound. It is obvious $\chi^{'}(G) \leq \chi_d^{'t}(G)$. Let $f = (E_1, E_2, \ldots, E_l)$ be a minimal TDEC of G, where $l = \chi_d^{'t}(G)$ and $D = \{e_1, e_2, \ldots, e_l\}$, $e_i \in E_i (1 \leq i \leq l)$. According to the definition of TDEC, we know that any edge e of G dominates at least a color class and is adjacent to at least one edge in D. And by the definition of TEDS, it is easy to see that D is a TEDS of G, so $\gamma_t^{'}(G) \leq \chi_d^{'t}(G)$.

Now we prove $\chi_d^{'t}(G) \leq \gamma_t^{'}(G) + \chi^{'}(G)$. Let $c : E(G) \to \{1, 2, \ldots, k\}$ $(k \in N)$ be a proper edge coloring of G. We assign the colors $k+1, k+2, \ldots, k+\gamma_t^{'}$ to the edges of a $\gamma_t^{'}(G)$-set of G and the other edges are colored as before. Clearly, this is a TDEC of G.

Furthermore, the lower bound is sharp if $G \cong P_3$, we know that

$$\chi_d^{'t}(P_3) = 2 = \chi^{'}(P_3) = \gamma_t^{'}(P_3).$$

The upper bound is sharp if $G \cong P_9$, we know that

$$\chi_d^{'t}(P_9) = 6 = \gamma_t^{'}(P_9) + \chi^{'}(P_9).$$

■

By Lemma 3 and Theorem 1, the following Corollary is immediate.

Corollary 1. *Let n be any positive integer and $n \geq 2$. Then $2n-1 \leq \chi_d^{'t}(K_{2n}) \leq \lfloor \frac{4n}{3} \rfloor + 2n - 1$ and $2n-1 \leq \chi_d^{'t}(K_{2n-1}) \leq \lfloor \frac{4n-2}{3} \rfloor + 2n - 1$.*

Next, we assign a unique color to each edge in the TEDS of G, then a proper edge coloring to the remaining edges.

Theorem 2. *Let G be a connected graph. Then*

$$\chi_d^{'t}(G) \leq \gamma_t^{'}(G) + \min_D \chi^{'}(G - D),$$

where $D \subseteq E(G)$ is a $\gamma_t^{'}(G)$-set of G. Also the bound is sharp.

Proof: Let $p = \min\{\chi^{'}(G-D) \mid D$ is a $\gamma_t^{'}(G)$-set of $G.\}$, $S = \{e_1, e_2, \ldots, e_q\}$ be a $\gamma_t^{'}(G)$ -set such that $\chi^{'}(G - S) = p$, and $f : E(G) - S \to \{1, 2, \ldots, p\}$ be a proper edge coloring of $G - S$. We define $g : E(G) \to \{1, 2, \ldots, p+q\}$ such that

$$g(e) = \begin{cases} p+i & \text{if } e = e_i \in S, \\ f(e) & \text{if } e \notin S. \end{cases}$$

Since S is a TEDS of G, g is a TDEC of G. Hence, $\chi_d^{'t}(G) \leq q + p = \gamma_t^{'}(G) + \min\{\chi^{'}(G-D) \mid D$ is a $\gamma_t^{'}(G)$-set of $G.\}$.

Furthermore, the bound is sharp if $G \cong K_{1,n}$, we know that

$\chi_d^{'t}(K_{1,n}) = n = \gamma_t^{'}(K_{1,n}) + \chi^{'}(K_{1,n-2})$
$= \gamma_t^{'}(K_{1,n}) + \min\{\chi^{'}(K_{1,n} - D) \mid D\ is\ a\ \gamma_t^{'}(K_{1,n}) - set.\}$. ■

Now, we study the TDEC-number for a disconnected graph without isolated edges.

Theorem 3. *Let G be a disconnected graph without isolated edges and $G_1, G_2, \ldots, G_s$ be all connected components of G, where $s \geq 2$. Then*

$$\max_{1 \leq i \leq s} \chi_d^{'t}(G_i) + 2s - 2 \leq \chi_d^{'t}(G) \leq \sum_{i=1}^{s} \chi_d^{'t}(G_i).$$

Also the bounds are sharp.

Proof: Let f_i be a TDEC of G_i with $\chi_d^{'t}(G_i)$ colors, for $1 \leq i \leq s$. Without loss of generality, we have $\chi_d^{'t}(G) \leq \sum_{i=1}^{s} \chi_d^{'t}(G_i)$. Let $\chi_d^{'t}(G_j) = \max\{\chi_d^{'t}(G_i) \mid 1 \leq i \leq s\}$, for some j, $1 \leq j \leq s$. Since the edges of G_i need at least two new colors, where $i \neq j$, we obtain $\max_{1 \leq i \leq s} \chi_d^{'t}(G_i) + 2s - 2 \leq \chi_d^{'t}(G)$.

Furthermore, the bounds are sharp if $G \cong kP_3 (k \geq 2)$, where kP_3 is the disjoint union of k copies of P_3. ■

Let M be an independent edges set of a graph G such that $G - M$ has no isolated edge or every isolated edge is adjacent to all edges in M.

Theorem 4. *Let G be a connected graph of size m and without isolated edges. Then $\chi_d^{'t}(G) \leq m - |M| + 1$.*

Proof: Let M be a maximum independent edges set, f be an edge coloring of G as following. Assign $m-|M|$ colors to $m-|M|$ edges of $G-M$ respectively, and assign a new color to all edges of M. Clearly, f is a TDEC of G, so $\chi_d^{'t}(G) \le m-|M|+1$. ■

IV. TDEC-NUMBER OF SPECIAL GRAPHS

In this section, we determine the TDEC-number of some special graphs. First, we state the following observation.

Observation 2. *For* $3 \le n \le 19$, *we have*

$$\chi_d^{'t}(P_n) = \begin{cases} \gamma_t^{'}(P_n) & \text{for } n \in \{3,4,7\}, \\ \gamma_t^{'}(P_n)+2 & \text{for } n \in \{9,13,14,16,17,18\}, \\ \gamma_t^{'}(P_n)+1 & \text{for } \textit{other}. \end{cases}$$

We have the following result when $n \ge 20$.

Theorem 5. *For any positive integer* $n \ge 20$, $\chi_d^{'t}(P_n) = \gamma_t^{'}(P_n)+2$.

Proof: We prove the theorem by induction on size of G. Let G be a path P_n with the vertex set $\{v_i \mid 1 \le i \le n\}$ and the edge set $\{e_i \mid e_i = v_iv_{i+1}, 1 \le i \le n-1\}$. It is easy to verify that the theorem holds when $n = 20, 21, 22, 23$.

By inductive hypothesis, we have $\chi_d^{'t}(P_{n'}) = \gamma_t^{'}(P_{n'})+2$ when $20 \le n^{'} < n$. Let f be a TDEC of G, $G^{'} = G - \{v_1, v_2, v_3, v_4\}$ and c be the restriction of f to the edges in $G^{'}$. It's easy to know that the edge e_2 is assigned a unique color in f and the two edges in $N(e_2) = \{e_1, e_3\}$ needs at least a new color. Hence, f has at least two more color classes than c, and we will discuss the c as follows.

Case 1: Suppose that c is a TDEC of $G^{'}$. According to the inductive hypothesis, we see that c has at least $\chi_d^{'t}(G^{'}) = \gamma_t^{'}(G^{'})+2 = \gamma_t^{'}(P_{n-4})+2 = \gamma_t^{'}(P_n) = \gamma_t^{'}(G)$ color classes. It means that f has at least $\gamma_t^{'}(G)+2$ color classes.

Case 2: Suppose that c is not a TDEC of $G^{'}$. Since f is a TDEC of G, only one edge e_5 in $G^{'}$ is not adjacent to all edges of some color class in c. Also, we see that e_6 is the only adjacent edge of e_5 in $G^{'}$, so the edge e_4 is assigned a unique color in f. Moreover, the color class that contains e_6 has at least two edges. Furthermore, let $c^{'}$ be a coloring of $G^{'}$, which is obtained by recoloring e_6 from c with the color of e_4 in f, then $c^{'}$ is a TDEC of $G^{'}$ and $c^{'}$ has at least $\chi_d^{'t}(G^{'}) = \gamma_t^{'}(G^{'})+2 = \gamma_t^{'}(P_{n-4})+2 = \gamma_t^{'}(P_n) = \gamma_t^{'}(G)$ color classes. Since e_1, e_2, e_3 are assigned at least two colors in f, f has at least two more color classes than $c^{'}$, so f has at least $\gamma_t^{'}(G)+2$ color classes.

According to the above discussions, we have $\chi_d^{'t}(G) \ge \gamma_t^{'}(G)+2$. In addition, by Theorem 1, $\chi_d^{'t}(G) \le \gamma_t^{'}(G)+2$. Hence, $\chi_d^{'t}(G) = \gamma_t^{'}(G)+2$. ■

By Lemma 1, Observation 2 and Theorem 5, we have the following Corollary immediately.

Corollary 2. *If* P_n *is a path with* $n \ge 20$, *then*

$$\chi_d^{'t}(P_n) = \begin{cases} 2k+2 & \text{if } n = 4k, 4k+1, \\ 2k+3 & \text{if } n = 4k+2, \\ 2k+4 & \text{if } n = 4k+3. \end{cases}$$

Especially, $\chi_d^{'t}(P_3) = \chi_d^{'t}(P_4) = 2$, $\chi_d^{'t}(P_5) = 3$, $\chi_d^{'t}(P_6) = \chi_d^{'t}(P_7) = 4$, $\chi_d^{'t}(P_8) = 5$, $\chi_d^{'t}(P_9) = \chi_d^{'t}(P_{10}) = 6$, $\chi_d^{'t}(P_{11}) = \chi_d^{'t}(P_{12}) = 7$, $\chi_d^{'t}(P_{13}) = 8$, $\chi_d^{'t}(P_{14}) = \chi_d^{'t}(P_{15}) = 9$, $\chi_d^{'t}(P_{16}) = \chi_d^{'t}(P_{17}) = 10$, $\chi_d^{'t}(P_{18}) = \chi_d^{'t}(P_{19}) = 11$.

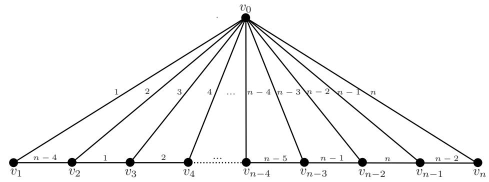

Fig. 1. A total dominator edge coloring of F_n.

Analogously, we have some results about cycles.

Observation 3. *For* $3 \le n \le 14$, *we have*

$$\chi_d^{'t}(C_n) = \begin{cases} \gamma_t^{'}(C_n) & \text{for } n \in \{4,6\}, \\ \gamma_t^{'}(C_n)+1 & \text{for } n \in \{3,5,7,9,10,14\}, \\ \gamma_t^{'}(C_n)+2 & \text{for } n \in \{8,11,12,13\}. \end{cases}$$

Theorem 6. *For* $n \ge 15$, $\chi_d^{'t}(C_n) = \gamma_t^{'}(C_n)+2$.

Proof: According to Theorem 5, we can similarly prove that $\chi_d^{'t}(C_n) \ge \gamma_t^{'}(C_n)+2$ when $n \ge 15$. Next we show that $\chi_d^{'t}(C_n) \le \gamma_t^{'}(C_n)+2$. Let C_n be a cycle with the edge set $E(C_n) = \{e_1, e_2, \ldots, e_n\}$, F be a $\gamma_t^{'}(C_n)$-set of C_n and $\gamma_t^{'}(C_n) = k$. We define $f: E \to \{1, 2, \ldots, k+2\}$ as an edge coloring of C_n, such that $f(e_i) = i$ for any $e_i \subseteq F$ and use the remaining two colors to give a proper edge coloring to $C_n - F$. Clearly, this is a TDEC of C_n, so $\chi_d^{'t}(C_n) \le \gamma_t^{'}(C_n)+2$. ■

By Lemma 2, Observation 3 and Theorem 6, we have the following Corollary immediately.

Corollary 3. *If* C_n *is a cycle with* $n \ge 15$, *then*

$$\chi_d^{'t}(C_n) = \begin{cases} 2k+2 & \text{if } n = 4k, \\ 2k+3 & \text{if } n = 4k+1, \\ 2k+4 & \text{if } n = 4k+2, 4k+3. \end{cases}$$

Especially, $\chi_d^{'t}(C_3) = 3$, $\chi_d^{'t}(C_4) = 2$, $\chi_d^{'t}(P_5) = \chi_d^{'t}(P_6) = 4$, $\chi_d^{'t}(P_7) = 5$, $\chi_d^{'t}(P_8) = \chi_d^{'t}(P_9) = 6$, $\chi_d^{'t}(P_{10}) = 7$, $\chi_d^{'t}(P_{11}) = \chi_d^{'t}(P_{12}) = 8$, $\chi_d^{'t}(P_{13}) = \chi_d^{'t}(P_{14}) = 9$.

A fan graph F_n is a graph with $n+1$ vertices and $2n-1$ edges, see Figure 1.

Theorem 7. *For* $n \ge 3$, $\chi_d^{'t}(F_n) = n$.

Proof: This result is obvious when $3 \le n \le 6$. For $n \ge 7$, there is a TDEC of F_n, see Figure 1. Actually, we assign the color j to the edge v_0v_j $(1 \le j \le n)$, the color $n-4$ to the edge v_1v_2, the color i to the edge $v_{i+1}v_{i+2}$ $(1 \le i \le n-5)$, the colors $n-1$, n, $n-2$ to the edges $v_{n-3}v_{n-2}$, $v_{n-2}v_{n-1}$, $v_{n-1}v_n$ respectively. So $\chi_d^{'t}(F_n) \le n$, and since $\Delta(F_n) = n$, $\chi_d^{'t}(F_n) \ge n$. Hence $\chi_d^{'t}(F_n) = n$. ■

A double star graph $S_{n,m}$ is a tree, obtained by connecting the centers of two disjoint stars $K_{1,n}$ and $K_{1,m}$, for $n, m \ge 2$, see Figure 2.

Theorem 8. *For* $n, m \ge 2$, $\chi_d^{'t}(S_{n,m}) = \max\{n, m\}+1$.

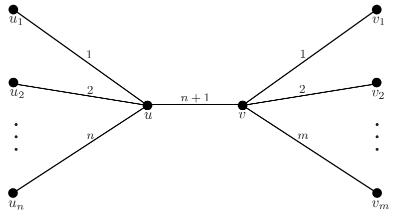

Fig. 2. A total dominator edge coloring of $S_{n,m}$ when $n \geq m$.

Proof: Without loss of generality, suppose that $n \geq m$ $(n, m \geq 2)$. Since $\Delta(S_{n,m}) = n+1$, a TDEC of $S_{n,m}$ contains at least $n+1$ colors. Assigning the color i to the edge uu_i $(1 \leq i \leq n)$, the color j to the edge vv_j $(1 \leq j \leq m)$ and the color $n+1$ to the edge uv, see Figure 2. Obviously, this is a TDEC of $S_{n,m}$, so $\chi_d^{'t}(S_{n,m}) = \max\{n, m\} + 1$. ■

V. The stability (bondage) number of TDE-coloring

In this section, we study the stability (bondage) number of a TDEC of a graph.

The TDC-stability (TDC-bondage) number of a graph G, is the minimum number of vertices (edges) of G, whose removal changes the TDC-number of G. Motivated by the stability (bondage) number of a TDC of a graph G, we naturally consider the stability (bondage) number of a TDEC of G.

The total dominator edge chromatic stability number of G, briefly TDEC-stability number, denoted by $St_d^{'t}(G)$, is $\min\{|X| \mid X \subseteq V(G)\}$, such that $\chi_d^{\prime t}(G) \neq \chi_d^{\prime t}(G - X)$ and $G - X$ is connected. The total dominator edge chromatic bondage number of G, briefly TDEC-bondage number, denoted by $Bt_d^{'t}(G)$, is $\min\{|Y| \mid Y \subseteq E(G)\}$, such that $\chi_d^{\prime t}(G) \neq \chi_d^{\prime t}(G - Y)$.

First, we show that the TDEC-stability numbers of paths, cycles, fan graphs and double star graphs.

Observation 4. *For* $5 \leq n \leq 21$,

$$St_d^{'t}(P_n) = \begin{cases} 2 & \text{for } n \in \{7, 10, 12, 15, 17, 19, 21\}, \\ 1 & \text{for } other. \end{cases}$$

Theorem 9. *For* $n \geq 22$,

$$St_d^{'t}(P_n) = \begin{cases} 2 & \text{if } n = 4k, \\ 3 & \text{if } n = 4k+1, \\ 1 & \text{if } n = 4k+2, 4k+3. \end{cases}$$

Proof: We consider the following four cases.

When $n = 4k$. By Corollary 2, we have $\chi_d^{'t}(P_{4k}) = \chi_d^{'t}(P_{4k-1}) = 2k+2$, so we need to remove at least two vertices. Removing the vertices v_{4k+3}, v_{4k+4}, we obtain P_{4k+2} and $\chi_d^{'t}(P_{4k+2}) = 2k+3$. So $St_d^{'t}(P_{4k+4}) = 2$.

When $n = 4k+1$. We have $\chi_d^{'t}(P_{4k+1}) = \chi_d^{'t}(P_{4k}) = \chi_d^{'t}(P_{4k-1}) = 2k+2$, so we need to remove at least three vertices. Removing the vertices $v_{4k-1}, v_{4k}, v_{4k+1}$, we obtain P_{4k-2} and $\chi_d^{'t}(P_{4k-2}) = 2k+1$. So $St_d^{'t}(P_{4k+1}) = 3$.

When $n = 4k+2$. We have $\chi_d^{'t}(P_{4k+2}) = 2k+3$. Removing the pendant vertex v_{4k+2}, we obtain P_{4k+1} and $\chi_d^{'t}(P_{4k+1}) = 2k+2$. So $St_d^{'t}(P_{4k+2}) = 1$.

When $n = 4k+3$. The proof is similar to the case $n = 4k+2$. So $St_d^{'t}(P_{4k+3}) = 1$. ■

Proposition 1. *For* $5 \leq n \leq 20$, $St_d^{'t}(C_n) = 1$.

Proof: Removing one vertex of C_n, we know that C_n becomes P_{n-1}. By Corollary 2 and Corollary 3, we have $\chi_d^{'t}(P_{n-1}) < \chi_d^{'t}(C_n)$, so $St_d^{'t}(C_n) = 1$ when $5 \leq n \leq 20$. ■

Theorem 10. *For* $n \geq 21$,

$$St_d^{'t}(C_n) = \begin{cases} 2 & \text{if } n = 4k. \\ 1 & \text{if } n = 4k+1, 4k+2, 4k+3. \end{cases}$$

Proof: We consider the following cases.

When $n = 4k$. By Corollary 3, we have $\chi_d^{'t}(C_{4k}) = \chi_d^{'t}(P_{4k-1}) = 2k+2$, so we need to remove at least two vertices. Removing two consecutive vertices of C_{4k+4}, we obtain P_{4k+2} and $\chi_d^{'t}(P_{4k+2}) = 2k+3$. So $St_d^{'t}(C_{4k+4}) = 2$.

When $n = 4k+1$. We have $\chi_d^{'t}(C_{4k+1}) = 2k+3$. Removing the vertex v_{4k+1}, we obtain P_{4k} and $\chi_d^{'t}(P_{4k}) = 2k+2$. So $St_d^{'t}(C_{4k+1}) = 1$.

When $n = 4k+2$ or $n = 4k+3$. the proof is similar to the case $n = 4k+1$. So $St_d^{'t}(C_{4k+2}) = St_d^{'t}(C_{4k+3}) = 1$. ■

Theorem 11. *For* $n \geq 3$, $St_d^{'t}(F_n) = 1$.

Proof: Obviously, removing the vertex v_0 in F_n, see Figure 1, we have $\chi_d^{'t}(P_n) < \chi_d^{'t}(F_n)$. So $St_d^{'t}(F_n) = 1$. ■

Theorem 12. *For* $n, m \geq 2$,

$$St_d^{'t}(S_{n,m}) = \begin{cases} 1 & \text{if } n \neq m, \\ 2 & \text{if } n = m. \end{cases}$$

Proof: We consider the following two cases.

If $n \neq m$. Without loss of generality, let $n > m$. Removing the pendant vertex u_i in $S_{n,m}$, see Figure 2. And by Theorem 8, we have $\chi_d^{'t}(S_{n,m}) > \chi_d^{'t}(S_{n-1,m})$. So $St_d^{'t}(S_{n,m}) = 1$.

If $n = m$. It is easy to see that $\chi_d^{'t}(S_{n,m})$ does not change by removing a pendant vertex in $S_{n,m}$. Therefore, we need to remove two pendant vertices u_i and v_j, where $1 \leq i \leq n, 1 \leq j \leq m$, and $\chi_d^{'t}(S_{n-1,m-1}) < \chi_d^{'t}(S_{n,m})$. So $St_d^{'t}(S_{n,m}) = 2$. ■

Next, we investigate the TDEC-bondage numbers of these graphs.

Observation 5. *For* $5 \leq n \leq 24$,

$$Bt_d^{'t}(P_n) = \begin{cases} 2 & \text{for } n = 7, 15, \\ 1 & \text{for } other. \end{cases}$$

Theorem 13. *For* $n \geq 25$,

$$Bt_d^{'t}(P_n) = \begin{cases} 2 & \text{if } n = 4k, \\ 1 & \text{if } n = 4k+1, 4k+2, 4k+3. \end{cases}$$

Proof: We consider the following four cases.

When $n = 4k$. By Corollary 2, we have $\chi_d^{'t}(P_{4k}) = 2k+2$. It is easy to see that $\chi_d^{'t}(P_{4k})$ does not change by removing a pendant edge of P_{4k}. Therefore, we consider that removing one edge e of P_{4k}, we obtain P_{4k-s} and P_s as the components of $P_{4k} - \{e\}$, where $3 \leq s \leq 2k$. There is a minimum TDEC of $P_{4k} - \{e\}$ as following. Giving a TDEC to P_{4k-s}, we have

$$\chi_d^{'t}(P_{4k-s}) = \begin{cases} \frac{4k-s}{2}+2 & \text{if } 4k-s \equiv 0,2(mod\ 4), \\ \lfloor \frac{4k-s}{2} \rfloor +2 & \text{if } 4k-s \equiv 1(mod\ 4), \\ \lceil \frac{4k-s}{2} \rceil +2 & \text{if } 4k-s \equiv 3(mod\ 4). \end{cases}$$

And giving a TDEC to P_s, we know that the TDEC contains $f(s)$ new colors and other colors use free colors of P_{4k-s}. We have

$$f(s) = \begin{cases} \frac{s}{2} & \text{if } s \equiv 0,2(mod\ 4), \\ \lfloor \frac{s}{2} \rfloor & \text{if } s \equiv 1(mod\ 4), \\ \lceil \frac{s}{2} \rceil & \text{if } s \equiv 3(mod\ 4). \end{cases}$$

Clearly, $\chi_d^{'t}(P_{4k} - \{e\}) = \chi_d^{'t}(P_{4k-s}) + f(s) = 2k+2 = \chi_d^{'t}(P_{4k})$. So we need to remove at least two edges in P_{4k}. Removing two pendant edges, we obtain P_{4k-2} and $\chi_d^{'t}(P_{4k-2}) = 2k+1$. Hence, $Bt_d^{'t}(P_{4k}) = 2$.

When $n = 4k+1$. We have $\chi_d^{'t}(P_{4k+1}) = 2k+2$. We can obtain P_{4k-2} and P_3 by removing one edge of P_{4k+1}. Since $\chi_d^{'t}(P_{4k-2}) = 2k+1$ and $\chi_d^{'t}(P_3) = 2$, $Bt_d^{'t}(P_{4k+1}) = 1$.

When $n = 4k+2$. We have $\chi_d^{'t}(P_{4k+2}) = 2k+3$. Removing a pendant edge of P_{4k+2}, we obtain P_{4k+1} and $\chi_d^{'t}(P_{4k+1}) = 2k+2$. So $Bt_d^{'t}(P_{4k+2}) = 1$.

When $n = 4k+3$. The proof is similar to the case $n = 4k+2$. So $Bt_d^{'t}(P_{4k+3}) = 1$. ■

Observation 6. *For* $5 \le n \le 20$

$$Bt_d^{'t}(C_n) = \begin{cases} 2 & \text{for } n \in \{6,9,14,16,20\}, \\ 1 & \text{for } other. \end{cases}$$

Theorem 14. *For* $n \ge 21$,

$$Bt_d^{'t}(C_n) = \begin{cases} 3 & \text{if } n = 4k, \\ 1 & \text{if } n = 4k+1, 4k+2, \\ 2 & \text{if } n = 4k+3, 24. \end{cases}$$

Proof: We consider the following four cases.

When $n = 4k$. By Corollary 2 and Corollary 3, it is easy to see that $\chi_d^{'t}(C_{4k}) = \chi_d^{'t}(P_{4k}) = 2k+2$, so we need to remove at least two edges in C_{4k}. If $n = 24$. We can obtain P_{19} and P_5 by removing two inconsecutive edges of C_{24}, and use 13 colors to give them a TDEC. We know that $\chi_d^{'t}(C_{24}) = 14$, so $Bt_d^{'t}(C_{24}) = 2$.

When $n = 4k$ and $n \neq 24$, we have two subcases:

i) Removing two consecutive edges of C_{4k}, we obtain P_{4k-1} and $\chi_d^{'t}(P_{4k-1}) = 2k+2 = \chi_d^{'t}(C_{4k})$.

ii) Removing two inconsecutive edges e, $e^{'}$ of C_{4k}, we obtain P_{4k-m} and P_m as the components of $C_{4k} - \{e, e^{'}\}$, where $3 \le m \le 2k$. There is a minimum TDEC of $C_{4k} - \{e, e^{'}\}$ as following. Giving a TDEC to P_{4k-m}, we have

$$\chi_d^{'t}(P_{4k-m}) = \begin{cases} \frac{4k-m}{2}+2 & \text{if } 4k-m \equiv 0,2(mod\ 4), \\ \lfloor \frac{4k-m}{2} \rfloor +2 & \text{if } 4k-m \equiv 1(mod\ 4), \\ \lceil \frac{4k-m}{2} \rceil +2 & \text{if } 4k-m \equiv 3(mod\ 4). \end{cases}$$

And giving a TDEC to P_m, we know that the TDEC contains $f(m)$ new colors and other colors use free colors of P_{4k-m}, we have

$$f(m) = \begin{cases} \frac{m}{2} & \text{if } m \equiv 0,2(mod\ 4), \\ \lfloor \frac{m}{2} \rfloor & \text{if } m \equiv 1(mod\ 4), \\ \lceil \frac{m}{2} \rceil & \text{if } m \equiv 3(mod\ 4). \end{cases}$$

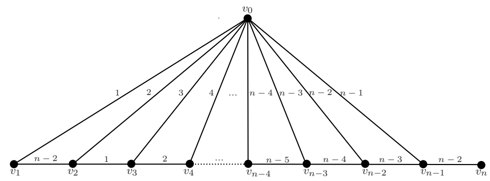

Fig. 3. A total dominator edge coloring of $F_n - \{v_0v_n\}$.

Clearly, $\chi_d^{'t}(C_{4k} - \{e, e^{'}\}) = \chi_d^{'t}(P_{4k-m}) + f(m) = 2k+2 = \chi_d^{'t}(C_{4k})$. Hence, the value of $\chi_d^{'t}(C_{4k})$ does not change by removing two edges. We remove three consecutive edges of C_{4k}, obtain P_{4k-2} and $\chi_d^{'t}(P_{4k-2}) = 2k+1$. So $Bt_d^{'t}(C_{4k}) = 3$.

When $n = 4k+1$. We have $\chi_d^{'t}(C_{4k+1}) = 2k+3$. Removing any edge of C_{4k+1}, we obtain P_{4k+1} and $\chi_d^{'t}(P_{4k+1}) = 2k+2$. So $Bt_d^{'t}(C_{4k+1}) = 1$.

When $n = 4k+2$. It is similar to the case $n = 4k+1$. So $Bt_d^{'t}(C_{4k+2}) = 1$.

When $n = 4k+3$. We have $\chi_d^{'t}(C_{4k+3}) = \chi_d^{'t}(P_{4k+3}) = 2k+4$, so the value of $\chi_d^{'t}(C_{4k+3})$ does not change by removing one edge. We need to remove at least two edges. Removing two consecutive edges of C_{4k+3}, we obtain P_{4k+2} and $\chi_d^{'t}(P_{4k+2}) = 2k+3$. So $Bt_d^{'t}(C_{4k+3}) = 2$. ■

Theorem 15. *For* $n \ge 5$, $Bt_d^{'t}(F_n) = 1$. *Especially,* $Bt_d^{'t}(F_2) = Bt_d^{'t}(F_3) = 1$, $Bt_d^{'t}(F_4) = 2$.

Proof: Removing any edge of F_n, we see that the value of $\chi_d^{'t}(F_n)$ does not change when $n = 4$. By removing v_0v_4, v_3v_4 in F_n, we have $\chi_d^{'t}(F_3) = 3 < \chi_d^{'t}(F_4)$. So $Bt_d^{'t}(F_4) = 2$. Removing the edge v_0v_n in F_n, we have $\chi_d^{'t}(F_n - \{v_0v_n\}) = n-1$ when $n \ge 5$, see Figure 3. So $Bt_d^{'t}(F_n) = 1$. ■

Theorem 16. *For* $n, m \ge 2$, $Bt_d^{'t}(S_{n,m}) = 1$.

Proof: We consider the following two cases.

If $n \neq m$. Without loss of generality, let $n > m$. Removing a pendant edge uu_i $(1 \le i \le n)$ in $S_{n,m}$, see Figure 2, we have $\chi_d^{'t}(S_{n,m}) > \chi_d^{'t}(S_{n-1,m})$. So $Bt_d^{'t}(S_{n,m}) = 1$.

If $n = m$. It is easy to see that $\chi_d^{'t}(S_{n,m})$ does not change by removing a pendant edge in $S_{n,m}$. However, removing the edge uv, we obtain two star graphs $K_{1,n}$ and $K_{1,m}$ as the components of $S_{n,m} - \{uv\}$, see Figure 2. Clearly, $\chi_d^{'t}(S_{n,m} - \{uv\}) = n+2$ and $\chi_d^{'t}(S_{n,m}) = n+1$. So $Bt_d^{'t}(S_{n,m}) = 1$. ■

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