Agents, Activity Levels and Utility Distributing Mechanism: Game-theoretical Viewpoint

Yu-Hsien Liao, Chia-Hung Li, Yen-Chin Chen, Li-Yang Tsai, Yu-Chen Hsu and Chih-Kuan Chen

***Abstract*—In general, agents and its activity levels might be two essential factors under real-world situations. Thus, we propose a consistent solution to analyze utility distributing mechanism by focusing on the agents and its activity levels at the same time. Two existing concepts from traditional game theory are also applied to reinterpret in this paper. First, by applying consistency which related to an extended reduction, two axiomatic results are offered to discuss the rationality of this solution. Second, based on excess mapping, two dynamic processes are constructed to illustrate that this solution can be attained by agents who begin from an arbitrary efficient outcome and make succeeding modifications.**

***Index Terms*—Utility distributing mechanism, reduction, axiomatic result, dynamic process.**

I. Introduction

In most interactive systems (economic systems, management systems, operating systems, environmental systems, etc.), models, or programs, attention is usually paid to the optimal or equilibrium state of an allocation or processing concept, often called the solution. To highlight the advantages of a solution, simply proclaiming how superior it is does not necessarily lead to a majority of people accepting this solution. In *axiomatization*, various mathematical theories are often used to analyze and propose the optimal or equilibrium states of these solutions. Axiomatization is a mathematical theory that first uses various mathematical theories to model systems, models, procedures, and certain fair, just, and well-recognized properties based on game theory. A relevant solution is then proposed, analyzed, and proven to be the only solution that satisfies certain fair, just, and well-recognized properties. It is important that these properties are indispensable. In this way, agreeing with these fair, just and well-recognized properties is equivalent to agreeing with the solution. On the other hand, if an allocation or processing concept is not agreed upon by the majority simultaneously, usually some form of communication, debate or negotiation will be used to revise the concept so that it gradually becomes one that can be agreed upon by the majority – such is the so-called *dynamic process*.

Manuscript received June 15, 2021; revised August 31, 2021.

Yu-Hsien Liao is a professor of Department of Applied Mathematics, National Pingtung University, 900 Pingtung, Taiwan. (Corresponding author. email: twincos@ms25.hinet.net)

Chia-Hung Li is an assistant professor of Office of Physical Activities, National Pingtung University, 900 Pingtung, Taiwan. (Co-corresponding author. email: Pippen0917@mail.nptu.edu.tw)

Yen-Chin Chen is a student of Department of Physical Education, National Pingtung University, 900 Pingtung, Taiwan. (email: nh10128@gmail.com)

Li-Yang Tsai is a student of Department of Physical Education, National Pingtung University, 900 Pingtung, Taiwan. (email: aa0911896771@gmail.com)

Yu-Chen Hsu is a student of Department of Physical Education, National Pingtung University, 900 Pingtung, Taiwan. (email: a7262296@gmail.com)

Chih-Kuan Chen is a student of Department of Physical Education, National Pingtung University, 900 Pingtung, Taiwan. (email: stasd4567@gmail.com)

Under the theory of traditional cooperative games, a characteristic mapping is determined over whole the subsets of the collection of agents. This implies that the choices available for each agent are either to partake completely or not to partake at all. Under real-world situations, however, agents might take different activity levels to partake. Therefore, it is reasonable that the agents could be allowed to adopt different activity levels of participation in a coalition. A *multi-choice game* could be looked upon as a natural analogue of a traditional game in which each agent takes several activity levels. For example, an application appears in a large enterprise with many sections, where the income-making depends on its expressions. This forms a multi-choice condition in which the agents are the sections and the worth of a coalition where each section functions at a certain level is the corresponding income occured by the enterprise. As a result, the domain of the characteristic mapping is extended to allow *multi-choice coalition*.

In the axiomatic formulation for solutions in game theory, *consistency* is an important property. Consistency declares the independence of an outcome with respect to fixing several agents with its alloted payoffs. It claims that the proposal made for any issue should always assent with the proposal made in the subissue that arises if the payoffs of several agents are settled on. It has been defined in distinct situations depending upon how the payoffs of the agents that ”leave the bargaining” are determined. This axiom has been probed under various issues by applying *reductions*. In addition to axiomatic analysis, *dynamic processes* could be depicted that lead the agents to that solution, starting from an efficient outcome.

This article focuses on the solution concept of the *equal allocation of non-separable costs* (EANSC, Ransmeier [12]). Based on the notion of the EANSC, all agents firstly obtain its marginal contributions from the grand coalition, and further allot the rest of utilities equally. Under traditional games, Moulin [10] introduced a notion of reduction and related consistency to show that the EANSC could provide a fair rule for distributing utility. By determining overall outcomes for a given agents on multi-choice games, Liao [5] proposed an extended EANSC by applying the *maximal marginal contributions* of agents. Inspired by the notion of *replication* due to Nouweland et al. [11], Liao [6] introduce the duplicate EANSC to compute overall values for a given agents. Related researches also could be refered to Cheng et al. [1], Hwang and Liao [3], Liao et al. [8], and so on. In real-world situations, however, agents and its activity levels might be two essential factors at the same time. These mentioned above raise one motivation under multi-choice consideration:

- whether the EANSC could extended by considering agents and its activity levels at the same time.

This article is aimed at answering above motivation. Some existing results of traditional game theory would be extended in this article. The main results are as follows.

1) By considering the agents and the activity levels on multi-choice games simultaneously, a generalization of the traditional EANSC, the *multi-choice equal allocation of non-separable costs* (MCEANSC), is defined in Section 2.
2) By extending the reduction proposed by Moulin [10] to multi-choice games, we provide two axiomatic results to present the rationality of the MCEANSC in Section 3. We show that
   - the MCEANSC is the unique allocation matching the properties of bilateral consistency and standard for two-person games;
   - the MCEANSC is the unique allocation matching the properties of efficiency, bilateral consistency, symmetry and zero-independence.
3) A solution could be analyzed by axiomatic justification. Alternatively, *dynamic processes* could be depictd that lead the agents to that solution, starting from an efficient outcome. The basis of a dynamic analysis was laid by Stearns [14]. Different from dynamic result due to Maschler and Owen [9], the *excess mapping* is introduced to analyze dynamic processes leading to the MCEANSC in Section 4, starting from an efficient outcome. Some more applications, comparisons, connections and statements are also discussed throughout this article.

II. Preliminaries

Let U be the universe of agents. For $i \in U$, one could set $G_i = \{0, 1, \cdots, g_i\}$ as the activity level collection of agent i and $G_i^+ = G_i \setminus \{0\}$, where $g_i \in \mathbb{N}$ and level 0 implies not partaking. For $A \subseteq U$, $A \neq \emptyset$, let $G^A = \prod_{i\in A} G_i$ be the product set of the activity level collections for agents in A. A **multi-choice coalition** is a vector $\zeta \in G^A$. The i-th coordinate ζ_i of ζ is the activity level of agent i in ζ. A multi-choice coalition could be depicted as a set of economic agents, i.e., agents, who deliver fractions of its representation to a group strategy maker, the multi-choice coalition. The term multi-choice coalition also appears if the importance for examining the qualification of a agent in a coalition is pondered. Denote the zero vector in $\mathbb{R}^A$ to be 0_A. The multi-choice coalition 0_A corresponds to the empty agent-coalition. Let $\zeta \in G^A$ and $K \subseteq A$. $|K|$ is the amount of agents in K, $N(\zeta) = \{i \in A | \zeta_i \neq 0\}$ and $\zeta_K \in \mathbb{R}^K$ is the restriction of ζ to K.

A **multi-choice transferable-utility (TU) game** is (A, g, u), where $A \neq \emptyset$ is finite set of agents and $u : G^A \to \mathbb{R}$ is a characteristic mapping which assigns to each $\zeta = (\zeta_i)_{i\in A} \in G^A$ the value that the agents can get when each agent i partakes at level ζ_i with $u(0_A) = 0$. The mapping u assigns to each multi-choice coalition $\zeta = (\zeta_i)_{i\in A} \in G^A$ a value, explaining what such a coalition can accomplish in cooperation.

Denote the class of all multi-choice TU games by Δ. Given $(A, g, u) \in \Delta$, let $P^A = \{(i, k_i) \mid i \in A, k_i \in G_i^+\}$. A **solution** on Δ is a mapping κ assorting to each $(A, g, u) \in \Delta$ a vector

$$\kappa(A, g, u) = \big(\kappa_{i,k_i}(A, g, u)\big)_{(i,k_i)\in P^A} \in \mathbb{R}^{P^A}.$$

Here $\kappa_{i,k_i}(A, g, u)$ is the value of the agent i if it participates in a coalition with membership k_i in u. For convenience, one could define $\kappa_{i,0}(A, g, u) = 0$ for each $i \in A$.

Next, a multi-choice generalization of the equal allocation of non-separable costs is provided as follows.

Definition 1: The **multi-choice equal allocation of non-separable costs (MCEANSC)**, $\overline{\theta}$, is the mapping on Δ which associates to each $(A, g, u) \in \Delta$, each agent $i \in A$ and each $k_i \in G_i$ the value

$$\begin{aligned} & \overline{\theta}_{i,k_i}(A, g, u) \\ = \quad & \theta_{i,k_i}(A, g, u) + \tfrac{1}{|A|} \cdot \Big[u(g) - \sum_{j\in A} \theta_{j,g_j}(A, g, u)\Big], \end{aligned}$$

where $\theta_{i,k_i}(A, g, u) = u\big(g_{A\setminus\{i\}}, k_i\big) - u\big(g_{A\setminus\{i\}}, 0\big)$ is the **level-marginal contribution** of the agent i and its activity level k_i. Under $\overline{\theta}$, all agents get its level-marginal contributions respectively, and further allot the rest of utility equally.

Subsequently, one would like to demonstrate that the MCEANSC could provide "*optimal or balanced allocating mechanisms*" among all agents, in the sense that this agency can obtain payoff from each combination of operational levels of all agents under multi-choice performances.

III. Reduction and axiomatic results

This section would show that there exists a specific reduction that could be applied to characterize the MCEANSC.

Let κ be a solution. κ matches **efficiency (EFF)** if for all $(A, g, u) \in \Delta$, $\sum_{i\in A} \kappa_{i,g_i}(A, g, u) = u(g)$. κ matches **standard for two-person games (STPG)** if for all $(A, g, u) \in \Delta$ with $|A| \leq 2$, $\kappa(A, g, u) = \overline{\theta}(A, g, u)$. κ matches **symmetry (SYM)** if for all $(A, g, u) \in \Delta$ with $u(\zeta, k_i, g_j) - u(\zeta, 0, g_j) = u(\zeta, g_i, k_j) - u(\zeta, g_i, 0)$ for some $(i, k_i), (j, k_j) \in P^A$ and for all $\zeta \in G^{A\setminus\{i,j\}}$, $\kappa_{i,k_i}(A, g, u) = \kappa_{j,k_j}(A, g, u)$. κ matches **zero-independence (ZI)** if for all $(A, g, u), (A, g, v) \in \Delta$ with $u(\zeta) = v(\zeta) + \sum_{i\in N(\zeta)} c_{i,\zeta_i}$ for some $c \in \mathbb{R}^{P^A}$ and for all $\zeta \in G^A$, $\kappa(A, g, u) = \kappa(A, g, v) + c$.

Axioms EFF is famous and diffusely acceptable. EFF claims that all agents allot the usability completely if all agents partake at full steam. Axiom STPG is a generalized analogue of Hart and Mas-Colell's [2] two-person standardness property, as generated for the Shapley value [13]. STPG claims that each agent obtains the payoff based on $\overline{\theta}$ in two-person situations. Axiom SYM claims that the payoffs of two agents should be the same if the marginal contributions among them are equal. Axiom ZI could be explained as a mighty weak analogue of additivity. By Definition 2.1, it is easy to examine that the MCEANSC matches EFF, STPG, SYM and ZI.

Given a subdivision of a group of agents, and an outcome vector assigned by a solution under some game. Moulin [10] defined the reduction as that in which each alliance in this subdivision could attain outcomes to its elements only if they are consonant with the initial outcomes to "all" the elements outside of this subdivision. In the following a multi-choice analogue of the Moulin's reduction is defined. Let $(A, g, u) \in \Delta$, $S \subseteq A$ and κ be a solution. The **reduction** (S, g_S, u^{κ}_S) is defined as that for all $\zeta \in G^S$,

$$\begin{aligned} & u^{\kappa}_S(\zeta) \\ = & \begin{cases} 0 & \text{, if } \zeta = 0_S, \\ u\big(\zeta, g_{A\setminus S}\big) - \sum\limits_{i\in A\setminus S} \kappa_{i,g_i}(A, g, u) & \text{, otherwise.} \end{cases} \end{aligned}$$

The *bilateral consistency* prerequisite could be depicted briefly as follows. For arbitrary group of two agents under a multi-choice game, one could introduces a "reduced mechanism" among them by considering the measures remaining after the rest of the agents are given the outcomes stipulated via a solution κ on Δ. κ is *bilateral consistent* if it emerges the same outcomes as under the initial situation when it is applied to arbitrary reduction. Officially, a solution κ satisfies **bilateral consistency (BCON)** if for every $(A, g, u) \in \Delta$ with $|A| \geq 3$, for every $S \subseteq A$ with $|S| = 2$ and for every $(i, k_i) \in P^S$, $\kappa_{i,k_i}(A, g, u) = \kappa_{i,k_i}(S, g_S, u^{\kappa}_S)$.

Lemma 1: The MCEANSC $\overline{\theta}$ matches BCON.

Proof: Let $(A, g, u) \in \Delta$ with $|A| \geq 3$ and $S \subseteq A$ with $|S| = 2$. Assume that $S = \{i, j\}$. By definition of $\overline{\theta}$, for every $(p, k_p) \in P^S$,

$$\begin{aligned} & \overline{\theta}_{p,k_p}(S, g_S, u^{\overline{\theta}}_S) \\ = & \theta_{p,k_p}(S, g_S, u^{\overline{\theta}}_S) + \tfrac{1}{|S|}\Big[u^{\overline{\theta}}_S(g_S) - \sum_{t\in S} \theta_{t,g_t}(S, g_S, u^{\overline{\theta}}_S)\Big]. \end{aligned} \quad (1)$$

By definitions of θ and $u^{\overline{\theta}}_S$, for every $k_i \in G_k$,

$$\begin{aligned} & \theta_{i,k_i}(S, g_S, u^{\overline{\theta}}_S) \\ = & u^{\overline{\theta}}_S(k_i, g_j) - u^{\overline{\theta}}_S(0, g_j) \\ = & u\big(g_{A\setminus\{i\}}, k_i\big) - u\big(g_{A\setminus\{i\}}, 0\big) \\ = & \theta_{i,k_i}(A, g, u). \end{aligned} \quad (2)$$

By definitions of $u^{\overline{\theta}}_S$ and $\overline{\theta}$ and equations (1), (2),

$$\begin{aligned} & \overline{\theta}_{i,k_i}(S, g_S, u^{\overline{\theta}}_S) \\ = & \theta_{i,k_i}(A, g, u) + \tfrac{1}{|S|} \cdot \Big[u^{\overline{\theta}}_S(g_S) - \sum_{t\in S} \theta_{t,g_t}(A, g, u)\Big] \\ = & \theta_{i,k_i}(A, g, u) + \tfrac{1}{|S|} \cdot \Big[u(g) - \sum_{t\in A\setminus S} \overline{\theta}_{t,g_t}(A, g, u) \\ & \qquad - \sum_{t\in S} \theta_{t,g_t}(A, g, u)\Big] \\ = & \theta_{i,k_i}(A, g, u) + \tfrac{1}{|S|} \cdot \Big[\sum_{t\in S} \overline{\theta}_{t,g_t}(A, g, u) \\ & \qquad - \sum_{t\in S} \theta_{t,g_t}(A, g, u)\Big] \\ & \big(\text{by EFF of } \overline{\theta}\big) \\ = & \theta_{i,k_i}(A, g, u) + \tfrac{1}{|S|} \cdot \Big[\tfrac{|S|}{|A|} \cdot \big[u(g) \\ & \qquad - \sum_{t\in A} \theta_{t,g_t}(A, g, u)\big]\Big] \\ & \big(\text{by Definition } 1\big) \\ = & \theta_{i,k_i}(A, g, u) + \tfrac{1}{|A|} \cdot \Big[u(g) - \sum_{t\in A} \theta_{t,g_t}(A, g, u)\Big] \\ = & \overline{\theta}_{i,k_i}(A, g, u). \end{aligned}$$

Hence, the MCEANSC matches BCON. ■

Next, the MCEANSC would be characterized by means of the properties of bilateral consistency and two-person standardness.

Theorem 1: Solution κ matches BCON and STPG if and only if $\kappa = \overline{\theta}$.

Proof: It's shown that $\overline{\theta}$ matches BCON by Lemma 1. Clearly, $\overline{\theta}$ matches STPG.

To present uniqueness, assume that κ matches BCON and STPG on Δ. By BCON and STPG of κ, it is easy to conclude that κ matches EFF. Let $(A, g, u) \in \Delta$. By STPG of κ, $\kappa(A, g, u) = \overline{\theta}(A, g, u)$ if $|A| = 2$. Similar to traditional games by putting a "dummy" agent to one-person situation, the proof is done if $|A| = 1$. The condition $|A| > 2$. Let $i \in A$ and $S = \{i, j\}$ for some $j \in A \setminus \{i\}$, then for all $k_i \in G_i, k_j \in G_j$,

$$\begin{aligned} & \kappa_{i,k_i}(A, g, u) - \kappa_{j,k_j}(A, g, u) \\ = & \kappa_{i,k_i}(S, g_S, u^{\kappa}_S) - \kappa_{j,k_j}(S, g_S, u^{\kappa}_S) \\ & \big(\text{by BCON of } \kappa\big) \\ = & \overline{\theta}_{i,k_i}(S, g_S, u^{\kappa}_S) - \overline{\theta}_{j,k_j}(S, g_S, u^{\kappa}_S) \\ & \big(\text{by STPG of } \kappa\big) \\ = & \theta_{i,k_i}(S, g_S, u^{\kappa}_S) - \theta_{j,k_j}(S, g_S, u^{\kappa}_S) \\ = & \Big[u^{\kappa}_S(k_i, g_j) - u^{\kappa}_S(0, g_j)\Big] \\ & \quad - \Big[u^{\kappa}_S(g_i, k_j) - u^{\kappa}_S(g_i, 0)\Big] \\ = & \Big[u(g_{A\setminus\{i\}}, k_i) - u(g_{A\setminus\{i\}}, 0)\Big] \\ & \quad - \Big[u(g_{A\setminus\{j\}}, k_j) - u(g_{A\setminus\{j\}}, 0)\Big]. \end{aligned} \quad (3)$$

Similarly, $\overline{\theta}$ instead of κ in equation (3), one could derive that

$$\begin{aligned} & \overline{\theta}_{i,k_i}(A, g, u) - \overline{\theta}_{j,k_j}(A, g, u) \\ = & \Big[u(g_{A\setminus\{i\}}, k_i) - u(g_{A\setminus\{i\}}, 0)\Big] \\ & \quad - \Big[u(g_{A\setminus\{j\}}, k_j) - u(g_{A\setminus\{j\}}, 0)\Big] \end{aligned} \quad (4)$$

By equations (3), (4),

$$\begin{aligned} & \kappa_{i,k_i}(A, g, u) - \kappa_{j,k_j}(A, g, u) \\ = & \overline{\theta}_{i,k_i}(A, g, u) - \overline{\theta}_{j,k_j}(A, g, u). \end{aligned}$$

This implies that $\kappa_{i,k_i}(A, g, u) - \overline{\theta}_{i,k_i}(A, g, u) = w$ for every (i, k_i). It remains to verify that $w = 0$. By EFF of κ and $\overline{\theta}$,

$$\begin{aligned} 0 & = \sum_{i\in A} \Big[\kappa_{i,g_i}(A, g, u) - \overline{\theta}_{i,g_i}(A, g, u)\Big] \\ & = |A| \cdot w. \end{aligned}$$

Thus, $w = 0$. ■

Next, the MCEANSC would be characterized by means of related properties of efficiency, bilateral consistency, symmetry and zero-independence.

Lemma 2: A solution κ matches STPG if it matches EFF, SYM and ZI.

Proof: Suppose that a solution κ matches EFF, SYM and ZI. Let $(A, g, u) \in \Delta$ with $A = \{i, j\}$ for $i \neq j$. Define $(A, g, v) \in \Delta$ to be as for all $\zeta \in G^A$,

$$v(\zeta) = u(\zeta) - \sum_{i\in N(\zeta)} \theta_{i,\zeta_i}(A, g, u).$$

By the definition of v, for all $k_i \in G_i$,

$$\begin{aligned} & v(k_i, g_j) - v(0, g_j) \\ = & u(k_i, g_j) - \theta_{i,k_i}(A, g, u) - \theta_{j,g_j}(A, g, u) \\ & \quad - u(0, g_j) + \theta_{j,g_j}(A, g, u) \\ = & u(k_i, g_j) - \theta_{i,k_i}(A, g, u) - u(0, g_j) \\ = & u(k_i, g_j) - u(0, g_j) - \theta_{i,k_i}(A, g, u) \\ = & \theta_{i,k_i}(A, g, u) - \theta_{i,k_i}(A, g, u) \\ = & 0. \end{aligned}$$

Similarly, $v(g_i, k_j) - v(g_i, 0) = 0$ for all $k_j \in G_j$. Since $v(g_i, g_j) - w(0, g_j) = w(g_i, g_j) - w(g_i, 0) = 0$, by SYM of

κ, $\kappa_{i,g_i}(A,g,v)=\kappa_{j,g_j}(A,g,v)$. By EFF of κ,

$$\begin{aligned} v(g) &= \kappa_{i,g_i}(A,g,v)+\kappa_{j,g_j}(A,g,v) \\ &= 2\cdot \kappa_{i,g_i}(A,g,v). \end{aligned} \quad (5)$$

By definition of v and equation (5),

$$\begin{aligned} & \kappa_{i,g_i}(A,g,v) \\ = & \frac{v(g)}{2} \\ = & \frac{1}{2}\cdot\big[u(g)-\theta_{i,g_i}(A,g,u)-\theta_{j,g_j}(A,g,u)\big]. \end{aligned}$$

By ZI of κ,

$$\begin{aligned} & \kappa_{i,k_i}(A,g,u) \\ = & \kappa_{i,k_i}(A,g,v)+\theta_{i,k_i}(A,g,u) \\ = & \frac{1}{2}\cdot\big[u(g)-\theta_{i,g_i}(A,g,u)-\theta_{j,g_j}(A,g,u)\big] \\ & +\theta_{i,k_i}(A,g,u) \\ = & \overline{\theta}_{i,k_i}(A,g,u). \end{aligned}$$

Similarly, $\kappa_{j,k_j}(A,g,u)=\overline{\theta}_{j,k_j}(A,g,u)$ for all $k_j\in G_j$. Hence, κ matches STPG. ■

Theorem 2: Solution κ matches EFF, BCON, SYM and ZI if and only if $\kappa=\overline{\theta}$.

Proof: $\overline{\theta}$ matches EFF, SYM and ZI by Definition 1. The rest of proofs follow from Lemmas 1, 2 and Theorem 1. ■

The following examples could examine that each of the axioms applied in Theorems 1 and 2 is logically independent of the rest of axioms.

Example 1: Define a solution κ^1 to be as for every $(A,g,u)\in\Delta$ and for every $(i,k_i)\in P^A$,

$$\kappa^1_{i,k_i}(A,g,u)=\begin{cases}\overline{\theta}_{i,k_i}(A,g,u) & \text{, if } |A|\le 2,\\ 0 & \text{, otherwise.}\end{cases}$$

Clearly, κ^1 matches STPG, but it violates BCON.

Example 2: Define a solution κ^2 to be as for every $(A,g,u)\in\Delta$ and for every $(i,k_i)\in P^A$, $\kappa^2_{i,k_i}(A,g,u)=\theta_{i,k_i}(A,g,u)$. Clearly, κ^2 matches BCON, SYM and ZI, but it violates STPG and EFF.

Example 3: Define a solution κ^3 to be as for every $(A,g,u)\in\Delta$ and for every $(i,k_i)\in P^A$, $\kappa^3_{i,k_i}(A,g,u)=\frac{u(g)}{|A|}$. Clearly, κ^3 matches EFF, BCON and SYM, but it violates ZI.

Example 4: Define a solution κ^4 to be as for every $(A,g,u)\in\Delta$ and for every $(i,k_i)\in P^A$,

$$\begin{aligned} & \kappa^4_{i,k_i}(A,g,u) \\ = & \big[u(g)-u(g_{A\setminus\{i\}},0)\big] \\ & +\frac{1}{|A|}\cdot\Big[u(g)-\sum_{k\in A}\big[u(g)-u(g_{A\setminus\{k\}},0)\big]\Big]. \end{aligned}$$

Clearly, κ^4 matches EFF, BCON and ZI, but it violates SYM.

Example 5: Define a solution κ^5 to be as for every $(A,g,u)\in\Delta$ and for every $(i,k_i)\in P^A$,

$$\begin{aligned} \kappa^5_{i,k_i}(A,g,u) &= \sum_{\substack{S\subseteq A\\ i\in S}}\frac{(|S|-1)!(|A|-|S|)!}{|A|!}\big[u\big((g_{A\setminus\{i\}},k_i)_S,0_{A\setminus S}\big) \\ &\quad -u\big((g_{A\setminus\{i\}},0)_S,0_{A\setminus S}\big)\big]. \end{aligned}$$

Clearly, κ^5 matches EFF, SYM and ZI, but it violates BCON.

IV. EXCESS MAPPING AND DYNAMIC RESULTS

Different from existing outcomes due to Maschler and Owen [9], this section introduces the excess mapping to present two dynamic results that leads the agents to the MCENASC, starting from an efficient outcome.

The set of efficient outcomes under a game (A,g,u) is defined to be $X(A,g,u)=\{\kappa(A,g,u)|\ \kappa \text{ is a efficient solution on } \Delta\}$. Let $(A,g,u)\in\Delta$ and $\kappa(A,g,u)\in X(A,g,u)$. The **excess** of a level vector $\zeta\in G^A$ at κ is $EC(\zeta,u,\kappa)=u(\zeta)-\sum_{i\in A}\kappa_{i,\zeta_i}(A,g,u)$. $EC(\zeta,u,\kappa)$ can be regarded to be the **discontent** of level vector ζ if all agents obtain its outcomes from κ in (A,g,u).

Let $(A,g,u)\in\Delta$ with $A\ge 3$, $\kappa(A,g,u)\in X(A,g,u)$ and $t>0$. Define the *correction mapping* $r: X(A,g,u)\to X(A,g,u)$ to be as for all $(i,k_i)\in P^A$,

$$\begin{aligned} & r_{i,k_i}(\kappa(A,g,u)) \\ = & \kappa_{i,k_i}(A,g,u)+t\cdot\sum_{j\in A\setminus\{i\}}\Big(EC\big((g_{A\setminus\{j\}},0),u,\kappa\big) \\ & \qquad -EC\big((g_{A\setminus\{i\}},0),u,\kappa\big)\Big), \end{aligned}$$

where t is a regular positive calculation, which makes known the assumption that agent i does not require for full modification (if $t=1$) but only (frequently) a part of it. The calculation t behaves how much the excess is revised. When agents partake in a game, some mutations or discontents may be arisen from different conditions. The correction mappings are based on the idea that, each agent cuts down the discontent relating to its own and others' non-participation, and applys these modifications to regulate the initial outcome.

The following result displays that the correction mapping is well-defined, i.e., $r(\kappa(A,g,u))\in X(A,g,u)$ if $\kappa(A,g,u)\in X(A,g,u)$. This result plays an important role to generate the necessary convergence result.

Lemma 3: Let $(A,g,u)\in\Delta$ with $|A|\ge 3$ and $\kappa(A,g,u)\in X(A,g,u)$. Then for all $i\in A$,

$$\begin{aligned} & \sum_{j\in A\setminus\{i\}}\Big(EC\big((g_{A\setminus\{j\}},0),u,\kappa\big)-EC\big((g_{A\setminus\{i\}},0),u,\kappa\big)\Big) \\ = & |A|\cdot\Big(\overline{\theta}_{i,g_i}(A,g,u)-\kappa_{i,g_i}(A,g,u)\Big) \\ \text{and} & \sum_{i\in A}\sum_{j\in A\setminus\{i\}}\Big(EC\big((g_{A\setminus\{j\}},0),u,\kappa\big)-EC\big((g_{A\setminus\{i\}},0),u,\kappa\big)\Big) \\ = & 0. \end{aligned}$$

Proof: Let $(A,g,u)\in\Delta$, $i,j\in A$ and $\kappa(A,g,u)\in X(A,g,u)$.

$$\begin{aligned} & \sum_{j\in A\setminus\{i\}}\Big(EC\big((g_{A\setminus\{j\}},0),u,\kappa\big)-EC\big((g_{A\setminus\{i\}},0),u,\kappa\big)\Big) \\ = & \sum_{j\in A\setminus\{i\}}\Big(u(g_{A\setminus\{j\}},0)-\sum_{k\in A\setminus\{j\}}\kappa_{k,g_k}(A,g,u) \\ & \quad -u(g_{A\setminus\{i\}},0)+\sum_{k\in A\setminus\{i\}}\kappa_{k,g_k}(A,g,u)\Big) \\ = & \sum_{j\in A\setminus\{i\}}\Big(u(g_{A\setminus\{j\}},0)-u(g_{A\setminus\{i\}},0) \\ & \quad -\kappa_{i,g_i}(A,g,u)+\kappa_{j,g_j}(A,g,u)\Big). \end{aligned} \quad (6)$$

By definition of $\overline{\theta}$,

$$\begin{aligned} & \overline{\theta}_{i,g_i}(A,g,u)-\overline{\theta}_{j,g_j}(A,g,u) \\ = & u\big(g_{A\setminus\{j\}},0\big)-u\big(g_{A\setminus\{i\}},0\big). \end{aligned} \quad (7)$$

By equations (6) and (7),

$$\begin{aligned}
&\sum_{j\in A\setminus\{i\}} \Big(EC\big((g_{A\setminus\{j\}},0),u,x\big) \\
&\qquad - EC\big((g_{A\setminus\{i\}},0),u,x\big)\Big) \\
= &\sum_{j\in A\setminus\{i\}} \Big(\overline{\theta}_{i,g_i}(A,g,u) - \overline{\theta}_{j,g_j}(A,g,u) \\
&\qquad - \kappa_{i,g_i}(A,g,u) + \kappa_{j,g_j}(A,g,u)\Big) \\
= &(|A|-1)\Big[\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big] \\
&- \sum_{j\in A\setminus\{i\}} \overline{\theta}_{j,g_j}(A,g,u) + \sum_{j\in A\setminus\{i\}} \kappa_{j,g_j}(A,g,u) \\
= &|A|\Big[\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big] \\
&-u(g)+u(g) \\
&\big(\text{by EFF of } \overline{\theta} \text{ and } \kappa\big) \\
= &|A|\Big[\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big].
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sum_{i\in A}\sum_{j\in A\setminus\{i\}} \Big(EC\big((g_{A\setminus\{j\}},0),u,\kappa\big) \\
&\qquad - EC\big((g_{A\setminus\{i\}},0),u,\kappa\big)\Big) \\
= &\sum_{i\in A} |A|\cdot\Big[\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big] \\
= &|A|\cdot\Big[u(g)-u(g)\Big] \\
&\big(\text{by EFF of } \overline{\theta} \text{ and } \kappa\big) \\
= &0.
\end{aligned}$$

■

Let $(A,g,u)\in\Delta$ and $\kappa(A,g,u)\in X(A,g,u)$. We define the dynamic sequences $\{\kappa^q(A,g,u)\}_{q=1}^{\infty}$ for every $q\in\mathbb{N}$ as follows.

$$\kappa^0(A,g,u)=\kappa(A,g,u),\cdots,\kappa^q(A,g,u)=r(\kappa^{q-1}(A,g,u)).$$

Theorem 3: Let $(A,g,u)\in\Delta$. If $0<t<\frac{2}{|A|}$, then $\{\kappa^q_{i,g_i}(A,g,u)\}_{q=1}^{\infty}$ converges to $\overline{\theta}_{i,g_i}(A,g,u)$ for each $\kappa(A,g,u)\in X(A,g,u)$ and for all $i\in A$.

Proof: Let $(A,g,u)\in\Delta$, $i\in A$ and $\kappa(A,g,u)\in X(A,g,u)$. By Lemma 3 and the definition of h,

$$\begin{aligned}
&r_{i,g_i}(\kappa(A,g,u)) - \kappa_{i,g_i}(A,g,u) \\
= &t\cdot\sum_{j\in A\setminus\{i\}}\Big(EC\big((g_{A\setminus\{j\}},0),u,\kappa\big) - EC\big((g_{A\setminus\{i\}},0),u,\kappa\big)\Big) \\
= &t\cdot|A|\cdot\Big[\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\overline{\theta}_{i,g_i}(A,g,u) - r_{i,g_i}(\kappa(A,g,u)) \\
= &\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u) + \kappa_{i,g_i}(A,g,u) \\
&-r_{i,g_i}(\kappa(A,g,u)) \\
= &\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u) \\
&\quad - t\cdot|A|\cdot\Big(\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big) \\
= &\Big(1-t\cdot|A|\Big)\cdot\Big[\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big].
\end{aligned}$$

For all $q\in\mathbb{N}$,

$$\begin{aligned}
&\overline{\theta}_{i,g_i}(A,g,u) - \kappa^q_{i,g_i}(A,g,u) \\
= &\Big(1-t\cdot|A|\Big)^q\cdot\Big[\overline{\theta}_{i,g_i}(A,g,u) - \kappa_{i,g_i}(A,g,u)\Big].
\end{aligned}$$

If $0<t<\frac{2}{|A|}$, then $-1<\Big(1-t\cdot|A|\Big)<1$ and $\{\kappa^q_{i,g_i}(A,g,u)\}_{q=1}^{\infty}$ converges to $\overline{\theta}_{i,g_i}(A,g,u)$. ■

Efficiency of a solution is essential in the techniques of dynamic analysis. In Theorem 3, some values among the MCEANSC of a game could not be reached by applying efficiency and related dynamic analysis. Similar to Liao [4], a different definition of efficiency is offered on as follows. Let $(A,g,u)\in\Delta$. A solution κ on Δ matches **plurality-efficiency (PEFF)** in (A,g,u) if for all $(i,k_i)\in P^A$,

$$\kappa_{i,k_i}(A,g,u)+\sum_{j\in A\setminus\{i\}}\kappa_{j,g_j}(A,g,u)=u\big(g_{A\setminus\{i\}},k_i\big).$$

It also matches EFF in (A,g,u) if there exists (A,g,u) such that a solution matches PEFF in (A,g,u). An interpretation of plurality-efficiency due to Liao [4] is stated as follows. Under a traditional game (A,u^{TU}), the foremost supposition is that the grand alliance A forms, and then that $u^{TU}(A)$ is the utility that has to be allocated. Hence, a solution is a mapping κ^{TU} appointing to (A,u^{TU}) an outcome $\kappa(A,u^{TU})=(\kappa_i(A,u^{TU}))_{i\in A}\in\mathbb{R}^A$ where $\kappa_i(A,u^{TU})$ is the value assigned to agent i, and *TU-efficiency* claims that $\sum_{i\in A}\kappa_i(A,u^{TU})=u^{TU}(A)$, all the incomes (maybe losses) are to be allocated among the agents. In a multi-choice game (A,g,u), the foremost supposition is still that the grand coalition A takes shape, and then there exist various cooperative aspects of A. This implies that for each $\zeta\in G^A$ with $\zeta_i\neq 0$ for every $i\in A$, it is probable that $u(\zeta)$ is the utility that has to be allocated. In order to attain the maximal and beneficial result of "identity", each individual agent expects that all other agents are assumed to partake at its maximum level of energy if it partakes at level ζ_i, which is also most significant condition.

Theorem 4: Let $(A,g,u)\in\Delta$ such that the solution $\overline{\theta}$ matches PEFF in (A,g,u). If $0<t<\frac{2}{|A|}$, then for each solution κ which matches PEFF in (A,g,u), $\{\kappa^q(A,g,u)\}_{q=1}^{\infty}$ converges to $\overline{\theta}(A,g,u)$.

Proof: The proof of this theorem is direct analogue of the proof of Theorem 3, therefore it would be omitted. ■

V. Discussion

The meritss of the method in this article are that the MCEANSC of a multi-choice game continuously exists and to determine an outcome for a given agent partaking at a given level that different from the general type with multi-choice games, which determining a type of entire outcome for a given agent by gathering the contribution of this agent among total levels. In order to explain how the MCEANSC can be applied and to cause its implication clear, an application due to Liao [8] is quoted as follows. **(Liao [8])** *Let $(A,g,u)\in\Delta$ and A be a set of investors. Suppose that the capital of each $i\in A$ is c_i. In this model the capital of a agent can be non-positive; in fact, some agents may be in need of capital (in this case an investment of a negative capital is a financing process). For all $\zeta\in G^A$, ζ could be treated as a multi-choice coalition. A multi-choice coalition ζ is seen as an organization meant to achieve some goals, which are common to its members. The endowment of a multi-choice coalition ζ with the capital it needs for its activities is done by the members and the degree of membership of agent $i\in A$ to multi-choice coalition ζ is measured by the level of capital c_i agent i invests in the multi-choice coalition ζ. Observe that this way of measuring the degree of membership is different from the more usual one in which the degree of membership is measured by the share of coalitional capital a agent owns. It better reflects* *the risks agents are ready to take over when investing in a specific organization and also their personal interest in realizing the goals the organization is meant to achieve: if a agent with a capital of $100 and another agent with a capital of $10000 invest the same amount of $100 in organization ζ, it means that the first agent is much more interested in ζ and, consequently, more personally involved and assuming a higher risk than the second agent for the realization of the goals of ζ. In that follows we interpret the membership degree of a agent to a multi-choice coalition as a measure of the risk the agent assumes by transferring a part of its capital to the coalition considered as a collective decision maker.*

Another application for resource-distributing under a sports organization is also stated as follows. Let $A = \{1, 2, \cdots, a\}$ be a collection of all elements of the operational committee of a sports organization. In the operational committee, all elements are picked by recommendation or voting from sections of the sports organization. All elements have the authority to raise, consult, originate, and veto all projects for resource allocating. All elements dedicate distinct levels of observation and involvement to different projects depending on its professional expertise and the common observation they represent. The level of affection is also closely connected with the coalitional decision constituted for the interests of different divisions. For the foregoing reasons, decisions applied by each element of the operational committee present distinct levels of involvement and particular metes of multiplicity. The mapping u could be regarded as an affect mapping which appoints to each level vector $\zeta = (\zeta_i)_{i\in A} \in G^A$ the affect that the elements can contribute if each element i partakes at operational decision $\zeta_i \in G_i$. Modeled under this notion, the decision-making processes of the operational committee of a sports organization (A, g, u) could be regarded as a multi-choice TU game, with u being a characteristic mapping and G_i being the collection of all operational decisions of the element i. To evaluate the affect of each element in the operational committee, applying the power indexes this article proposed, one could first assess the affect each element under each level over previous resource-distributing project meetings based on various performances, which is the the level-marginal contribution defined in Definition 1. The rest of shared affect should also be equally distributed, which is the MCEANSC defined in Definition 1.

Subsequently, one would explore the realistic implications of the properties presented in Section 3. In this way, we can also explore whether the MCEANSC can be regarded as an appropriate allocation and processing principle in real-world situations.

- **Efficiency (EFF)** represents the situation where resources are completely allocated, which in real-world situations usually means ”resources must be used completely and properly”.
- **Symmetry (SYM)** represents the situation where, if two people make the same amount of marginal contribution, they should eventually receive the same pay, which in real-world situations usually means ”equal pay for equal work”.
- **Zero-independence (ZI)** represents the situation where any conditions that occur during the game must be reflected in the final allocation, which in real-world conditions usually implies that production and allocation must be synchronized and proportional.
- **Standard for two-person games (STPG)** represent a self-sufficient situation if there is only one agent in the game, but if there are two agents in the game, each of them first receive what they could have occurred alone, and they partake all the rest of losses and profits at the tail of the game. In reality, many concepts of allocation and processing usually depend on the characteristics of individual behavior and the states of interaction between two people. In real-world situations, STPG usually represents ”self-sufficiency in the case of one person, and helping both yourself and each other in the case of two people”. As stated in the nature of STPG, the state of interaction between two people has a decisive influence on the overall situation of allocation. No allocation concept will match everyone.
- What **bilateral consistency (BCON)** suggests in real-world situations is that if any two people are dissatisfied, they are allowed to restart the process and perform another allocation with the best conditions possible, and if the result of the allocation turns out to be the same as the original result, then the allocation concept has a stable and consistent criterion.

With the above statement, together with the relevant results presented for the MCEANSC in Section 3, we can clearly summarize what can be considered an appropriate allocation and processing principle for the MCEANSC in real-world situations.

VI. Concluding Remarks

1) Cheng et al. [1], Hwang and Liao [3], Liao [5], [6], and Liao et al. [8] proposed several extensions of the Banzhaf-Owen index, the EANSC, the PEANSC and related results by respectively considering different notions on multi-choice games. By both considering the agents and the activity levels, this article proposed an extension of the EANSC and related results on multi-choice games. One might compare these published results with the results of this article. Several major differences are as follows:
   - Cheng et al. [1] defined the *multi-choice normalized Banzhaf-Owen index* to determine a type of entire outcome for a given agent by applying the *maximal-utilities* related to the sizes of coalitions. Differing from the results due to Cheng et al. [1], this article proposed the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Cheng et al. [1]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].
   - Hwang and Liao [3] defined three extensions of the EANSC to determine an outcome of a given level for a given agent by allocating the rest of utility among all levels. Differing from the results due to Hwang and Liao [3], this article proposed

the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Hwang and Liao [3]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].
- Liao [5] defined the *maximal EANSC* to determine a type of entire outcome for a given agent by applying the *maximal marginal contributions* of agents among all levels. Differing from the results due to Liao [5], this article proposed the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Liao [5]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].
- Liao [6] proposed the *duplicate EANSC* to determine a type of entire outcome for a given agent by applying the *replicated behavior* of agents among all levels. Differing from the results due to Liao [6], this article proposed the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Liao [6]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].
- Liao et al. [8] defined the *multi-choice pseudo equal allocation of non-separable costs* to determine a value of a given level for a given agent by extending the PEANSC. Differing from the results due to Liao [8], this article proposed the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Liao [8]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].
- Inspired by Maschler and Owen [9], Liao [4] adopted the plurality-efficiency to offer the dynamic result of a solution. Inspired by Liao [4], this article proposed two dynamic results of the MCEANSC. In view of the correcting mappings due to Maschler and Owen [9] and Liao [4], the "reduction" is a key factor. However, the correcting mapping of this paper is generated from "excess mapping".

2) This paper offer several axiomatic and two dynamic results of the MCEANSC respectively. Due to bilateral consistency property, this article present two axiomatic results which are analogues of the results of Hart and Mas-Colell [2] and Moulin [10]. Due to efficiency property, some outcomes of the MCEANSC of a game could not be generated from dynamic results. One would try to investigate axiomatic results by discarding consistency and investigate dynamic results by discarding efficiency.
3) This article has combined proofs with mathematical theories, statements with practical examples, and cross arguments between them to derive an allocation and processing principle that can be applied in real-world situations. Some might wonder whether the concept of allocation in other game theories can also be applied in real-world situations. This article leaves it to the researches to explore this in future researches.

REFERENCES

[1] C.Y. Cheng, E.C. Chi, K. Chen and Y.H. Liao, "A Power Mensuration and its Normalization under Multicriteria Situations," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp262-267, 2020

[2] S. Hart and A. Mas-Colell, "Potential, Value and Consistency," *Econometrica*, vol. 57, pp589-614, 1989

[3] Y.A. Hwang and Y.H. Liao, "Note: Natural Extensions of the Equal Allocation of Non-Separable Costs," *Engineering Letters*, vol. 28, no. 4, pp1325-1330, 2020

[4] Y.H. Liao, "A Dynamic Approach to a Consistent Value under Plurality-efficiency," *Economics Bulletin*, vol. 3, no. 40, pp1-8, 2007

[5] Y.H. Liao, "The Maximal Equal Allocation of Non-separable Costs on Multi-choice Games," *Economics Bulletin*, vol. 3, no. 70, pp1-8, 2008

[6] Y.H. Liao, "The Duplicate Extension for the Equal Allocation of Non-separable Costs," *Operational Research: An International Journal*, vol. 13, pp385-397, 2012

[7] Y.H. Liao, "The Equal Allocation of Nonseparable Costs: An Extension under Fuzzy Behavior," *Fuzzy Sets and Systems*, vol. 231, pp84-94, 2013

[8] Y.H. Liao, P.T. Liu, R.R. Huang, K.C. Lee and E.C. Chi, "Resource Allocation under Sports Management Systems: Game-theoretical Methods," *Engineering Letters*, vol. 28, no. 4, pp1027-1212, 2020

[9] M. Maschler and G. Owen, "The Consistent Shapley Value for Hyperplane Games," *International Journal of Game Theory*, vol. 18, pp389-407, 1989

[10] H. Moulin, "The Separability Axiom and Equal-sharing Methods," *Journal of Economic Theory*, vol. 36, pp120-148, 1985

[11] A van den. Nouweland, J. Potters, S. Tijs and J.M. Zarzuelo, "Core and Related Solution Concepts for Multi-choice Games," *ZOR-Mathematical Methods of Operations Research*, vol. 41, pp289-311, 1995

[12] J.S. Ransmeier, "The Tennessee Valley Authority," Nashville, Vanderbilt University Press, 1942

[13] L.S Shapley, "A Value for A-person Game," in: Kuhn, H.W., Tucker, A.W.(Eds.), *Contributions to the Theory of Games* **II**, Princeton, 1953, pp307-317

[14] R.E. Stearns, "Convergent Transfer Schemes for A-person Games," *Transactions of the American Mathematical Society*, vol. 134, pp449-459, 1968