Inferences on Stress-Strength Reliability from Inverse Weibull Distribution based on First-Failure Progressively Unified Hybrid Censoring Schemes

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Abstract—**In this paper, we discuss the estimation of the stress-strength parameter $\delta = P(X < Y)$ based on first-failure progressively unified hybrid censored samples, where X and Y are two independent inverse Weibull distributions with the same shape parameters but having different scale parameters. The maximum likelihood estimate of δ is obtained. The asymptotic distribution of δ is used to construct the asymptotic MLE confidence interval. Two bootstrap confidence intervals are constructed. We obtain the Bayes estimate and corresponding HPD credible interval of δ is also constructed using the Metropolis-Hasting algorithm. The performance of the proposed estimate methods are compared by Monte Carlo simulations. Finally, analysis of two data sets have been presented for illustrative purposes.**

Index Terms—**Stress-strength model, inverse Weibull distribution, maximum likelihood estimator; Bayes estimator, Metropolis-Hasting method, first-failure progressively unified hybrid censoring schemes.**

I. Introduction

THIS problem of estimation the stress-strength model $\delta = P(Y < X)$ arises in mechanical reliability systems. Suppose that X represents the strength of a component with a stress Y, then δ can be considered as a measure of system performance. The system becomes out of control if the stress is greater than its strength. The model is used in many fields such as medicine, physics, mechanics, finance and so on. Estimation of δ has received considerable attention in the statistical literature. The stress-strength model has been extensively studied in the past few decades. Downton [6] derived the minimum variance unbiased estimator of the $P(Y < X)$ under the assumption that X and Y are independent normal distributions. Kundu [7], [8] obtained estimation of $P(Y < X)$ for generalized exponential distributions and Weibull distributions. Raqab [9] considered the estimation δ when X and Y are independent three-parameter generalized exponential distributions. Kunda and Raqab [10] considered the estimation of δ when X and Y are independent three-parameter Weibull distributions. Ali et al. [11] studied the estimation of the δ, when X and Y belonged to different distributions, with X following a Levy distribution, while Y is uniform distribution or exponential distribution. Hajebi et al. [12] constructed confidence intervals for δ based on the asymptotic maximum likelihood method and bootstrap method, when X and Y are two independent generalized exponential random variables. AL-Mutairi et al. [13] studied the estimation of the stress-strength reliability from Lindley distributions. Soliman et al. [14] considered the estimation of the δ for modified Weibull distributions. Hussian [15] considered the estimation of the δ for generalized inverted exponential distributions. Pak et al. [16] studied the estimation of the δ for bivariate Rayleigh distribution. Kundu and Raqab [17] considered the estimation of the δ for three-parameter generalized Rayleigh distributions. Al-Mutairi [18] studied the estimation of the δ, when X and Y are two independent weighted Lindley random variables. Rao et al. [19] studied the estimation of the δ from exponentiated Fréchet distributions.

Manuscript received October 9, 2020; revised April 1, 2021. This work was supported by the National Natural Science Foundation of China (No. 11861049, 81860605), Natural Science Foundation of Inner Mongolia (2020LH01002, 2018LH01012, 2018MS01027, 2020MS01001), Foundation of Inner Mongolia University of Technology (Grant No. BS2020029).

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The above studies on the δ are based on complete samples. Censoring schemes are common in life tests because of time limits and other restrictions on data collection. Therefore, many researchers have studied the estimation of reliability under censored samples. For example, Saracoğlu et al. [20] studied the estimation of the δ, when X and Y are independent exponential random variables under progressive type-II censoring samples. Lio and Tsai [21] studied the estimation of the δ for two-parameter Burr-XII distributions under the progressively first failure censored samples. Lin and Ke [22] studied the estimation of the δ for location-scale distributions under joint progressively type-II right censoring. Valiollahi et al. [23] considered the estimation of the δ for Weibull distributions under progressive type-II censored samples. Shoaee and Khorram [24] considered stress-strength reliability of two-parameter bathtub-shaped lifetime distributions under progressively censored samples. Shoaee and Khorram [25] studied the estimation of the δ for Weibull distributions under type-II progressively hybrid censored data. Chiang et al. [26] studied the estimation of the δ for Burr-XII distributions with record samples. Krishna et al. [27] considered the estimation of the δ for generalized inverted exponential distributions based on progressively first-failure-censored samples. Babayi and Khorram [28] studied the estimation of the δ for the type-II generalized logistic distributions under progressively type-II censored samples. Guo and Gui [29] discussed the estimation of the maximum likelihood and Bayes estimators of δ when X and Y are independent generalized exponential random variables based on progressive type-II censoring samples. Kohansal and Rezakhah [30] derived the UMVUE and maximum likelihood estimator of δ for two-parameter Rayleigh distributions based on progressively censored samples. Peng et al. [31] considered the maximum likelihood and Bayes estimators of δ based on progressively type-II censored samples, when X and Y are independent Birnbaum-Saunders distributions.

Jia et al. [32] introduced the first-failure progressively unified hybrid censoring schemes (FFPUHCS). The FFPUHCS can be described as follows: assume that $k \times n$ items are put on test in n independent groups with k items in each group. The integers $r < m <= n$ is fixed at the beginning of the experiment and the progressive censoring scheme $R = (R_1, R_2, \cdots, R_m)$ is prefixed. The time points T_1 and T_2 with $T_1 < T_2$ are also fixed constant before the experiment. Let D_1 and D_2 denote the number of observed failures up to time T_1 and T_2, respectively. Also, let d_1 and d_2 are the observed value of D_1 and D_2, respectively. When the first-failure of unit (say $X^R_{1:m:n:k}$) occurs, we remove that group in which first failure occurrs and R_1 additional groups randomly from the remaining $n-1$ groups in the experiment. As soon as second failure (say $X^R_{2:m:n:k}$) takes place we remove that group and additional R_2 groups randomly from remaining $n - R_1 - 2$ groups and so on. If $X^R_{m:m:n:k} < T_1$, then instead of terminating the test by withdrawing the remaining R_m groups after the mth failure. We continue to observe failures (without removing the other groups) up to T_1. Therefore, $R_m = R_{m+1} = \cdots = R_{d_1} = 0$. If $T_1 < X^R_{m:m:n:k} < T_2$, the test is terminated at $X^R_{m:m:n:k}$. If $X^R_{r:m:n:k} < T_2 < X^R_{m:m:n:k}$, the test is terminated at T_2. If $T_2 < X^R_{r:m:n:k}$, the test is terminated at $X^R_{r:m:n:k}$ and $R_r = n - r - R_1 - \cdots - R_{r-1}$. For the FFPUHCS, we have one of the following types of observations:

Case I: $X^R_{1:m:n:k}, X^R_{2:m:n:k}, \cdots, X^R_{m:m:n:k}, X^R_{m+1:n:k}, \cdots,$
$X^R_{d_1:n:k},\ X^R_{m:m:n:k} < T_1, d_1 \le m + R_m,$

Case II: $X^R_{1:m:n:k}, X^R_{2:m:n:k}, \cdots, X^R_{m:m:n:k},$
$T_1 < X^R_{m:m:n:k} < T_2,$

Case III: $X^R_{1:m:n:k}, X^R_{2:m:n:k}, \cdots, X^R_{d_2:m:n:k},$
$X^R_{r:m:n:k} < T_2 < X^R_{m:m:n:k}, 0 < d_2 < m.$

Case IV: $X^R_{1:m:n:k}, X^R_{2:m:n:k}, \cdots, X^R_{r:m:n:k},$
$X^R_{r:m:n:k} > T_2, 0 < r < m.$

Then $X^R_{1:m:n:k}, X^R_{2:m:n:k}, \cdots, X^R_{D:m:n:k}$ is a first-failure progressive unified hybrid censored order statistics from a population with probability density function (PDF) $f(x)$ and cumulative distribution function (CDF) $F(x)$, respectively. From [32] shows that FFPUHCS is a generalization of progressive censored scheme. Therefore, the likelihood function based on FFPUHCS (see [32]) is given by

$$L(\Theta|x) = C_D k^D \prod_{i=1}^{D} f(x_i)\bar{F}(x_i)^{k(R_i+1)-1} \bar{F}(T_1)^{kR_{d_1'}} \bar{F}(T_2)^{kR_{d_2'}} \quad (1)$$

Here, $D = d_1, R_{d_1'} = R_{d_1}, R_{d_2'} = 0$, for case I, $D = m, R_{d_1'} = 0, R_{d_2'} = 0$, for case II, $D = d_2, R_{d_1'} = 0, R_{d_2'} = R_{d_2}$, for case III and $D = r, R_{d_1'} = 0, R_{d_2'} = 0$, for case IV.

The inverse Weibull (IW) distribution has a long right tail compared to other known distributions. The hazard function of the IW distribution has upside down bathtub shape. Keller and Kamath [33] show IW distribution has been derived as a good fit to describe the degradation phenomena of mechanical components of diesel engines such as pistons, crankshafts and main bearings. The PDF and the CDF of IW distribution are given by

$$f(x;\alpha,\lambda) = \alpha\lambda x^{-\alpha-1} e^{-\lambda x^{-\alpha}}, x > 0, \alpha, \lambda > 0, \quad (2)$$

$$F(x;\alpha,\lambda) = e^{-\lambda x^{-\alpha}}, x > 0, \alpha, \lambda > 0. \quad (3)$$

where λ is a scale parameter and α is a shape parameter of the model. In this paper, we discuss the estimation of δ, when $X \sim IW(\alpha, \lambda_1)$ and $Y \sim IW(\alpha, \lambda_2)$ are independent random variables based FFPUHC data.

The rest of this paper is organized as follows. We obtain the MLE of δ in Section II. The asymptotic, Boot-p, Boot-t confidence intervals are constructed in Section III. In Section IV, the Bayes estimate and the corresponding HPD credible interval are obtained. In Section V, we compare different proposed methods using numerical simulation, and the results are presented. Analysis of two data sets are presented for illustrative purposes in Section VI. Finally, we conclude the paper in Section VII.

II. Maximum likelihood estimation of δ

Let $X \sim IW(\alpha, \lambda_1)$ and $Y \sim IW(\alpha, \lambda_2)$, then we obtain

$$\delta = P(Y < X) = \int_0^\infty \int_0^x f_X(x|\alpha,\lambda_1) f_Y(y|\alpha,\lambda_2) \mathrm{d}y\mathrm{d}x = \frac{\lambda_1}{\lambda_1+\lambda_2}. \quad (4)$$

Suppose that $X = (X^{R_x}_{1:m_1:n_1:k_1}, X^{R_x}_{2:m_1:n_1:k_1}, \cdots, X^{R_x}_{D_x:m_1:n_1:k_1})$ be a first-failure progressive unified hybrid censored sample from $IW(\alpha, \lambda_1)$ distribution, with a censoring scheme R_x, and $Y = (Y^{R_y}_{1:m_2:n_2:k_2}, Y^{R_y}_{2:m_2:n_2:k_2}, \cdots, Y^{R_y}_{D_y:m_2:n_2:k_2})$ be a first-failure progressive unified hybrid censored sample from $IW(\alpha, \lambda_2)$ distribution, with a censoring scheme R_y. Where D_x and D_y denote the number of the observed failures of X and Y, respectively, until the end of the experiment. For notation simplicity, we will write $(X_1, X_2, \cdots, X_{D_x})$ for $(X^{R_x}_{1:m_1:n_1:k_1}, X^{R_x}_{2:m_1:n_1:k_1}, \cdots, X^{R_x}_{D_x:m_1:n_1:k_1})$ and $(Y_1, Y_2, \cdots, Y_{D_y})$ for $(Y^{R_y}_{1:m_2:n_2:k_2}, Y^{R_y}_{2:m_2:n_2:k_2}, \cdots, Y^{R_y}_{D_y:m_2:n_2:k_2})$.

$$\begin{aligned} L(\alpha,\lambda_1,\lambda_2) = {} & [C_{D_x} k_1^{D_x} \prod_{i=1}^{D_x} \alpha\lambda_1 x_i^{-(\alpha+1)} e^{-\lambda_1 x_i^{-\alpha}} \\ & (1-e^{-\lambda_1 x_i^{-\alpha}})^{k_1(R_{xi}+1)-1} \\ & (1-e^{-\lambda_1 T_{x1}^{-\alpha}})^{k_1 R_{d'_{x1}}} (1-e^{-\lambda_1 T_{x2}^{-\alpha}})^{k_1 R_{d'_{x2}}}] \\ & \times [C_{D_y} k_2^{D_y} \prod_{j=1}^{D_y} \alpha\lambda_2 y_j^{-(\alpha+1)} e^{-\lambda_2 y_j^{-\alpha}} \\ & (1-e^{-\lambda_2 y_j^{-\alpha}})^{k_2(R_{yj}+1)-1} (1-e^{-\lambda_2 T_{y1}^{-\alpha}})^{k_2 R_{d'_{y1}}} \\ & (1-e^{-\lambda_2 T_{y2}^{-\alpha}})^{k_2 R_{d'_{y2}}}]. \end{aligned} \quad (5)$$

$C_D = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\cdots(n - R_1 - \cdots - R_{D-1} - D + 1)$, $R_{d_1} = n - d_1 - \sum_{j=1}^{d_1-1} R_j, R_{d_2} = n - d_2 - \prod_{j=1}^{d_2} R_j$, $\bar{F}(\cdot) = 1 - F(\cdot)$ denotes the survival function. Therefore, from Equation (5) the log-likelihood function of

α, λ_1 and λ_2 is given by

$$
\begin{aligned}
l(\alpha,\lambda_1,\lambda_2) &\propto (D_x+D_y)\log(\alpha)+D_x\log(\lambda_1)+D_y\log(\lambda_2)\\
&-(\alpha+1)(\sum_{i=1}^{D_x}\log(x_i)+\sum_{j=1}^{D_y}\log(y_j))-\lambda_1\sum_{i=1}^{D_x}x_i^{-\alpha}\\
&-\lambda_2\sum_{j=1}^{D_y}y_j^{-\alpha}+\sum_{i=1}^{D_x}[k_1(R_{xi}+1)-1]\log(1-e^{-\lambda_1 x_i^{-\alpha}})\\
&+k_1R_{d'_{x1}}\log(1-e^{-\lambda_1 T_{x1}^{-\alpha}})+k_1R_{d'_{x2}}\log(1-e^{-\lambda_1 T_{x2}^{-\alpha}})\\
&+\sum_{j=1}^{D_y}[k_2(R_{yj}+1)-1]\log(1-e^{-\lambda_2 y_j^{-\alpha}})\\
&+k_2R_{d'_{y1}}\log(1-e^{-\lambda_2 T_{y1}^{-\alpha}})+k_2R_{d'_{y2}}\log(1-e^{-\lambda_2 T_{y2}^{-\alpha}}).
\end{aligned}
\tag{6}
$$

The MLEs $\hat{\alpha}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ of the parameters α, λ_1 and λ_2, respectively, are the solutions of the following likelihood equations:

$$
\begin{aligned}
\frac{\partial l}{\partial\alpha} &= \frac{D_x+D_y}{\alpha}-(\sum_{i=1}^{D_x}\log(x_i)+\sum_{j=1}^{D_y}\log(y_j))\\
&+\lambda_1\sum_{i=1}^{D_x}x_i^{-\alpha}\log(x_i)+\lambda_2\sum_{j=1}^{D_y}y_j^{-\alpha}\log(y_j)\\
&-k_1\lambda_1R_{d'_{x1}}\frac{T_{x1}^{-\alpha}e^{-\lambda_1T_{x1}^{-\alpha}}\log(T_{x1})}{1-e^{-\lambda_1T_{x1}^{-\alpha}}}\\
&-\lambda_1\sum_{i=1}^{D_x}[k_1(R_{xi}+1)-1]\frac{x_i^{-\alpha}e^{-\lambda_1x_i^{-\alpha}}\log(x_i)}{1-e^{-\lambda_1x_i^{-\alpha}}}\\
&-k_1\lambda_1R_{d'_{x2}}\frac{T_{x2}^{-\alpha}e^{-\lambda_1T_{x2}^{-\alpha}}\log(T_{x2})}{1-e^{-\lambda_1T_{x2}^{-\alpha}}}\\
&-k_2\lambda_2R_{d'_{y1}}\frac{T_{y1}^{-\alpha}e^{-\lambda_2T_{y1}^{-\alpha}}\log(T_{y1})}{1-e^{-\lambda_2T_{y1}^{-\alpha}}}\\
&-\lambda_2\sum_{j=1}^{D_y}[k_2(R_{yj}+1)-1]\frac{y_j^{-\alpha}e^{-\lambda_2y_j^{-\alpha}}\log(y_j)}{1-e^{-\lambda_2y_j^{-\alpha}}}\\
&-k_2\lambda_2R_{d'_{y2}}\frac{T_{y2}^{-\alpha}e^{-\lambda_yT_{y2}^{-\alpha}}\log(T_{y2})}{1-e^{-\lambda_yT_{y2}^{-\alpha}}},
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial\lambda_1} &= \frac{D_x}{\lambda_1}-\sum_{i=1}^{D_x}x_i^{-\alpha}+\sum_{i=1}^{D_x}[k_1(R_{xi}+1)-1]\frac{x_i^{-\alpha}e^{-\lambda_1x_i^{-\alpha}}}{1-e^{-\lambda_1x_i^{-\alpha}}}\\
&+k_1R_{d'_{x1}}\frac{T_{x1}^{-\alpha}e^{-\lambda_1T_{x1}^{-\alpha}}}{1-e^{-\lambda_1T_{x1}^{-\alpha}}}+k_1\lambda_1R_{d'_{x2}}\frac{T_{x2}^{-\alpha}e^{-\lambda_1T_{x2}^{-\alpha}}}{1-e^{-\lambda_1T_{x2}^{-\alpha}}},
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial\lambda_2} &= \frac{D_y}{\lambda_2}-\sum_{j=1}^{D_y}y_j^{-\alpha}+\sum_{j=1}^{D_y}[k_2(R_{yj}+1)-1]\frac{y_j^{-\alpha}e^{-\lambda_2y_j^{-\alpha}}}{1-e^{-\lambda_2y_j^{-\alpha}}}\\
&+k_2R_{d'_{y1}}\frac{T_{y1}^{-\alpha}e^{-\lambda_2T_{y1}^{-\alpha}}}{1-e^{-\lambda_2T_{y1}^{-\alpha}}}+k_2\lambda_2R_{d'_{y2}}\frac{T_{y2}^{-\alpha}e^{-\lambda_2T_{y2}^{-\alpha}}}{1-e^{-\lambda_2T_{y2}^{-\alpha}}},
\end{aligned}
\tag{9}
$$

It is observed from (7), (8) and (9) that the MLEs of the parameter α, λ_1 and λ_2 cannot be obtained in closed form. The approximate ones can be obtained by numerical method such as New-Raphson. Once we obtain $\hat{\alpha}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ the parameters of α, λ_1 and λ_2, the MLE of δ can be obtained using invariance property of MLEs as

$$
\hat{\delta}=\frac{\hat{\lambda}_1}{\hat{\lambda}_1+\hat{\lambda}_2}
\tag{10}
$$

III. Interval Estimation

In this section, we construct δ asymptotic confidence interval based on the asymptotic distribution of $\hat{\delta}$. Also, two confidence intervals are proposed based on the bootstrap method.

A. Asymptotic confidence interval

In this subsection, the asymptotic distribution of $\Theta = (\alpha, \lambda_1, \lambda_2)$ and $\hat{\delta}$ are obtained. Based on the asymptotic distribution $\hat{\delta}$, we can obtain the asymptotic confidence interval of δ. The asymptotic variances and covariances of the MLEs, $\hat{\alpha}, \hat{\lambda}_1$ and $\hat{\lambda}_2$, are given by the entries of the Fisher information matrix $I(\Theta) = E[-(\partial^2(\Theta)/\partial\theta_i\partial\theta_j)]$, where $i, j = 1, 2, 3$. Unfortunately, the exact closed forms for above expectations are difficult to obtain. Therefore, the observed Fisher information matrix $I_0(\Theta) = [-(\partial^2(\Theta)/\partial\theta_i\partial\theta_j)]_{\Theta=\hat{\Theta}}$, for $i, j = 1, 2, 3$ replace the Fisher information matrix $I(\Theta)$. The asymptotic variance of $\hat{\delta}$ is constructed by delta method and is given by

$$
\widehat{Var}(\hat{\delta}) = [h^t I_0^{-1}(\Theta)h]_{\Theta=\hat{\Theta}},
$$

where $h = (\frac{\partial\delta}{\partial\alpha}, \frac{\partial\delta}{\partial\lambda_1}, \frac{\partial\delta}{\partial\lambda_2})^t$. The partial derivations of the Fisher information matrix are given as following:

$$
\begin{aligned}
\frac{\partial^2 l}{\partial\alpha^2} &= -\frac{D_x+D_y}{\alpha^2}-\lambda_1\sum_{i=1}^{D_x}x_i^{-\alpha}\log^2(x_i)-\lambda_2\sum_{j=1}^{D_y}y_j^{-\alpha}\log^2(y_j)\\
&-k_1\lambda_1R_{d'_{x1}}\frac{T_{x1}^{-\alpha}e^{-\lambda_1T_{x1}^{-\alpha}}\log^2(T_{x1})(e^{-\lambda_1T_{x1}^{-\alpha}}+\lambda_1T_{x1}^{\alpha}-1)}{(1-e^{-\lambda_1T_{x1}^{-\alpha}})^2}\\
&-\lambda_1\sum_{i=1}^{D_x}[k_1(R_{xi}+1)-1]\\
&\frac{\log^2(x_i)x_i^{-\alpha}e^{-\lambda_1x_i^{-\alpha}}(e^{-\lambda_1x_i^{-\alpha}}+\lambda_1x_i^{-\alpha}-1)}{(1-e^{-\lambda_1x_i^{-\alpha}})^2}\\
&-k_1\lambda_1R_{d'_{x2}}\frac{T_{x2}^{-\alpha}e^{-\lambda_1T_{x2}^{-\alpha}}\log^2(T_{x2})(e^{-\lambda_1T_{x2}^{-\alpha}}+\lambda_1T_{x2}^{\alpha}-1)}{(1-e^{-\lambda_1T_{x2}^{-\alpha}})^2}\\
&-k_2\lambda_2R_{d'_{y1}}\frac{T_{y1}^{-\alpha}e^{-\lambda_2T_{y1}^{-\alpha}}\log^2(T_{y1})(e^{-\lambda_2T_{y1}^{-\alpha}}+\lambda_2T_{y1}^{-\alpha}-1)}{(1-e^{-\lambda_2T_{y1}^{-\alpha}})^2}\\
&-\lambda_2\sum_{j=1}^{D_y}[k_2(R_{yj}+1)-1]\\
&\frac{\log^2(y_j)y_j^{-\alpha}e^{-\lambda_2y_j^{-\alpha}}(e^{-\lambda_12y_j^{-\alpha}}+\lambda_2y_j^{-\alpha}-1)}{(1-e^{-\lambda_2y_j^{-\alpha}})^2}\\
&-k_2\lambda_2R_{d'_{y2}}\frac{T_{y2}^{-\alpha}e^{-\lambda_2T_{y2}^{-\alpha}}\log^2(T_{y2})(e^{-\lambda_2T_{y2}^{-\alpha}}+\lambda_2T_{y2}^{-\alpha}-1)}{(1-e^{-\lambda_2T_{y2}^{-\alpha}})^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l}{\partial\lambda_1^2} &= -\frac{D_x}{\lambda_1^2}-\sum_{i=1}^{D_x}[k_1(R_{xi}+1)-1]\frac{x_i^{-2\alpha}e^{-\lambda_1x_i^{-\alpha}}}{(1-e^{-\lambda_1x_i^{-\alpha}})^2}\\
&-k_1R_{d'_{x1}}\frac{T_{x1}^{-2\alpha}e^{-\lambda_1T_{x1}^{-\alpha}}}{(1-e^{-\lambda_1T_{x1}^{-\alpha}})^2}-k_1\lambda_1R_{d'_{x2}}\frac{T_{x2}^{-2\alpha}e^{-\lambda_1T_{x2}^{-\alpha}}}{(1-e^{-\lambda_1T_{x2}^{-\alpha}})^2},
\end{aligned}
$$

$$\frac{\partial^2 l}{\partial\lambda_2^2} = -\frac{D_y}{\lambda_2^2} - \sum_{j=1}^{D_y}[k_2(R_{yj}+1)-1]\frac{y_j^{-2\alpha}e^{-\lambda_2 y_j^{-\alpha}}}{(1-e^{-\lambda_2 y_j^{-\alpha}})^2}$$
$$-k_2 R_{d'_{y1}}\frac{T_{y1}^{-2\alpha}e^{-\lambda_2 T_{y1}^{-\alpha}}}{(1-e^{-\lambda_2 T_{y1}^{-\alpha}})^2} - k_2\lambda_2 R_{d'_{y2}}\frac{T_{y2}^{-2\alpha}e^{-\lambda_2 T_{y2}^{-\alpha}}}{(1-e^{-\lambda_2 T_{y2}^{-\alpha}})^2},$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda_1} = \sum_{i=1}^{D_x} x_i^{-\alpha}\log(x_i) - k_1 R_{d'_{x1}}\frac{T_{x1}^{-\alpha}e^{-\lambda_1 T_{x1}^{-\alpha}}\log(T_{x1})}{1-e^{-\lambda_1 T_{x1}^{-\alpha}}}$$
$$+ k_1\lambda_1 R_{d'_{x1}}\frac{T_{x1}^{-2\alpha}e^{-\lambda_1 T_{x1}^{-\alpha}}\log(T_{x1})}{(1-e^{-\lambda_1 T_{x1}^{-\alpha}})^2}$$
$$-\sum_{i=1}^{D_x}[k_1(R_{xi}+1)-1]\frac{x_i^{-\alpha}e^{-\lambda_1 x_i^{-\alpha}}\log(x_i)}{1-e^{-\lambda_1 x_i^{-\alpha}}}$$
$$+\lambda_1\sum_{i=1}^{D_x}[k_1(R_{xi}+1)-1]\frac{x_i^{-2\alpha}e^{-\lambda_1 x_i^{-\alpha}}\log(x_i)}{(1-e^{-\lambda_1 x_i^{-\alpha}})^2}$$
$$-k_1 R_{d'_{x2}}\frac{T_{x2}^{-\alpha}e^{-\lambda_1 T_{x2}^{-\alpha}}\log(T_{x2})}{1-e^{-\lambda_1 T_{x2}^{-\alpha}}}$$
$$+ k_1\lambda_1 R_{d'_{x2}}\frac{T_{x2}^{-2\alpha}e^{-\lambda_1 T_{x2}^{-\alpha}}\log(T_{x2})}{(1-e^{-\lambda_1 T_{x2}^{-\alpha}})^2}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda_2} = \sum_{j=1}^{D_y} y_j^{-\alpha}\log(y_j) - k_2 R_{d'_{y1}}\frac{T_{y1}^{-\alpha}e^{-\lambda_2 T_{y1}^{-\alpha}}\log(T_{y1})}{1-e^{-\lambda_2 T_{y1}^{-\alpha}}}$$
$$+ k_2\lambda_2 R_{d'_{y1}}\frac{T_{y1}^{-2\alpha}e^{-\lambda_2 T_{y1}^{-\alpha}}\log(T_{y1})}{(1-e^{-\lambda_2 T_{y1}^{-\alpha}})^2}$$
$$-\sum_{j=1}^{D_y}[k_2(R_{yj}+1)-1]\frac{y_j^{-\alpha}e^{-\lambda_2 y_j^{-\alpha}}\log(y_j)}{1-e^{-\lambda_2 y_j^{-\alpha}}}$$
$$+\lambda_2\sum_{j=1}^{D_y}[k_2(R_{yj}+1)-1]\frac{y_j^{-2\alpha}e^{-\lambda_2 y_j^{-\alpha}}\log(y_j)}{(1-e^{-\lambda_2 y_j^{-\alpha}})^2}$$
$$-k_2 R_{d'_{y2}}\frac{T_{y2}^{-\alpha}e^{-\lambda_y T_{y2}^{-\alpha}}\log(T_{y2})}{1-e^{-\lambda_y T_{y2}^{-\alpha}}}$$
$$+ k_2\lambda_2 R_{d'_{y2}}\frac{T_{y2}^{-2\alpha}e^{-\lambda_y T_{y2}^{-\alpha}}\log(T_{y2})}{(1-e^{-\lambda_y T_{y2}^{-\alpha}})^2},$$
$$\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} = 0.$$

Thus, $(\hat{\delta}-\delta) \sim N(0,1)$, see Lawless [34]. This results yields the asymptotic $100(1-\gamma)\%$ confidence interval for δ as

$$\hat{\delta} \pm Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{\delta})}, \tag{11}$$

where $Z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.

B. Bootstrap confidence interval

In this subsection, two confidence intervals are proposed based on the parametric bootstrap methods: (1) percentile bootstrap method (Boot-p) based on the idea of Efron [35], and (2) bootstrap-t method (Boot-t) based on the idea of Hall [36]. The following algorithms are used for two parametric bootstrap confidence intervals for the δ.

(i) Boot-p method

Step 1. Generate two independent FFPUHC samples $\underline{x} = (x_1, x_2, \cdots, x_{D_x})$ with fixed $n_1, m_1, R_x, r_x, T_{x1}, T_{x2}$ from IW(α, λ_1) and $\underline{y} = (y_1, y_2, \cdots, y_{D_y})$ with fixed $n_2, m_2, R_y, r_y, T_{y1}, T_{y2}$ from IW(α, λ_2). Compute the MLEs $\hat{\alpha}, \hat{\lambda}_1$ and $\hat{\lambda}_2$.

Step 2. For fixed R_x, r_x, T_{x1} and T_{x2} generate a bootstrap FFPUHC sample $\underline{x}^* = (x_1^*, x_2^*, \cdots, x_{D_x}^*)$ from IW$(\hat{\alpha}, \hat{\lambda_1})$, and for fixed R_y, r_y, T_{y1} and T_{y2} generate a bootstrap FFPUHC sample $\underline{y}^* = (y_1^*, y_2^*, \cdots, y_{D_y}^*)$ from IW$(\hat{\alpha}, \hat{\lambda_2})$. Compute the bootstrap estimate $\hat{\delta}^*$ of δ using Equation (10).

Step 3. Repeat step 2, NBOOT times.

Step 4. Let $H(x) = p(\hat{\delta}^* < x)$ be the CDF of $\hat{\delta}^*$. For a given x define $\hat{\delta}^*_{Bp}(x) = H^{-1}(x)$, The approximate $100(1-\gamma)\%$ confidence interval of δ is given by

$$(\hat{\delta}^*_{Bp}(\frac{\gamma}{2}),\ \hat{\delta}^*_{Bp}(1-\frac{\gamma}{2})).$$

Bootstrap-t method

Steps (1)–(2) are similar to boot-p method as discussed above.

(3) Compute the following statistic:

$$T^* = \frac{\hat{\delta}^* - \hat{\delta}}{\sqrt{Var(\hat{\delta}^*)}}.$$

(4) Repeat (2-3) NBOOT times.

(5) Let $G(x) = p(\hat{T}^* < x)$ be the CDF of $\hat{T}^*$. For a given x define $\hat{\delta}^*_{Bt}(x) = \hat{\delta} + G^{-1}(x)\sqrt{Var(\hat{\delta})}$. The approximate $100(1-\gamma)\%$ confidence intervals of δ is given

$$(\hat{\delta}^*_{Bt}(\frac{\gamma}{2}),\ \hat{\delta}^*_{Bt}(1-\frac{\gamma}{2}))$$

.

IV. Bayes estimation of δ

In this section, we discuss the Bayes estimation of δ. It is assumed the α, λ_1 and λ_2 have independent gamma priors with PDFs:

$$\pi(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha}, \quad \alpha>0, a_1>0, b_1>0, \tag{12}$$

$$\pi(\lambda_1) = \frac{b_2^{a_2}}{\Gamma(a_2)}\lambda_1^{a_2-1}e^{-b_2\lambda_1}, \quad \lambda_1>0, a_2>0, b_2>0, \tag{13}$$

$$\pi(\lambda_2) = \frac{b_3^{a_3}}{\Gamma(a_3)}\lambda_2^{a_3-1}e^{-b_3\lambda_2}, \quad \lambda_2>0, a_3>0, b_3>0, \tag{14}$$

where a_1, b_1, a_2, b_2, a_3 and b_3 are the hyper-parameters. Then, the joint posterior density of α, λ_1 and λ_2 is given by

$$\pi(\alpha,\lambda_1,\lambda_2) = \frac{L(\alpha,\lambda_1,\lambda_2)\pi(\alpha)\pi(\lambda_1)\pi(\lambda_2)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\lambda_1,\lambda_2)\pi(\alpha)\pi(\lambda_1)\pi(\lambda_2)d\alpha d\lambda_1 d\lambda_2}$$
$$\Rightarrow \pi(\alpha,\lambda_1,\lambda_2) \propto \alpha^{D_x+D_y+a_1-1}\lambda_1^{D_x+a_2-1}\lambda_2^{D_y+a_3-1}e^{-b_1\alpha}e^{-b_2\lambda_1}$$
$$e^{-b_3\lambda_2}\prod_{i=1}^{D_x} x_i^{-(\alpha+1)}e^{-\lambda_1 x_i^{-\alpha}}(1-e^{-\lambda_1 x_i^{-\alpha}})^{k_1(R_{xi}+1)-1}$$
$$(1-e^{-\lambda_1 T_{x1}^{-\alpha}})^{k_1 R_{d'_{x1}}}(1-e^{-\lambda_1 T_{x2}^{-\alpha}})^{k_1 R_{d'_{x2}}}$$
$$\prod_{j=1}^{D_y} y_j^{-(\alpha+1)}e^{-\lambda_2 y_j^{-\alpha}}(1-e^{-\lambda_2 y_j^{-\alpha}})^{k_2(R_{yj}+1)-1}$$
$$(1-e^{-\lambda_2 T_{y1}^{-\alpha}})^{k_2 R_{d'_{y1}}}(1-e^{-\lambda_2 T_{y2}^{-\alpha}})^{k_2 R_{d'_{y2}}} \tag{15}$$

Since the posterior distribution in Equation (15) can not be obtained analytically, the Gibbs sampling technique is adopted to compute the Bayes estimate of δ and the corresponding highest posterior density (HPD) credible interval of δ is constructed. The joint posterior PDFs of α, λ_1 and λ_2 be written as follows:

$$g_1(\alpha|\lambda_1,\lambda_2,data) \propto \alpha^{D_x+D_y+a_1-1}e^{-b_1\alpha} \prod_{i=1}^{D_x} x_i^{-(\alpha+1)}e^{-\lambda_1 x_i^{-\alpha}}(1-e^{-\lambda_1 x_i^{-\alpha}})^{k_1(R_{xi}+1)-1} (1-e^{-\lambda_1 T_1^{-\alpha}})^{k_1 R_{d'_{x1}}}(1-e^{-\lambda_1 T_2^{-\alpha}})^{k_1 R_{d'_{x2}}} \prod_{j=1}^{D_y} y_j^{-(\alpha+1)}e^{-\lambda_2 y_j^{-\alpha}}(1-e^{-\lambda_2 y_j^{-\alpha}})^{k_2(R_{yj}+1)-1} (1-e^{-\lambda_2 T_1^{-\alpha}})^{k_2 R_{d'_{y1}}}(1-e^{-\lambda_2 T_2^{-\alpha}})^{k_2 R_{d'_{y2}}} \quad (16)$$

$$g_2(\lambda_1|\alpha,\lambda_2,data) \propto \lambda_1^{D_x+a_2-1}e^{-b_2\lambda_1}(1-e^{-\lambda_1 T_1^{-\alpha}})^{k_1 R_{d'_{x1}}} \prod_{i=1}^{D_x} e^{-\lambda_1 x_i^{-\alpha}}(1-e^{-\lambda_1 x_i^{-\alpha}})^{k_1(R_{xi}+1)-1}(1-e^{-\lambda_1 T_2^{-\alpha}})^{k_1 R_{d'_{x2}}} \quad (17)$$

$$g_3(\lambda_2|\alpha,\lambda_1,data) \propto \lambda_2^{D_y+a_3-1}e^{-b_3\lambda_2}(1-e^{-\lambda_2 T_1^{-\alpha}})^{k_2 R_{d'_{y1}}} \prod_{j=1}^{D_y} e^{-\lambda_2 y_j^{-\alpha}}(1-e^{-\lambda_2 y_j^{-\alpha}})^{k_2(R_{yj}+1)-1}(1-e^{-\lambda_2 T_2^{-\alpha}})^{k_2 R_{d'_{y2}}} \quad (18)$$

The posterior PDFs of α, λ_1 and λ_2 are not in well-known form, so we use the Metropolis-Hastings method with normal proposal distribution to generate random numbers from these distributions. Hence, the sample generation process is the following:

Step 1. Set an initial values $(\alpha^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)})$;
Step 2. Let $t = 1$;
Step 3. $\alpha^{(t)}$ from $g_1(\alpha^{(t)}|\lambda_1^{(t-1)}, \lambda_2^{(t-1)})$ using Metropolis-Hastings algorithm;
Step 4. $\lambda_1^{(t)}$ from $g_2(\lambda_1^{(t)}|\alpha^{(t)}, \lambda_2^{(t-1)})$ using Metropolis-Hastings algorithm;
Step 5. $\lambda_2^{(t)}$ from $g_2(\lambda_1^{(t)}|\alpha^{(t)}, \lambda_2^{(t)})$ using Metropolis-Hastings algorithm;
Step 6. Compute $\delta^{(t)}$ from Equation (10);
Step 7. Set $t = t + 1$;
Step 8. Repeat 3-7, N times.

Then the approximate posterior mean and variance of δ become

$$\widehat{E}(\delta|data) = \frac{1}{N-M}\sum_{t=M+1}^{N}\delta^{(t)},$$

and

$$\widehat{Var}(\delta|data) = \frac{1}{N-M}\sum_{t=M+1}^{N}(\delta^{(t)} - \widehat{E}(\delta|data))^2.$$

Where M is the burn-in period. We can construct the $100(1-\gamma)\%$ HPD credible intervals of δ based on the method of Chen and Shao [37].

V. Simulation study

In this section, a Monte Carlo simulation is performed to compare the performance of the different estimation method based on the FFPUHCS. All programs are written in the R software. The performance of the different estimators in terms of biases and mean square errors (MSEs) are compared. The performance of the different intervals, namely asymptotic confidence interval, bootstrap confidence intervals and HPD credible interval, in terms of the average interval lengths (AIL) and coverage probabilities (CPs) are compared. In this simulation, we take $n_1 = n_2 = n, m_1 = m_2 = m, k_1 = k_2 = k, T_{x1} = T_{y1}, T_{2x} = T_{2y}, r_x = r_y = r$. The different values of n, m, k, T_1, T_2 and r are considered. We take $\alpha = 2, \lambda_1 = 1$ and $\lambda_2 = 1$ in all case. We use three different censoring schemes (CS):

Scheme I: $R_1 = R_m = \frac{n-m}{2}$, $R_i = 0$ for $i \neq 1$ and $i \neq m$.
Scheme II: $R_{\frac{m}{2}} = R_m = \frac{n-m}{2}$, $R_i = 0$ for $i \neq \frac{m}{2}$ and $i \neq m$.
Scheme III: $R_m = n - m$, $R_i = 0$ for $i \neq m$.

We compute the bootstrap confidence intervals based on 1000 re-samplings. We compute the Bayes estimates and 95% HPD credible intervals based on 5000 MCMC samples and discard the first 1000 values as "burn-in". In the Bayes estimates, non-informative and informative priors for the each value of $(\alpha, \lambda_1, \lambda_2)$ are considered. For non-informative prior, namely Prior 1, hyper-parameters are considered ($a_1 = a_2 = a_3 = 1, b_1 = b_2 = b_3 = 0$). For informative prior, namely Prior 2, we have chosen the hyper-parameters in such a way that the prior mean become the expected value of the corresponding parameter ($a_1 = 2, b_1 = a_2 = b_2 = 1 = a_3 = b_3 = 1$). We replicate the whole process 1000 times in each case. The biases and MSEs of the MLEs and Bayes estimates under squared errors loss functions are presented in Table I. Also, we obtain the average lengths of 95% confidence/HPD credible intervals and the CPs of the parameters based on simulation, and these results are presented in Table II.

The results of the Monte Carlo simulation study are given in Tables I-II. From the two tables the following conclusions are made:

(1) The biases and MSEs of δ in the case of MLE and Bayes estimation are very little for all cases.

(2) Bayes estimates based on informative prior are better than MLEs. Under non-informative prior, Bayes estimates are similar to MLEs.

(3) As k increases, MSEs of all estimates decrease. As m or n increases, MSEs decrease.

(4) HPD credible intervals provide the smallest average interval lengths for all cases. The asymptotic confidence interval is the second best.

(5) Boot-p confidence intervals perform better than the Boot-t confidence intervals.

(6) The asymptotic confidence intervals provide the most coverage probabilities in all cases.

(7) The HPD credible intervals based on non-informative prior have the least coverage probabilities in all cases.

(8) As k increases, the average length of the interval narrows. As m increases, the average length of the interval narrows in most cases. As n increases, the average length of the interval narrows.

TABLE I
BIASES AND MSES OF THE MLE AND BAYES ESTIMATION OF δ

		MLE		Prior1		Prior2	
(T_1, T_2, k, n, m, r)	CS	Biases	MSE	Biases	MSE	Biases	MSE
(1,3,2,30,10,8)	(I,I)	0.001137	0.003607	0.001170	0.003506	0.001074	0.003324
	(I,II)	0.0005973	0.003341	0.0008108	0.00324	0.0009555	0.003057
	(I,III)	0.0001758	0.002989	0.0013375	0.002919	0.0018854	0.002775
	(II,II)	-0.001349	0.003054	-0.001340	0.00295	-0.001313	0.002785
	(II,III)	-0.004124	0.002784	-0.003023	0.002703	-0.002417	0.002565
	(III,III)	-0.002464	0.002561	-0.002436	0.00251	-0.002397	0.00241
(1,3,2,30,20,10)	(I,I)	-0.004757	0.002761	-0.004717	0.002699	-0.004645	0.002574
	(I,II)	0.002186	0.002788	0.002178	0.002713	0.002251	0.002178
	(I,III)	-0.0009876	0.002596	-0.0008023	0.002534	-0.0006230	0.00242
	(II,II)	-0.004028	0.0026	-0.003964	0.002526	-0.003894	0.002419
	(II,III)	0.001667	0.00248	0.001748	0.002415	0.001737	0.002306
	(III,III)	0.0005808	0.002487	0.0005241	0.002421	0.0005671	0.002318
(1,3,2,50,20,10)	(I,I)	-0.001971	0.002027	-0.001992	0.001996	-0.002000	0.001935
	(I,II)	0.0004050	0.001866	0.0004740	0.001833	0.0005816	0.001779
	(I,III)	-0.003829	0.001728	-0.003211	0.001699	-0.002931	0.001641
	(II,II)	-0.003788	0.001673	-0.003775	0.001638	-0.003742	0.001591
	(II,III)	-6.546e-04	0.001572	-2.094e-04	0.00155	-2.788e-05	0.001501
	(III,III)	-0.0007795	0.001587	-0.0007105	0.001569	-0.0006977	0.001529
(1,3,2,50,30,15)	(I,I)	-0.0006453	0.001729	-0.0006869	0.001696	-0.0006386	0.00165
	(I,II)	0.001893	0.001719	0.001901	0.001687	0.001963	0.001647
	(I,III)	0.0004053	0.001463	0.0005810	0.001442	0.0006614	0.001398
	(II,II)	0.001829	0.00165	0.001782	0.001617	0.001735	0.001579
	(II,III)	0.001104	0.001575	0.001169	0.001545	0.001235	0.001509
	(III,III)	-0.001227	0.001403	-0.001231	0.001383	-0.001238	0.001346
(1,3,2,50,40,20)	(I,I)	0.0008959	0.001582	0.0008385	0.001562	0.0008476	0.001523
	(I,II)	0.001655	0.001564	0.001635	0.00154	0.001661	0.001507
	(I,III)	0.0009851	0.001467	0.0010520	0.001448	0.0010401	0.001416
	(II,II)	-0.0003019	0.001416	-0.0002606	0.001399	-0.0001792	0.001363
	(II,III)	0.001175	0.001471	0.001241	0.001452	0.001236	0.001417
	(III,III)	-0.001948	0.00153	-0.001923	0.001509	-0.001944	0.001476
(1,3,4,30,10,8)	(I,I)	0.002306	0.001969	0.002319	0.001918	0.002169	0.001848
	(I,II)	0.001655	0.001802	0.001613	0.001757	0.001625	0.001694
	(I,III)	-0.0011482	0.001656	-0.0003147	0.001621	-0.0000740	0.001569
	(II,II)	0.0003056	0.001717	0.0002425	0.001665	0.0002883	0.001608
	(II,III)	-8.383e-04	0.001571	-8.757e-07	0.001535	3.466e-04	0.001479
	(III,III)	-0.0001958	0.001473	-0.0002462	0.001446	-0.0001928	0.001409
(1,3,4,30,20,10)	(I,I)	-0.0008733	0.001763	-0.0008624	0.001726	-0.0008156	0.001673
	(I,II)	0.0001644	0.001653	0.0001675	0.00162	0.0001935	0.001571
	(I,III)	-0.0008700	0.001461	-0.0005599	0.00143	-0.0004834	0.001387
	(II,II)	-0.002353	0.001421	-0.002299	0.001393	-0.002287	0.001346
	(II,III)	-0.0012132	0.001398	-0.0009330	0.001367	-0.0007602	0.001331
	(III,III)	-0.0008197	0.001445	-0.0008286	0.001415	-0.0007621	0.001378
(1,3,4,50,20,10)	(I,I)	0.0006932	0.00113	0.0006655	0.001118	0.0006539	0.001093
	(I,II)	3.715e-05	0.001046	2.018e-05	0.001033	-7.146e-06	0.001012
	(I,III)	-0.0009370	0.001063	-0.0005711	0.001048	-0.0004373	0.001033
	(II,II)	-0.001479	0.001011	-0.001481	0.0009924	-0.001460	0.0009746
	(II,III)	0.0006692	0.0009681	0.0010829	0.0009567	0.0012065	0.0009385
	(III,III)	-0.0001667	0.0008177	-0.0001547	0.00081	-0.0001663	0.0007964
(1,3,4,50,30,15)	(I,I)	0.001391	0.0009459	0.001359	0.000934	0.001416	0.0009186
	(I,II)	-0.001821	0.0009868	-0.001823	0.0009739	-0.001773	0.000958
	(I,III)	0.001405	0.0009517	0.001595	0.0009411	0.001700	0.0009263
	(II,II)	9.284e-05	0.0008866	4.944e-05	0.0008723	6.383e-05	0.0008597
	(II,III)	-0.0008641	0.0009262	-0.0006073	0.0009177	-0.0005302	9e-04
	(III,III)	-2.265e-06	0.0008708	2.206e-06	0.0008605	-1.311e-05	0.0008484
(1,3,4,50,40,20)	(I,I)	0.0008156	0.0009371	0.0007898	0.0009254	0.0007971	0.0009092
	(I,II)	0.001784	0.0008477	0.001779	0.0008389	0.001699	0.0008228
	(I,III)	0.0006066	0.0008703	0.0006546	0.000859	0.0006748	0.0008484
	(II,II)	-1.237e-05	0.0008368	-2.616e-05	0.0008229	-4.439e-05	0.0008101
	(II,III)	0.0007653	0.0008331	0.0008364	0.0008234	0.0007942	0.0008092
	(III,III)	-0.001205	0.0008077	-0.001202	0.0007993	-0.001204	0.0007873

TABLE II
AVERAGE CONFIDENCE/CREDIBLE INTERVAL LENGTHS AND COVERAGE PROBABILITIES FOR ESTIMATORS OF δ

		MLE		Boot-p		Boot-t		Prior 1		Prior 2	
(T_1, T_2, k, n, m, r)	CS	AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
(1,3,2,30,10,8)	(I,I)	0.2506	0.96	0.3026	0.947	0.3415	0.932	0.2015	0.912	0.2116	0.936
	(I,II)	0.253	0.958	0.2722	0.932	0.3001	0.933	0.1946	0.903	0.2044	0.929
	(I,III)	0.2398	0.962	0.295	0.866	0.3741	0.821	0.1889	0.923	0.1992	0.943
	(II,II)	0.249	0.961	0.2418	0.934	0.2645	0.934	0.1879	0.913	0.1969	0.931
	(II,III)	0.2334	0.967	0.2622	0.871	0.3228	0.82	0.1822	0.923	0.1917	0.944
	(III,III)	0.2235	0.972	0.1856	0.941	0.1897	0.944	0.1751	0.92	0.1855	0.938
(1,3,2,30,20,10)	(I,I)	0.2195	0.962	0.2789	0.936	0.3123	0.931	0.1837	0.93	0.1951	0.948
	(I,II)	0.2223	0.956	0.2276	0.932	0.2345	0.923	0.18	0.915	0.1911	0.939
	(I,III)	0.2231	0.964	0.2788	0.948	0.3055	0.934	0.1791	0.923	0.1898	0.946
	(II,II)	0.2194	0.964	0.2239	0.936	0.2289	0.939	0.1758	0.916	0.1864	0.939
	(II,III)	0.2154	0.965	0.2757	0.948	0.3056	0.939	0.1747	0.917	0.1856	0.941
	(III,III)	0.2189	0.961	0.2879	0.953	0.3045	0.949	0.1737	0.925	0.1845	0.944
(1,3,2,50,20,10)	(I,I)	0.187	0.959	0.2738	0.943	0.2944	0.941	0.1559	0.905	0.165	0.932
	(I,II)	0.1884	0.966	0.2391	0.951	0.2531	0.948	0.15	0.906	0.1589	0.934
	(I,III)	0.1807	0.958	0.2596	0.849	0.2996	0.825	0.1465	0.913	0.1558	0.933
	(II,II)	0.182	0.97	0.2123	0.945	0.2256	0.935	0.1435	0.925	0.1515	0.935
	(II,III)	0.1714	0.955	0.233	0.795	0.2679	0.762	0.1402	0.762	0.1489	0.935
	(III,III)	0.1676	0.961	0.1513	0.93	0.1547	0.944	0.1361	0.905	0.145	0.929
(1,3,2,50,30,15)	(I,I)	0.1726	0.959	0.1967	0.952	0.1982	0.939	0.1465	0.925	0.1564	0.949
	(I,II)	0.1747	0.959	0.1944	0.94	0.1975	0.952	0.1422	0.914	0.1518	0.934
	(I,III)	0.1732	0.972	0.2569	0.95	0.2884	0.951	0.1414	0.939	0.1507	0.949
	(II,II)	0.1677	0.961	0.1929	0.944	0.1949	0.944	0.138	0.926	0.147	0.944
	(II,III)	0.168	0.964	0.2536	0.926	0.2841	0.919	0.1367	0.916	0.1457	0.938
	(III,III)	0.1667	0.974	0.2672	0.967	0.2825	0.97	0.1358	0.927	0.1446	0.949
(1,3,2,50,40,20)	(I,I)	0.1589	0.946	0.1552	0.939	0.1575	0.936	0.139	0.92	0.1488	0.943
	(I,II)	0.1591	0.948	0.1528	0.941	0.1528	0.934	0.1367	0.918	0.1465	0.939
	(I,III)	0.1592	0.966	0.1569	0.951	0.1576	0.945	0.1364	0.929	0.1461	0.949
	(II,II)	0.1568	0.964	0.1508	0.945	0.1518	0.945	0.1346	0.933	0.1441	0.952
	(II,III)	0.1572	0.958	0.1546	0.946	0.1561	0.944	00.1342	0.925	0.1438	0.951
	(III,III)	0.1578	0.957	0.1556	0.942	0.1567	0.941	0.1337	0.915	0.1433	0.944
(1,3,4,30,20,10)	(I,I)	0.1912	0.961	0.16	0.957	0.1652	0.96	0.1562	0.927	0.1638	0.946
	(I,II)	0.1915	0.971	0.1541	0.966	0.1589	0.96	0.1518	0.933	0.1589	0.951
	(I,III)	0.1815	0.964	0.1641	0.804	0.1863	0.79	0.1455	0.921	0.1532	0.943
	(II,II)	0.1845	0.968	0.1517	0.956	0.1546	0.944	0.1469	0.929	0.1533	0.954
	(II,III)	0.1731	0.97	0.1621	0.794	0.188	0.7667	0.1409	0.931	0.1481	0.948
	(III,III)	0.1665	0.964	0.136	0.936	0.1364	0.94	0.1338	0.91	0.1414	0.927
(1,3,4,30,20,10)	(I,I)	0.1769	0.963	0.193	0.953	0.205	0.955	0.1434	0.927	0.1512	0.942
	(I,II)	0.178	0.966	0.1847	0.945	0.1928	0.945	0.1407	0.913	0.1483	0.932
	(I,III)	0.1745	0.972	0.1923	0.919	0.2118	0.901	0.1387	0.928	0.1464	0.948
	(II,II)	0.1739	0.968	0.1759	0.962	0.1885	0.959	0.1378	0.931	0.1448	0.953
	(II,III)	0.1689	0.963	0.1839	0.915	0.2025	0.892	0.1359	0.937	0.1431	0.945
	(III,III)	0.1663	0.97	0.1526	0.956	0.1575	0.956	0.1339	0.939	0.1412	0.954
(1,3,4,50,20,10)	(I,I)	0.1436	0.963	0.1227	0.948	0.1227	0.945	0.1206	0.93	0.1273	0.947
	(I,II)	0.1444	0.975	0.1173	0.951	0.118	0.954	0.1166	0.931	0.1227	0.947
	(I,III)	0.1368	0.96	0.1194	0.698	0.1317	0.677	0.1125	0.913	0.119	0.933
	(II,II)	0.1345	0.961	0.1158	0.938	0.1165	0.936	0.1122	0.92	0.1179	0.928
	(II,III)	0.1293	0.956	0.1185	0.669	0.1361	0.643	0.1083	0.913	0.1143	0.936
	(III,III)	0.1271	0.968	0.1051	0.945	0.1054	0.94	0.1039	0.928	0.1099	0.942
(1,3,4,50,30,15)	(I,I)	0.1361	0.969	0.1497	0.956	0.1545	0.959	0.1141	0.937	0.1207	0.947
	(I,II)	0.1372	0.968	0.1386	0.957	0.1414	0.95	0.1112	0.934	0.1171	0.945
	(I,III)	0.1329	0.972	0.1426	0.804	0.1523	0.783	0.1092	0.933	0.1154	0.947
	(II,II)	0.1327	0.974	0.1277	0.954	0.1315	0.947	0.108	0.931	0.1136	0.944
	(II,III)	0.128	0.958	0.1305	0.826	0.14	0.804	0.1061	0.916	0.112	0.933
	(III,III)	0.1272	0.968	0.1055	0.945	0.1053	0.947	0.104	0.919	0.1101	0.942
(1,3,4,50,40,20)	(I,I)	0.1289	0.958	0.145	0.947	0.147	0.952	0.1082	0.927	0.1148	0.935
	(I,II)	0.1299	0.973	0.1425	0.962	0.1441	0.954	0.1067	0.932	0.113	0.952
	(I,III)	0.1288	0.975	0.1581	0.918	0.1709	0.913	0.106	0.926	0.1123	0.944
	(II,II)	0.1274	0.97	0.14	0.957	0.1414	0.949	0.1052	0.934	0.1113	0.946
	(II,III)	0.1265	0.96	0.1557	0.931	0.1671	0.925	0.1045	0.927	0.1106	0.944
	(III,III)	0.1263	0.974	0.154	0.963	0.1597	0.96	0.1039	0.931	0.11	0.954

VI. Application

In this section, we analyse two real data sets for illustrative purpose. The two data sets were measured by Xia et al [38]. The two data sets are listed in Tables III and IV. The two data sets represent the breaking strengths of jute fibre at two different gauge lengths. First, it is checked if inverse Weibull distribution is adequate to fit these two data sets. The K-S distances between empirical distribution and the fitted distributions and the corresponding p-values are calculated. The results are given in Table V. Table V shows IW distribution may well fit to two data sets. From Table V we observe that there is little difference between the two shape parameters. Therefore, it is assumed that the two shape parameters are equal. The results are given in Table VI. To verify that the hypothesis is correct, the likelihood ratio (LR) statistic can be used to test for hypothesis testing that the two shape parameters are equal. The following testing of hypothesis is performed:

$H_0: \ \alpha_1 = \alpha_2 = \alpha$ versus $H_1: \ \alpha_1 \neq \alpha_2$.

The LR statistic is

$$\omega = \frac{L(x, \hat{\alpha}^*, \hat{\lambda}_1^*, \hat{\lambda}_2^*)}{L(x, \hat{\alpha}_1, \hat{\lambda}_1, \hat{\alpha}_2, \hat{\lambda}_2)}.$$

Where $\hat{\alpha}^*, \hat{\lambda}_1^*, \hat{\lambda}_2^*, \hat{\alpha}_1, \hat{\lambda}_1, \hat{\alpha}_2$, and $\hat{\lambda}_2$ are MLEs of the parameters, respectively. It is known the $-2\log\omega$ follows a chi-square with one degree of freedom. Based on the numerical calculations, the value $-2\log\omega = 0.4$. Hence, the null hypothesis can not be rejected. Therefore, in this case the assumption of $\alpha_1 = \alpha_2$ is proper. Based completed data, the MLE and Bayes estimation of δ are 0.5796, 0.5736. The 95% asymptotic, Boot-p, Boot-t confidence and HPD credible intervals of δ are obtained as (0.4434, 0.6889), (0.4478, 0.7044), (0.4141, 0.6969) and (0.4403, 0.6879), respectively, under completed data.

For illustrative purpose, the FFPUHC samples have been generated from the above two data sets with $k_1 = k_2 = 1, m_1 = m_2 = 20, r_x = r_y = 8, T_{1x} = T_{1y} = 150, T_{2x} = T_{2y} = 700$ in all cases, and with different censored schemes. We compute MLE and Bayes estimates and asymptotic, Boot-p, Boot-t, confidence intervals, HPD credible interval of δ based on FFPUHC samples. The results are listed in Table VII. From Table VII, it is show that the estimates of δ based on complete and FFPUHC samples are very close.

TABLE III
Breaking strengths of jute fibre of gauge length 10 mm

693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48
108.94	50.16	671.49	183.16	257.44	727.23	291.27	101.15
376.42	163.40	141.38	700.74	262.90	353.24	422.11	43.93
590.48	212.13	303.90	506.60	530.55	177.25		

TABLE IV
Breaking strengths of jute fibre of gauge length 20 mm

71.46	419.02	284.64	585.57	456.60	113.85	187.85	688.16
662.66	45.58	578.62	756.70	594.29	166.49	99.72	707.36
765.14	187.13	145.96	350.70	547.44	116.99	375.81	581.60
119.86	48.01	200.16	36.75	244.53	83.55		

TABLE V
MLEs and K-S goodness-of-fit test with $\alpha_1 \neq \alpha_2$

Data set	α	λ	$-\log L$	K-S	P-value
1	1.183	491.647	209	0.17	0.3
2	1.085	228.510	207.5	0.16	0.4

TABLE VI
MLEs and K-S goodness-of-fit test with $\alpha_1 = \alpha_2$

Data set	α	λ	$-\log L$	K-S	P-value
1	1.166	459.061	209	0.18	0.3
2	1.166	332.913	210.3	0.17	0.3

TABLE VII
Point and interval estimations of δ under different methods

CS	different estimation	point estimation	interval estimtion
(I,I)	MLE	0.5769	(0.4466, 0.7072)
	Boot-p		(0.4474, 0.7060)
	Boot-t		(0.4707, 0.7556)
	Bayes	0.5841	(0.4472, 0.7054)
(I,II)	MLE	0.5509	(0.4151, 0.6866)
	Boot-p		(0.422, 0.678)
	Boot-t		(0.4486, 0.7166)
	Bayes	0.5562	(0.4231, 0.6776)
(I,III)	MLE	0.5343	(0.4007, 0.6679)
	Boot-p		(0.4044, 0.6747)
	Boot-t		(0.4342, 0.7243)
	Bayes	0.5390	(0.4072, 0.6616)
(II,II)	MLE	0.5732	(0.4455, 0.7009)
	Boot-p		(0.4499, 0.6984)
	Boot-t		(0.4728, 0.7395)
	Bayes	0.5792	(0.4501, 0.6947)
(II,III)	MLE	0.5559	0.4293 0.6826
	Boot-p		0.4322 0.6932
	Boot-t		0.4611 0.7392
	Bayes	0.5617	0.4326 0.6789
(III,III)	MLE	0.5621	(0.4410, 0.6833)
	Boot-p		(0.4369, 0.6886)
	Boot-t		(0.4639, 0.7249)
	Bayes	0.5675	(0.4391, 0.6840)

VII. Conclusion

In the paper, we have considered the problem of estimating $\delta = P(Y < X)$, when X and Y have independent inverse Weibull distributions with the same shape parameters but different scale parameters based on FFPUHC data. Our simulation study indicates that the following results are obtained.

For point estimation, we use ML and Bayes methods to estimate the δ under FFPUHC samples. Bayes estimation based on informative prior is better than MLE, under non-informative prior, Bayes estimates are similar to MLEs. If we know something about the parameters, we use Bayes estimation of the δ, otherwise we use maximum likelihood estimation.

For interval estimation, the asymptotic distribution of δ is used to construct the asymptotic confidence interval of δ, also, two bootstrap confidence intervals are proposed, HPD credible interval of δ can be obtained using Gibbs sampling. HPD credible intervals provide the smallest average interval lengths for all cases. The asymptotic confidence interval is the second best. But the asymptotic confidence intervals provide the most coverage probabilities in all cases.

Finally, we propose to use the maximum likelihood method to estimate the δ based on FFPUHC data, because the MLE calculation is small and effective.

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