Edge-magic Total Labeling Algorithm of Unicyclic Graphs

Bimei Wang, Jingwen Li

***Abstract*—A graph $G(p,q)$ is said to have an edge-magic total labeling if there exists a bijective function $f: V(G) \cup E(G) \to \{1,2,\dots,p+q\}$, such that for any edge $uv \epsilon E(G)$ the condition $f(u)+f(v)+f(uv)=k$ is satisfied, k is a constant. In this paper, a new algorithm, based on the graph generate algorithm, is designed to obtain the edge-magic total labeling of the unicyclic graphs. Some theorems about unicyclic graphs are also deduced from the algorithm's results. It's believed that the algorithm proposed is innovative and can be adopted by other researchers.**

***Index Terms*—Edge-magic total labeling, super edge-magic total labeling, unicyclic graphs, algorithm**

I. Introduction

HISTORICALLY graph labeling can be traced back to the mid-1960s; More than two thousand papers on graph labeling have been issued so far[1]. Especially, edge-magic total labeling is one of the most classical research types. The concept of magic labeling was introduced by Sedláček in 1963[2].

So far, the main focus of researchers has been on the labeling of special graphs. It has been proved in literature works [1, 3-6] that graphs such as $C_n, P_n+K_1, K_{m,n}, C_n \odot K_1$, fans and binary trees are edge-magic total labeling graphs. Reference [7] shows that C_n are edge-magic total labeling graphs if and only if n is odd. All caterpillar tree graphs have edge-magic total labeling[4]. In [8], it is known that some kinds of combination graphs have super edge-magic total labeling.

Some of the complex problems encountered in real world situations can't be reflected by the special graphs mentioned above. So far, scholars at home and aboard have done little research on general unicyclic graphs. In this paper, a new edge-magic total labeling algorithm based on the graph generation algorithm[9] is designed.

All of the graphs considered in this paper are simple, undirected and finite. There are two elements of $G(p,q)$, namely vertex set $V(G)$ and edge set $E(G)$, where p and q represent the numbers of vertices and edges respectively. In other words, $|V(G)|=p$, $|E(G)|=q$. The star graph S_n is constructed with $n-1$ leaf nodes and one center node[10]. When $p=q$, the graph $G(p,q)$ is called unicyclic graph.

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Some useful definitions and lemmas are as follows.

Definition 1[7]: Let $G(p,q)$ be a finite simple connected graph, a bijection f from V(G) ∪ E(G) to $\{1,2,\dots,p+q\}$ is called an edge-magic total labeling (EMTL for short) if there exists a constant k (called the magic number of f), such that, for any edge of G, $f(u)+f(v)+f(uv)=k$. The edge-magic total labeling of the graph, f, is further designated as super edge-magic total labeling (SEMTL for short) if $f(V(G)) \to \{1,2,\dots,p\}$ and $f(E(G)) \to \{\mathrm{p}+1,\mathrm{p}+2,\dots,p+q\}$.

It is easy to get the conclusion that SEMTL is a special case of EMTL.

Lemma 1[10]: A graph G with p vertices and q edges is a super edge magic graph if and only if there exists a bijective function $f: V(G) \cup E(G) \to \{1,2,\dots,p+q\}$ such that the set $S=\{f(u)+f(v) | uv \in E(G)\}$ consists of q consecutive integers. In such a case, f is further designated as the super edge-magic total labeling of G with the magic constant $k=p+q+s$, where $s=\min(S)$.

Definition 2: Let $\{u_1,u_2,\dots,u_n\}$ and $\{v_1,v_2,\dots,v_m\}$ be two vertex sets of C_n and S_m. Composite graphs $C_n \downarrow \mathrm{S}_m$ denote the unicyclic graphs obtained by connecting the central node v_1 of S_m to the vertex u_1 of C_n through an edge u_1v_1. It is not hard to see that $V(C_n \downarrow \mathrm{S}_m)=V(C_n) \cup V(S_m)$, $E(C_n \downarrow \mathrm{S}_m)=E(C_n) \cup E(S_m) \cup \{u_1v_1\}$, $|E(C_n \downarrow \mathrm{S}_m)|=|E(C_n)|+|E(S_m)|+1$. The structure of $C_n \downarrow \mathrm{S}_m$ is shown in Fig.1.

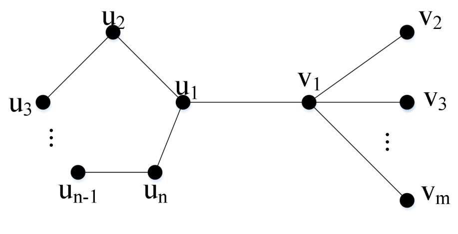


Fig.1. Composite graphs $C_n \downarrow \mathrm{S}_m$

Definition 3: Let $\{u_1,u_2,\dots,u_n\}$ and $\{v_1,v_2,\dots,v_m\}$ be two vertex sets of C_n and S_m. Composite graphs $C_n \uparrow \mathrm{S}_m$ denote the unicyclic graphs obtained by gluing the center node v_1 of S_m to the vertex u_1 of C_n. $V(C_n \uparrow \mathrm{S}_m)=V(C_n) \cup V(S_m) \setminus v$, $|V(C_n \uparrow \mathrm{S}_m)|=|V(C_n)|+|V(S_m)|-1$, $E(C_n \uparrow \mathrm{S}_m)=E(C_n) \cup E(S_m)$. The structure of $C_n \uparrow \mathrm{S}_m$ is as shown in Fig.2.

Definition 4: Composite graphs $C_3 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j$ denote the unicyclic graph obtained by gluing the three center nodes of the three star graphs to the three vertices of C_3. It is easy to see that $|V(C_3 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j)|=|V(\mathrm{S}_m)|+|V(\mathrm{S}_i)|+|V(\mathrm{S}_j)|$, $E(C_3 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j)=E(C_3) \uparrow E(S_m) \uparrow E(S_i) \uparrow E(S_j)$. The structure of $C_3 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j$ is as shown in Fig.3.

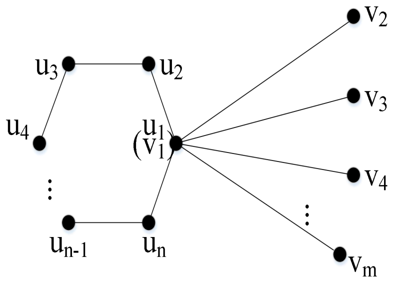


Fig.2. Composite graphs $C_n \uparrow \mathrm{S}_m$

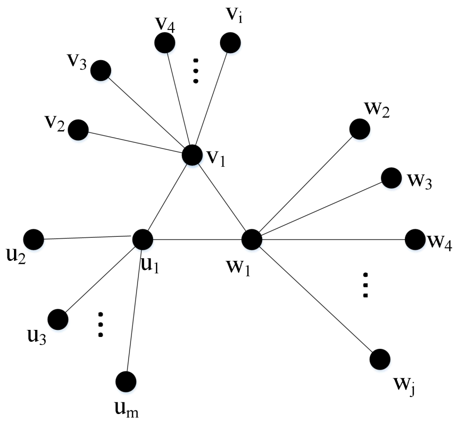


Fig 3. Composite graphs $C_3 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j$

Definition 5: Composite graphs $C_4 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_n \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j$ denote the unicyclic graph obtained by gluing the center nodes of star graphs to the four vertices of C_4. $|V(C_4 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_n \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j)| = |V(\mathrm{S}_m)| + |V(\mathrm{S}_n)| + |V(\mathrm{S}_i)| + |V(\mathrm{S}_j)|$, $E(C_4 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_n \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j) = E(C_4) \cup E(\mathrm{S}_m) \cup E(\mathrm{S}_n) \cup E(S_i) \cup E(S_j)$. The structure of $C_4 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_n \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j$ is as shown in Fig.4.

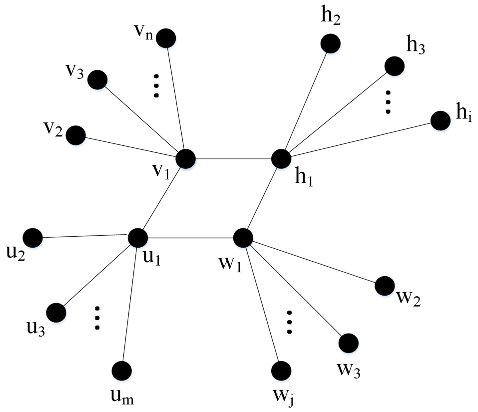


Fig.4. Composite graphs $C_4 \uparrow \mathrm{S}_m \uparrow \mathrm{S}_n \uparrow \mathrm{S}_i \uparrow \mathrm{S}_j$

Definition 6: Let $\{u_1, u_2, ..., u_n\}$, $\{v_1, v_2, ..., v_m\}$ and $\{w_1, w_2\}$ be three vertex sets of C_n, S_m and P_2. Graphs $P_2 \uparrow C_n \uparrow \mathrm{S}_m$ denote the unicyclic graph obtained by gluing vertex w_2 of P_2 to the vertex u_3 of $C_n \uparrow \mathrm{S}_m$ where the distance between u_1 and w_2 is the fastest in the graph $P_2 \uparrow C_n \uparrow \mathrm{S}_m$. $|V(P_2 \uparrow C_n \uparrow \mathrm{S}_m)| = |V(\mathrm{P}_2)| + |V(C_n)| + |V(\mathrm{S}_m)| - 2$, $E(P_2 \uparrow C_n \uparrow \mathrm{S}_m) = E(P_2) \cup E(C_n) \cup E(\mathrm{S}_m)$. The structure of $P_2 \uparrow C_n \uparrow \mathrm{S}_m$ is as shown in Fig.5.

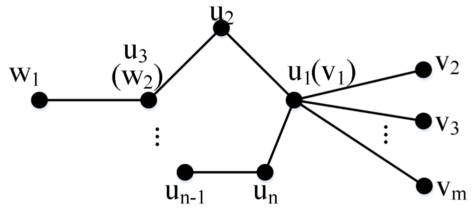


Fig.5. Composite graphs $P_2 \uparrow C_n \uparrow \mathrm{S}_m$

Definition 7: Let the elements of the tuples (x, y, z) represent respectively arbitrarily chosen two vertex labels and their edge label of unicycle graph $G(p, q)$. For each edge in graph G if there exists an integer k satisfying the following conditions: $x + y + z = k$, $x, y, z \epsilon [1, p + q]$, $x \neq y \neq z$ and $p + q + 3 \leq k \leq 2(p + q)$, then we call the group of tuples (x, y, z) the edge-magic total labeling solution space of graph $G(p, q)$, denoted by $\varphi(p, q, k)$. See Table 1 below.

TABLE I
EMTL SOLUTION SPACE $\varphi(p, q, k)$

k	Triple 1	Triple 2	Triple 3	...
p+q+3	(1,2,p+q)	(1,3,p+q-1)	(1,4,p+q-2)	...
p+q+4	(1,3,p+q)	(1,4,p+q-1)	(1,5,p+q-2)	...
⋮	...	...	...	...
2(p+q)	(1,p+q-1,p+q)	(2,p+q-2,p+q)	(3,p+q-3,p+q)	...

It is clear from Definition 1 that the following lemmas hold.

Lemma 2: If a $G(p, q)$ has EMTL, then all of the elements from tuples (x, y, z) belong to $\varphi(p, q, k)$.

Lemma 3: If a $G(p, q)$ doesn't have EMTL or SEMTL, we call it a non-edge magic total labeling graph(NEMTL graph for short).

II. EDGE MAGIC TOTAL LABELING ALGORITHM

A. Formula Derivation

According to the definition 1, it is known that if graph $G(p, q)$ has EMTL, then a map $f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, p + q\}$ exists, we can denote the sum of all vertices and edge labels of graph $G(p, q)$ by C.

$$C = L_p + L_q = \sum_{i=1}^{p+q} i = (p+q)(p+q+1)/2 \quad (1)$$

where L_p and L_q denote the sum of the labels of the vertices and edges, respectively.

For any edge uv of graph $G(p, q)$, we have $f(u) + f(v) + f(uv) = k$, because graph $G(p, q)$ has q edges, we get

$$qk = L_q + \sum_{i=1}^{p} deg\,(v_i)\, f(v_i)$$

where deg (v_i) represents the degree of any vertex v_i and $f(v_i)$ represents the label of any vertex v_i.

$$\sum_{i=1}^{p} \deg\,(v_i)\, f(v_i) = L_p + \sum_{i=1}^{p} (deg\,(v_i) - 1) f(v_i)$$

So

$$qk = C + \sum_{i=1}^{p} (deg\,(v_i) - 1) f(v_i) \quad (2)$$

Because $\mathrm{Dsc}(v_i) = \deg\,(v_i) - 1$, we have

$$qk = C + \sum_{i=1}^{p} Dsc(v_i)\, f(v_i)$$

we transformed formula (2) to the following formula.

$$qk = C + \sum_{i=1}^{p} Dsc(v_i)\, f(v_i) + \sum_{j=1}^{q} 0 * f(e_j)$$

where $f(e_j)$ represents the label of an arbitrary edge e_j. We can get the sum as:

$$Sum = \sum_{i=1}^{p} Dsc(v_i)\, f(v_i) + \sum_{j=1}^{q} 0 * f(e_j)$$

Then, we get

$$k = (C + Sum)/q \tag{3}$$

B. Algorithm Description

First, generate the non-homogeneous graphs of all unicyclic graphs with the number of vertices less than or equal to 16 and then store their matrixes in files.

Second, for each of the matrixes, calculate the corresponding constant C, $\deg(v_i)$, $Dsc(v_i)$, the range of k and the solution space $\varphi(p, q, k)$.

Third, in the EMTL solution space, initialize $f(v_i)$ by assigning labels to the vertices.

Forth, calculate the Sum and substitute it in (3). Invoke the distribution procedure if a positive integer k exists; otherwise, reorder the sequence of $Dsc(v_i)$ and edge label coefficients 0. The algorithm ends with either the accomplishment of the allocation or the consumption of all of the permutations. A graph is an EMTL graph if it can be labeled successfully, and the label matrix will be output; otherwise, it is a NEMTL graph.

C. Algorithm Pseudocode

Input: Adjacency matrix file of $G(p, q)$;

Output: EMTL, SEMTL matrix or NEMTL graph;

```
Begin
Input the adjacency matrix, Calculate C;
Get the corresponding deg(v_i)、Dsc(v_i) and the range of k;
Obtain the corresponding solution space φ(p, q, k);
isSuccess=false;
isContinue=true;
while(isContinue)
   Calculate Sum(Dsc, f(v_i), f(e_i));
   if((C + Sum)%q == 0)
      k = (C + Sum)/q;
      Apportion(k, f(v_i), f(e_i));
      if(isSuccess)
         break;
      end if
   end if
Permutation(Dsc);
end while
End
```

Lemma 4: For any unicyclic graph $G(p, q)$, after invoking the EMTL algorithm to search its solution space $\varphi(p, q, k)$ thoroughly, if a solution exists, it is an EMTL graph; otherwise, it is a NEMTL graph.

Proof: According to lemmas 2 and 3, it is evident that lemma 4 is true.

An example is given below to show the EMTL solution space more intuitively.

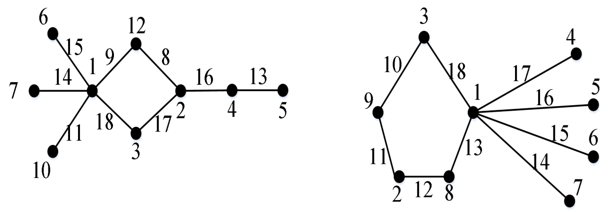
Example 1. Table 2 is the solution space $\varphi(9,9,k)$, and figures 3(1) and 3(2) serve as the examples of EMTL and SEMTL graphs.

TABLE II

EMTL SOLUTION SPACE $\varphi(9,9,k)$

k	Triple 1	Triple 2	Triple 3	Triple 4…
21	(1,2,18)	(1,3,17)	(1,4,16)	…
22	(1,2,19)	**(1,3,18)**	(1,4,17)	(1,5,16)
	(1,6,15)	**(1,7,14)**	(1,8,13)	**(1,9,12)**
	(1,10,11)	**(2,3,17)**	**(2,4,16)**	(2,5,15)
	(2,6,14)	(2,7,13)	**(2,8,12)**	(2,9,11)
	(3,4,15)	(3,5,14)	(3,6,13)	(3,7,12)
	(3,8,11)	(3,9,10)	**(4,5,13)**	(4,5,13)
	(4,6,12)	(4,7,11)	(4,8,10)	(5,6,11)
	(5,7,10)	(5,8,9)	(6,7,9)	
⋮	...	...	...	...
36	(1,17,18)	(2,16,18)	(3,15,18)	...

The triples shown in black-bold in Table 2 are the solution to Fig.6(1). The underlined triples in Table 2 are the solution to Fig.6(2).



(1) EMTL graph (2) SEMTL graph

Fig.6. EMTL and SEMTL graphs in atlas of $G(9,9)$

III. MAIN RESULTS

Employing the above algorithm, the EMTL and SEMTL of the unicyclic graphs with vertices less than or equal to 16 are mainly studied in this paper; several theorems are also obtained. The theorems are as follows.

Theorem 1: All unicyclic graphs have edge-magic total labeling when $3 \le p \le 16$.

Proof: (1) For unicyclic graphs, when $3 \le p \le 16$, it is known that $9 \le k \le 64$. The corresponding space is shown in Table 3.

TABLE III

EMTL SOLUTION SPACE OF UNICYCLIC GRAPHS$(3 \le p \le 16)$

k	Triple 1	Triple 2	Triple 3	Triple 4
9	(1,2,6)	(1,3,5)	(2,3,4)	
10	(1,3,6)	(1,4,5)	(2,3,5)	
⋮	...	...	...	...
63	(1,30,32)	(2,29,32)	(3,28,32)	...
64	(1,31,32)	(2,30,32)	(3,29,32)	...

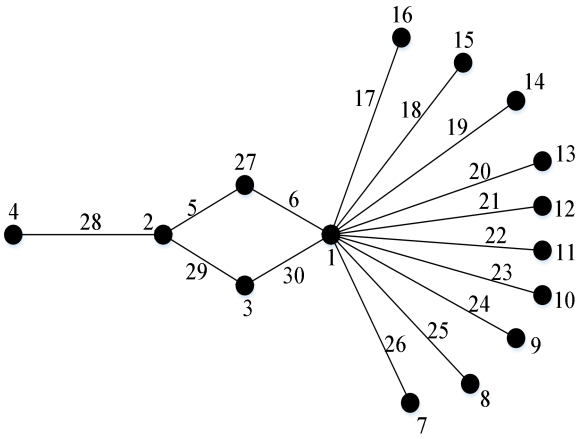
(2) Table 4 is the labeling results of the unicyclic graphs with vertices less than or equal to 16.

(3) As shown in Table 4, the number of the unicyclic graphs with vertices less than or equal to 16 is 482977, of which 482714 graphs are super edge-magic total labeling graphs, accounting for 99.946% of the total number of graphs. It can also be shown that when $3 \le p \le 16$, all the unicyclic graphs are EMTL graphs and most of them are also SEMTL graphs.

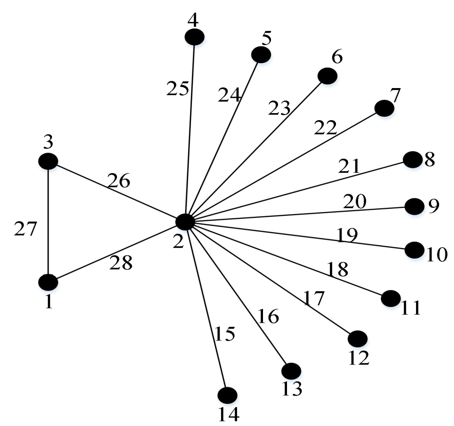
TABLE Ⅳ
ALGORITHM RESULTS OF UNICYCLIC GRAPHS ($3 \le p \le 16$)

(p,q)	TOTAL NUMBER	EMTL graph	SEMTL graph	The proportion of SEMTL /%
(3,3)	1	1	1	100
(4,4)	2	2	1	50
(5,5)	5	5	3	60
(6,6)	13	13	8	61.538
(7,7)	33	33	18	54.545
(8,8)	89	89	75	84.270
(9,9)	240	240	217	90.417
(10,10)	657	657	639	97.260
(11,11)	1806	1806	1768	97.896
(12,12)	5026	5026	5014	99.761
(13,13)	13999	13999	13950	99.650
(14,14)	39260	39260	39234	99.933
(15,15)	110381	110381	110343	99.966
(16,16)	311465	311465	311443	99.993

(4) Examples of EMTL and SEMTL graphs are as shown in Fig.7.



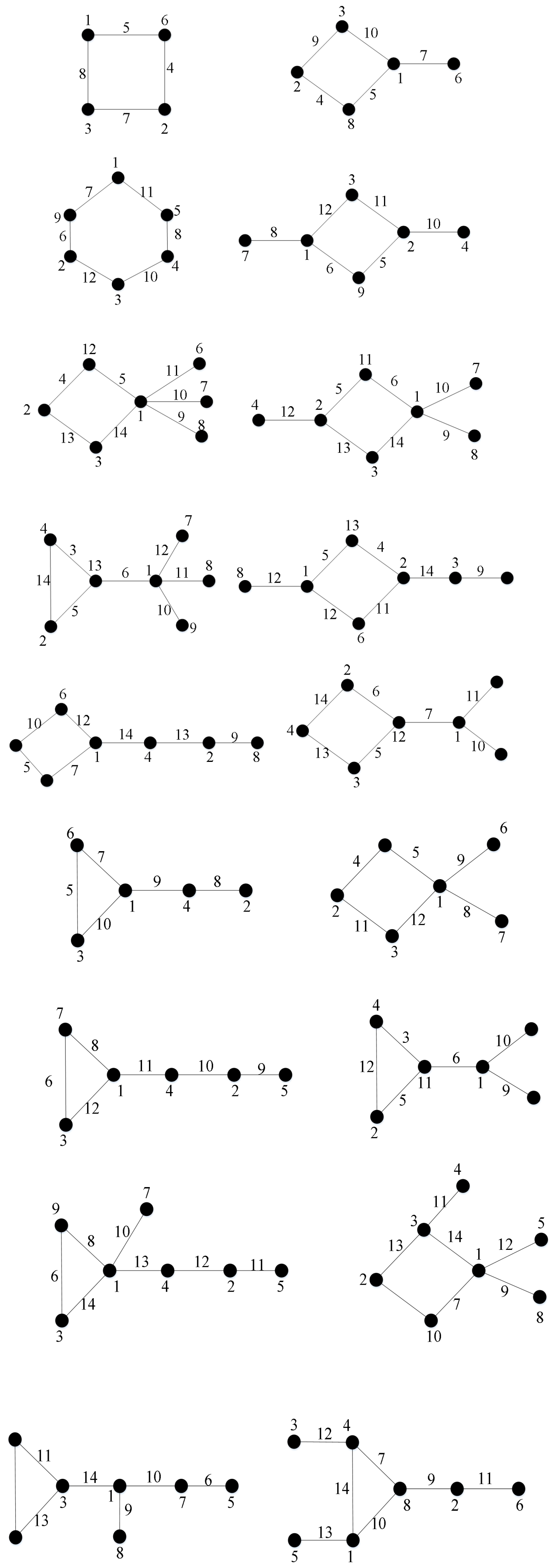
(1) G(15,15)



(2) G(14,14)

Fig.7. EMTL and SEMTL examples of unicyclic graphs

(5) All the graphs with vertices less than or equal to 7 have EMTL, but not have SEMTL. The results are as shown in Fig.8.



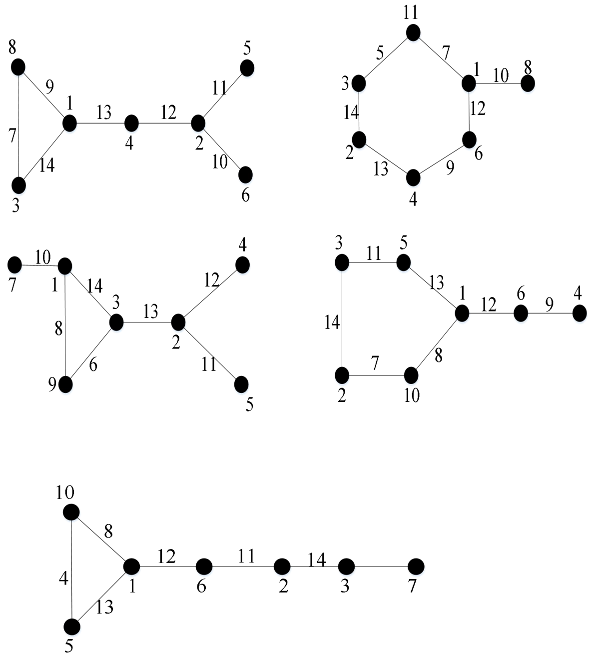


Fig.8. All unicyclic graphs having EMTL but not SEMTL($p \leq 7$)

(6) Conjecture 1: Graph $G(p,q)$ has EMTL when $p = q$ and $p \geq 17$.

Theorem 2: All unicyclic graphs $C_3 \downarrow S_m \downarrow S_n \downarrow S_j$ have SEMTL when $m, n, j \geq 2$.

Proof: Suppose that $m \leq n \leq j$, the structure of graphs $C_3 \downarrow S_m \downarrow S_n \downarrow S_j$ is as shown in Fig.9.

We know that

$$|V(G)| = |E(G)| = m + n + j$$

So, $k \in [2(m+n+j)+3, 4(m+n+j)]$.

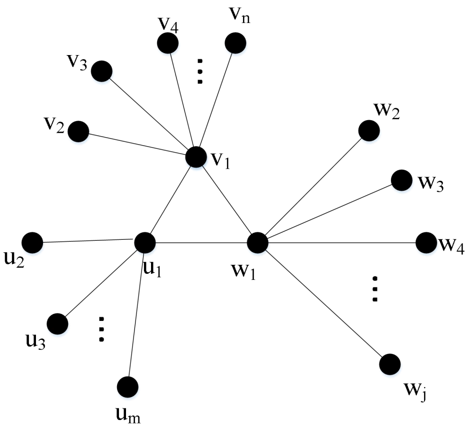


Fig.9. $C_3 \downarrow S_m \downarrow S_n \downarrow S_j$

The labels of the vertices and edges of $C_3 \downarrow S_m \downarrow S_n \downarrow S_j$ are as follows:

$$f(u_i) = \begin{cases} m+j+1, & i = 1 \\ i, & 2 \leq i \leq m \end{cases}$$

$$f(v_i) = \begin{cases} m+1, & i = 1 \\ m+j+i, & 2 \leq i \leq n \end{cases}$$

$$f(w_i) = \begin{cases} 1, & i = 1 \\ m+i, & 2 \leq i \leq j \end{cases}$$

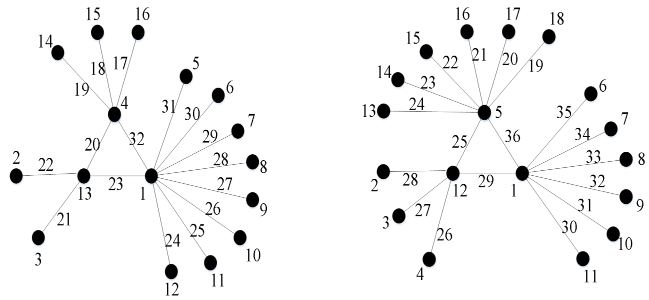
$$V(G) = \{1,2,\dots,m\} \cup \{m+1, m+2, \dots, m+j+n\}$$

The set $S = \{f(u)+f(v) | uv \in E(G)\} = \{m+2, m+3, \dots, m+j+1\} \cup \{m+j+2, m+j+3, \dots, 2m+j+1\} \cup \{2m+j+2, 2m+j+3, \dots, 2m+j+n+1\}$.

$s = \min(S) = f(v_1) + f(w_1) = m+1+1 = m+2$.

According to the lemma 1, we know that $k = p+q+s = (m+n+j)+(m+n+j)+m+2 = 3m+2n+2j+2$.

According to the definition of SEMTL, f is the super edge-magic total labeling of graphs $C_3 \downarrow S_m \downarrow S_n \downarrow S_j$. Some examples are shown in Fig.10.



(1) $C_3 \downarrow S_3 \downarrow S_4 \downarrow S_9$ (2) $C_3 \downarrow S_4 \downarrow S_7 \downarrow S_7$

Fig.10. Super edge-magic total labeling of $C_3 \downarrow S_m \downarrow S_n \downarrow S_j$

Theorem 3: All unicyclic graphs $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_m \downarrow S_n$ have SEMTL when $m, n \geq 2$.

Proof: Suppose that $2 \leq m \leq n$, the structure of graphs $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_m \downarrow S_n$ is as shown in Fig.11.

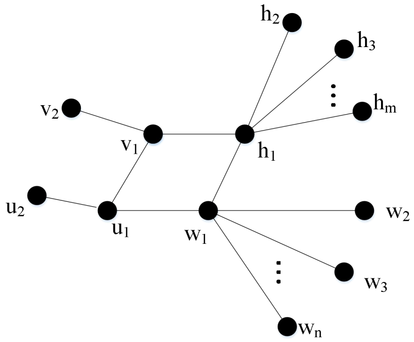


Fig.11. $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_m \downarrow S_n$

We know that

$$|V(G)| = |E(G)| = 2+2+m+n = m+n+4$$

So, $k \in [2(m+n+4)+3, 4(m+n+4)]$.

We define the labeling on the vertices and edges of $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_m \downarrow S_n$ as follows:

$$f(u_i) = \begin{cases} n+4, & i = 1 \\ 4, & i = 2 \end{cases}$$

$$f(v_i) = \begin{cases} 2, & i = 1 \\ n+5, & i = 2 \end{cases}$$

$$f(h_i) = \begin{cases} 3, & i = 1 \\ n+4+i, & 2 \leq i \leq m \end{cases}$$

$$f(w_i) = \begin{cases} 1, & i = 1 \\ i+3, & 2 \leq i \leq n \end{cases}$$

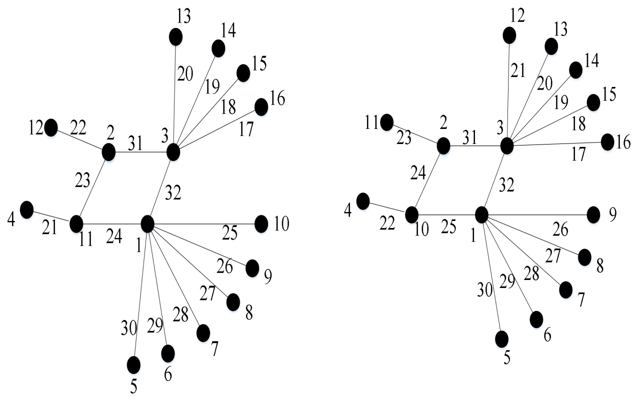
$$f(V) = \{1,2,3,4\} \cup \{5,6,\dots,n+3\} \cup \{n+4\} \cup \{n+5\} \cup \{n+6, n+7, \dots, n+m+4\}$$

The set $S = \{f(u)+f(v) | uv \in E(G)\} = \{4,5,\dots,n+4\} \cup \{n+5, n+6, n+7, n+8\} \cup \{n+9, n+10, \dots, n+m+7\}$.

$s = \min(S) = f(h_1) + f(w_1) = 1+3 = 4$.

According to the lemma 1, we know that $k = p+q+s = (m+n+4)+(m+n+4)+4 = 2m+2n+12$.

Because the set S consists of q consecutive integers, f is the super edge-magic total labeling of graphs $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_m \downarrow S_n$. Some examples are shown in Fig.12.



(1) $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_5 \downarrow S_7$ (2) $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_6 \downarrow S_6$

Fig.12. Super edge-magic total labeling of $C_4 \downarrow S_2 \downarrow S_2 \downarrow S_m \downarrow S_n$

Theorem 4: Unicyclic graphs $C_n \downarrow S_m$ have EMTL when $3 \le n \le 6$ and $m \ge 3$.

Proof: Let $\{u_1, u_2, ..., u_n\}$ and $\{v_1, v_2, ..., v_m\}$ be two vertex sets of C_n and S_m.

(1) When $n = 3$, the structure of $C_3 \downarrow S_m$ is as shown in Fig.13(1).

We know that

$$|V(G)| = |E(G)| = m + 3$$

So, $k \in [2m + 9, 4m + 12]$, when $k = 2m + 12$, the labels are obtained as follows.

The vertex labels of $C_3 \downarrow S_m$ are

$$f(u_i) = \begin{cases} 2m+5, \ i = 1 \\ 2, \ i = 2 \\ 4, \ i = 3 \end{cases}$$

$$f(v_i) = \begin{cases} 1, \ \mathrm{i} = 1 \\ i+5, \ 2 \le i \le m \end{cases}$$

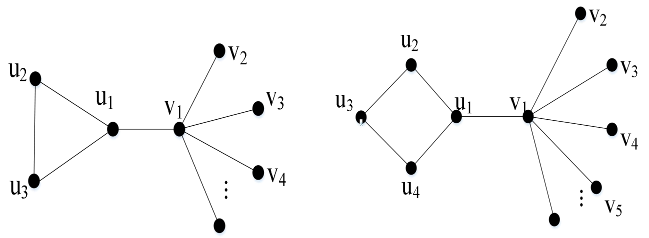
We know that $f(V) = \{1,2,4\} \cup \{7,8,\dots, m+5\} \cup \{2m+5\}$.

The set $S = \{f(u) + f(v) | uv \in E(G)\} = \{2m+6, 2m+7, 6, 2m+9\} \cup \{8,9,\dots, m+6\}$.

According to the definition 1, the label of an edge is equal to k minus the sum of the labels of its two vertices. Therefore, the edge label set of $C_3 \downarrow S_m$ is:

$$f(E) = \{6,5,2m+6,3\} \cup \{2m+4, 2m+3, ..., m+6\}$$

Since $f(V) \cup f(E) \to [1, 2m+6]$ and $f(V) \cap f(E) = \emptyset$, graphs $C_n \downarrow S_m$ have edge-magic total labeling.



(1) $C_3 \downarrow S_m$ (2) $C_4 \downarrow S_m$

Fig.13. $C_3 \downarrow S_m$ and $C_4 \downarrow S_m$

(2) When $n = 4$, the structure of $C_4 \downarrow S_m$ is as shown in Fig.13(2).

We know that

$$|V(G)| = |E(G)| = m + 4$$

So, $k \in [2m + 11, 4m + 16]$, when $k = 2m + 12$, we obtain the labels as follows.

The vertex labels of $C_4 \downarrow S_m$ are

$$f(u_i) = \begin{cases} 2m+6, \ i = 1 \\ 2, \ i = 2 \\ 4, \ i = 3 \\ 3, \ i = 4 \end{cases}$$

$$f(v_i) = \begin{cases} 1, \ i = 1 \\ i+6, \ 2 \le i \le m \end{cases}$$

Likewise, we know that $f(V) = \{1,2,3,4\} \cup \{8,9,\dots, m+6\} \cup \{2m+6\}$;

So we get $S = \{f(u) + f(v) | uv \in E(G)\} = \{2m+7, 2m+8, 6, 7, 2m+9\} \cup \{9, 10, \dots, m+7\}$;

$f(E) = \{7, 6, 2m+8, 2m+7, 5\} \cup \{2m+5, 2m+4, \dots, m+7\}$.

Since $f(V) \cup f(E) \to [1, 2m+8]$ and $f(V) \cap f(E) = \emptyset$, graph $C_4 \downarrow S_m$ have edge-magic total labeling.

(3) When $n = 5$, the structure of $C_5 \downarrow S_m$ is as shown in Fig.14(1).

We know that

$$|V(G)| = |E(G)| = m + 5$$

So, $k \in [2m + 13, 4m + 20]$, when $k = 2m + 16$, we obtain the labels as follows.

The vertex labels of $C_5 \downarrow S_m$ are

$$f(u_i) = \begin{cases} 2m+6, \ i = 1 \\ 3, \ i = 2 \\ 5, \ i = 3 \\ 2, \ i = 4 \\ 4, \ i = 5 \end{cases}$$

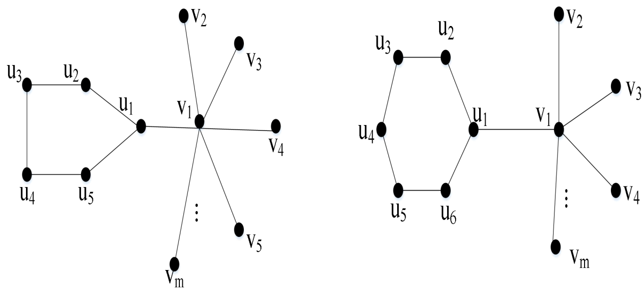
$$f(v_i) = \begin{cases} 1, \ i = 1 \\ 8, \ i = 2 \\ i+7, \ 3 \le i \le m \end{cases}$$

Likewise, we know that $f(V) = \{2m+6, 3, 5, 2, 4, 1, 8\} \cup \{10, 11, \dots, m+7\}$;

$S = \{f(u) + f(v) | uv \in E(G)\} = \{2m+7, 2m+9, 8, 7, 6, 2m+10, 9\} \cup \{11, 12, \dots, m+8\}$;

$f(E) = \{9, 7, 2m+8, 2m+9, 2m+10, 6, 2m+7\} \cup \{2m+5, 2m+4, \dots, m+8\}$.

Since $f(V) \cup f(E) \to [1, 2m+10]$ and $f(V) \cap f(E) = \emptyset$, graph $C_5 \downarrow S_m$ have edge-magic total labeling.



(1) $C_5 \downarrow S_m$ (2) $C_6 \downarrow S_m$

Fig.14. $C_5 \downarrow S_m$ and $C_6 \downarrow S_m$

(4) When $n = 6$, the structure of $C_6 \downarrow S_m$ is as shown in Fig.14(2).

We know that

$$|V(G)| = |E(G)| = m + 6$$

So, $k \in [2m + 15, 4m + 24]$, when $k = 2m + 18$, we obtain the labels as follows.

The vertex labels of $C_6 \downarrow S_m$ are

$$f(u_i) = \begin{cases} m+6, \ i = 1 \\ 3, \ i = 2 \\ 5, \ i = 3 \\ m+5, \ i = 4 \\ 6, \ i = 5 \\ 1, \ i = 6 \end{cases}$$

$$f(v_i)=\begin{cases}2,\ i=1\\4,\ i=2\\i+4,\ 3\le i\le m\end{cases}$$

Similarly, we know that $f(V)=\{m+6,3,5,m+5,6,1,2,4\}\cup\{7,8,\dots,m+4\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{m+7,m+8,m+9,m+10,m+11,7,8,6\}\cup\{9,10,\dots,m+6\}$;

$f(E)=\{m+11,m+10,m+9,m+8,m+7,2m+11,2m+10,2m+12\}\cup\{2m+9,2m+8,\dots,m+13\}$.

Since $f(V)\cup f(E)\to[1,2m+12]$ and $f(V)\cap f(E)=\emptyset$, graphs $C_6\downarrow \mathrm{S}_m$ have edge-magic total labeling.

Theorem 5: Unicyclic graphs $C_n\uparrow \mathrm{S}_m$ have EMTL when $3\le n\le 6$ and $m\ge 3$.

Proof: Let the vertex sets of C_n and S_m be $\{u_1,u_2,\dots,u_n\}$ and $\{v_1,v_2,\dots,v_m\}$.

(1) When $n=3$, the structure of $C_3\uparrow \mathrm{S}_m$ is as shown in Fig.15(1).

We know that

$$|V(G)|=|E(G)|=3+(m-1)\ =\ m+2$$

So, $k\in[2m+7,4m+8]$, when $k=2m+7$, we get the labels as follows.

The vertex labels of $C_3\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\2,\ i=2\\m+2,\ i=3\end{cases}$$

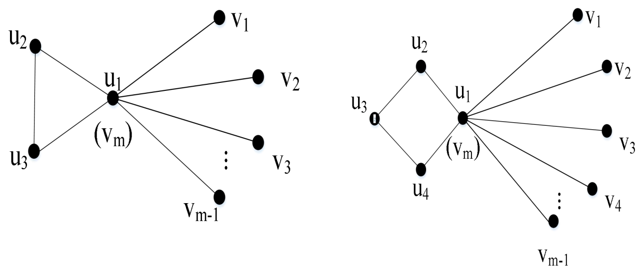
$$f(v_i)=i+2,\quad 1\le i\le m-1$$

Similarly, we obtain $f(V)=\{1,2,m+2\}\cup\{3,4,\dots,m+1\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{3,m+4,m+3\}\cup\{4,5,\dots,m+2\}$;

$f(E)=\{2m+4,m+3,m+4\}\cup\{2m+3,2m+2,\dots,m+5\}$.

Since $f(V)\cup f(E)\to[1,2m+4]$ and $f(V)\cap f(E)=\emptyset$, graphs $C_3\uparrow \mathrm{S}_m$ have edge-magic total labeling.



(1) $C_3\uparrow \mathrm{S}_m$ (2) $C_4\uparrow \mathrm{S}_m$

Fig.15. $C_3\uparrow \mathrm{S}_m$ and $C_4\uparrow \mathrm{S}_m$

(2) When $n=4$, the structure of $C_4\uparrow \mathrm{S}_m$ is as shown in Fig.15(2).

We know that

$|V(G)|=|E(G)|=4+(m-1)=[2m+9,4m+12]$, when $k=2m+10$, we get the labels as follows:

The vertex labels of $C_4\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\3,\ i=2\\2,\ i=3\\2m+4,\ i=4\end{cases}$$

$$f(v_i)=i+5,\quad 1\le i\le m-1$$

Likewise, we know that $f(V)=\{1,3,2,2m+4\}\cup\{6,7,\dots,m+4\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{4,5,2m+6,2m+5\}\cup\{7,8,\dots,m+5\}$;

$f(E)=\{2m+6,2m+5,4,5\}\cup\{2m+3,2m+2,\dots,m+5\}$.

Since $f(V)\cup f(E)\to[1,2m+6]$ and $f(V)\cap f(E)=\emptyset$, graphs $C_4\uparrow \mathrm{S}_m$ have edge-magic total labeling.

(3) When $n=5$, the structure of $C_5\uparrow \mathrm{S}_m$ is as shown in Fig.16(1).

We know that

$$|V(G)|=|E(G)|=5+(m-1)=m+4$$

So, $k\in[2m+11,4m+16]$, when $k=2m+12$, we get the labels as follows.

The vertex labels of $C_5\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\3,\ i=2\\m+4,\ i=3\\2,\ i=4\\m+4,\ i=5\end{cases}$$

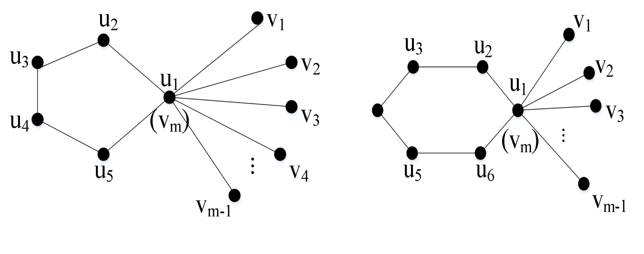
$$f(v_i)=i+3,\quad 1\le i\le m-1$$

Likewise, we know that $f(V)=\{1,3,m+4,2,m+3\}\cup\{4,5,\dots,m+2\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{4,m+7,m+6,m+5,m+4\}\cup\{5,6,\dots,m+3\}$;

$f(E)=\{2m+8,m+5,m+6,m+7,m+8\}\cup\{2m+7,2m+6,\dots,m+9\}$.

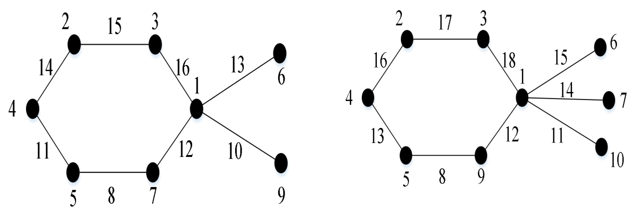
Since $f(V)\cup f(E)\to[1,2m+8]$ and $f(V)\cap f(E)=\emptyset$, graphs $C_5\uparrow \mathrm{S}_m$ have edge-magic total labeling.



(1) $C_5\uparrow S_m$ (2) $C_6\uparrow S_m$

Fig.16. $C_5\uparrow \mathrm{S}_m$ and $C_6\uparrow \mathrm{S}_m$

(4) When $n=6$, EMTL results of $C_6\uparrow \mathrm{S}_3$ and $C_6\uparrow \mathrm{S}_4$ are as shown in Fig.17(1) and 17(2) respectively.



(1) $C_6\uparrow S_3$ (2) $C_6\uparrow S_4$

Fig.17. The EMTL of graph $C_6\uparrow S_3$ and $C_6\uparrow S_4$

When $n=6$, $m\ge 5$, the structure of $C_6\uparrow \mathrm{S}_m$ is as shown in Fig.16(2).

We know that

$$|V(G)|=|E(G)|=6+(m-1)=m+5$$

So, $k\in[2m+13,4m+20]$, especially when $k=2m+14$, we get the labels as follows.

The vertex labels of $C_6\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\3,\ i=2\\2,\ i=3\\4,\ i=4\\5,\ i=5\\2m+1,\ i=6\end{cases}$$

$$f(u_i)=\begin{cases}6,\ i=1\\7,\ i=2\\9,\ i=3\\10,\ i=4\\11,\ i=5\\i+7, 6\le i\le m-1\end{cases}$$

Likewise, we know that $f(V)=\{1,2,3,4,5,6,7,9,10,11\}\cup\{13,14,\dots,\mathrm{m}+6\}\cup\{2m+1\}$;
$S=\{f(u)+f(v)|uv\in E(G)\}=\{4,5,6,7,8,910,11,12\}\cup\{14,15,\dots,m+7\}\cup\{2\mathrm{m}+2,2\mathrm{m}+6\}$;
$f(E)=\{2m+2,2m+3,\dots,2m+10\}\cup\{8,12\}\cup\{\mathrm{m}+7,\mathrm{m}+8,\dots,2\mathrm{m}\}$.
Since $f(V)\cup f(E)\to[1,2m+10]$ and $f(V)\cap f(E)=\emptyset$, graphs $C_6\uparrow \mathrm{S}_m$ have edge-magic total labeling.

Theorem 6: Unicycle graphs $C_n\uparrow \mathrm{S}_m$ have SEMTL when $7\le n\le 9$ and $m\ge 3$.
Proof: Let the vertex sets of C_n and S_m be $\{u_1,u_2,\dots,u_n\}$ and $\{v_1,v_2,\dots,v_m\}$.
(1) When $n=7$, the structure of $C_7\uparrow \mathrm{S}_m$ is as shown in Fig.18(1).
We know that

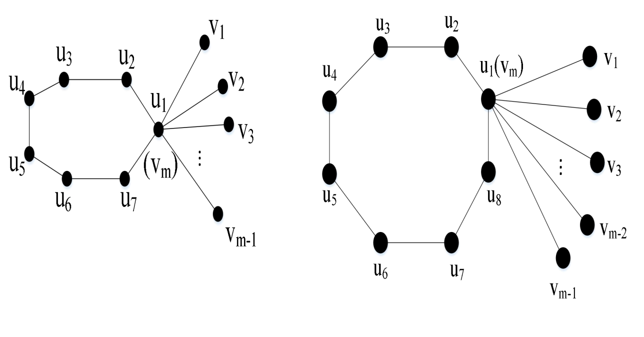
$$|V(G)|=|E(G)|=7+(m-1)=m+6$$

So, $k\in[2m+15,4m+24]$, when $k=2m+17$, we get the labels as follows:
The vertex labels of $C_7\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\\mathrm{m}+4,\ i=2\\2,\ i=3\\3,\ i=4\\\mathrm{m}+5,\ i=5\\4,\ i=6\\m+6,\ i=7\end{cases}$$

$$f(v_i)=i+4,\quad 1\le i\le m-1$$

Similarly, we know that $f(V)=\{1,2,3,4\}\cup\{5,6,\dots,\mathrm{m}+3\}\cup\{m+4,\mathrm{m}+5,\mathrm{m}+6\}$;
$S=\{f(u)+f(v)|uv\in E(G)\}=\{5,6,7,\dots,\mathrm{m}+4\}\cup\{\mathrm{m}+5,\mathrm{m}+6,\mathrm{m}+7,\mathrm{m}+8,\mathrm{m}+9,\mathrm{m}+10\}$;
$f(E)=\{2m+12,2m+11,\dots,m+13\}\cup\{\mathrm{m}+12,\mathrm{m}+11,\dots,\mathrm{m}+7\}$.
Since $f(V)\to[1,m+6]$ and $f(E)\to[\mathrm{m}+7,2m+12]$, graphs $C_7\uparrow \mathrm{S}_m$ have super edge-magic total labeling.

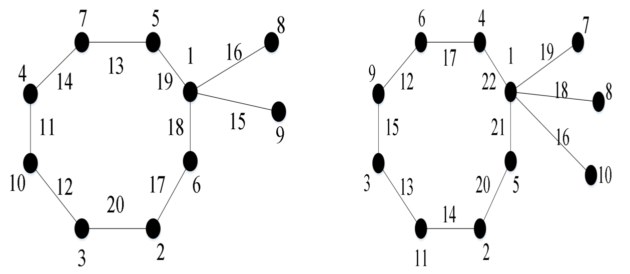


(1) $C_7\uparrow S_m$ (2) $C_8\uparrow S_m$
Fig.18. $C_7\uparrow \mathrm{S}_m$ and $C_8\uparrow \mathrm{S}_m$

(2) When $n=8$, EMTL results of graph $C_8\uparrow S_3$ and $C_8\uparrow S_4$ are as shown in Fig.19(1) and 19(2).
When $n=8$, $m\ge 5$, the structure of $C_8\uparrow \mathrm{S}_m$ is as shown in Fig.18(2).
We know that

$$|V(G)|=|E(G)|=8+(m-1)=m+7$$

So, $k\in[2m+17,4m+28]$, when $k=2m+19$, we get the labels as follows.



(1) $C_8\uparrow S_3$ (2) $C_8\uparrow S_4$
Fig.19. $C_8\uparrow \mathrm{S}_3$ and $C_8\uparrow \mathrm{S}_4$

The vertex labels of $C_8\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\5,\ i=2\\m+5,\ i=3\\3,\ i=4\\2,\ i=5\\m+7,\ i=6\\4,\ i=7\\m+2,\ i=8\end{cases}$$

$$f(u_i)\to\{\mathrm{m}+6,\mathrm{m}+4,\mathrm{m}+3\}\cup\{\mathrm{m}+1,\mathrm{m},\dots,6\}$$

Likewise, we know that $f(V)=\{1,2,3,4,5,6,\dots,\mathrm{m}+1\}\cup\{m+2,\mathrm{m}+3,\dots,\mathrm{m}+7\}$;
$S=\{f(u)+f(v)|uv\in E(G)\}=\{5,6,7,\dots,\mathrm{m}+7\}\cup\{\mathrm{m}+8,\mathrm{m}+9,\mathrm{m}+10,\mathrm{m}+11\}$;
$f(E)=\{2m+14,2m+13,\dots,m+12\}\cup\{\mathrm{m}+11,\mathrm{m}+10,\mathrm{m}+9,\mathrm{m}+8\}$.
Since $f(V)\to[1,m+7]$ and $f(E)\to[\mathrm{m}+8,2m+14]$, graphs $C_8\uparrow \mathrm{S}_m$ have super edge-magic total labeling.

(3) When $n=9$, the structure of $C_9\uparrow \mathrm{S}_m$ is as shown in Fig.20.

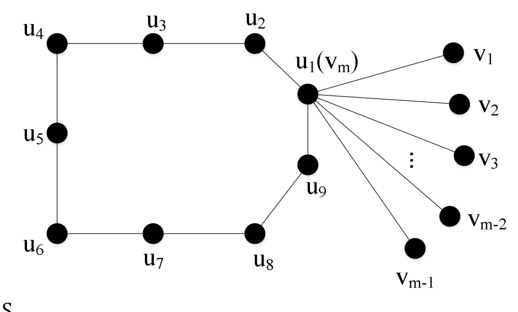


Fig.20. $C_9\uparrow \mathrm{S}_m$

We know that

$$|V(G)|=|E(G)|=9(m-1)=m+8$$

So, $k\in[2m+19,4m+32]$, especially, when $k=2m+22$, we get the labels as follows.
The vertex labels of $C_9\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\ m+5,\ i=2\\ 2,\ i=3\\ 4,\ i=4\\ \mathrm{m}+6,\ i=5\\ 3,\ i=6\\ m+8,\ i=7\\ 5,\ i=8\\ m+7,\ i=9\end{cases}$$

$$f(v_i)=i+5,\quad 1\le i\le m-1$$

Similarly, we know that $f(V)=\{1,2,3,4,5,6,\dots,\mathrm{m}+4\}\cup\{m+5,\mathrm{m}+6,\mathrm{m}+7,\mathrm{m}+8\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{6,7,\dots,\mathrm{m}+5\}\cup\{\mathrm{m}+6,\mathrm{m}+7,\dots,\mathrm{m}+13\}$;

$f(E)=\{2m+16,2m+15,\dots,m+17\}\cup\{\mathrm{m}+16,\mathrm{m}+15,\dots,\mathrm{m}+9\}$.

Since $f(V)\to[1,m+8]$ and $f(E)\to[\mathrm{m}+9,2m+16]$, graphs $C_9\uparrow \mathrm{S}_m$ have super edge-magic total labeling.

Theorem 7: Unicycle graphs $P_2\uparrow C_n\uparrow \mathrm{S}_m$ which n and m satisfying $3\le n\le 7$, $m\ge 3$. When n is odd, they have SEMTL; when n is even, they have EMTL.

Proof:

(1) When $n=3$, the structure of $P_2\uparrow C_3\uparrow \mathrm{S}_m$ is as shown in Fig.21(1).

We know that

$$|V(G)|=|E(G)|=1+3+(m-1)=m+3$$

So, $k\in[2m+9,4m+12]$, when $k=2m+9$, we obtain the labels as follows.

The vertex labels of $P_2\uparrow C_3\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\ m+2,\ i=2\\ 2,\ i=3\end{cases}$$

$$f(v_i)=i+1,\quad 2\le i\le m$$

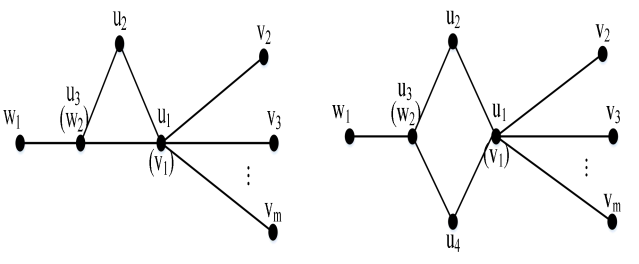
$$f(w_1)=m+3$$

Likewise, we know that $f(V)=\{1,2,3,4,\dots,\mathrm{m}+1\}\cup\{m+2,\mathrm{m}+3\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{3,4,5,\dots,\mathrm{m}+2\}\cup\{\mathrm{m}+3,\mathrm{m}+4,\mathrm{m}+5\}$;

$f(E)=\{2m+6,2m+5,2\mathrm{m}+4,\dots,m+7\}\cup\{\mathrm{m}+6,\mathrm{m}+5,\dots,\mathrm{m}+4\}$.

Since $f(V)\to[1,m+3]$ and $f(E)\to[\mathrm{m}+4,2m+6]$, graphs $P_2\uparrow C_3\uparrow \mathrm{S}_m$ have super edge-magic total labeling.



(1) $P_2\uparrow C_3\uparrow S_m$ (2) $P_2\uparrow C_4\uparrow S_m$

Fig.21. $P_2\uparrow C_3\uparrow S_m$ and $P_2\uparrow C_4\uparrow S_m$

(2) When $n=4$, the structure of $P_2\uparrow C_4\uparrow \mathrm{S}_m$ is as shown in Fig.21(2).

We know that

$$|V(G)|=|E(G)|=1+4+(m-1)=m+4$$

So, $k\in[2m+11,4m+16]$, when $k=2m+12$, we obtain the labels as follows:

The vertex labels of $P_2\uparrow C_4\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\ 3,\ i=2\\ 2,\ i=3\\ 2m+5,\ i=4\end{cases}$$

$$f(v_i)=i+5,\quad 2\le i\le m$$

$$f(w_1)=4$$

Similarly, we know that $f(V)=\{1,2,3,4\}\cup\{7,8,\dots,m+5\}\cup\{2m+5\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{4,5,6\}\cup\{8,9,\dots,\mathrm{m}+6\}\cup\{2\mathrm{m}+6,2\mathrm{m}+7\}$;

$f(E)=\{2m+8,2m+7,2\mathrm{m}+6\}\cup\{2\mathrm{m}+4,2\mathrm{m}+3,\dots,\mathrm{m}+6\}\cup\{6,5\}$.

Because $f(V)\cup f(E)\to[1,2m+8]$ and $f(V)\cap f(E)=\emptyset$, graphs $P_2\uparrow C_3\uparrow \mathrm{S}_m$ have edge-magic total labeling.

(3) When $n=5$, the structure of $P_2\uparrow C_5\uparrow \mathrm{S}_m$ is as shown in Fig.22(1).

We know that

$$|V(G)|=|E(G)|=1+5+(m-1)=m+5$$

So, $k\in[2m+13,4m+20]$, when $k=2m+14$, we obtain the labels as follows.

The vertex labels of $P_2\uparrow C_5\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\ m+3,\ i=2\\ 2,\ i=3\\ m+5,\ i=4\\ 3,\ i=5\end{cases}$$

$$f(v_i)=i+2,\quad 2\le i\le m$$

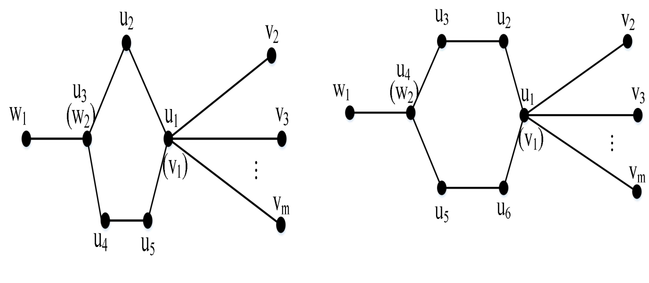
$$f(w_1)=m+4$$

Likewise, we know that $f(V)=\{1,2,3\}\cup\{4,5,\dots,m+2\}\cup\{m+3,m+4,m+5\}$;

$S=\{f(u)+f(v)|uv\in E(G)\}=\{4,5,6,\dots,m+3\}\cup\{m+4,m+5,\dots,m+8\}$,

$f(E)=\{2m+10,2m+9,\dots,m+11\}\cup\{m+10,m+9,\dots,m+6\}$.

Since $f(V)\to[1,m+5]$ and $f(E)\to[\mathrm{m}+6,2m+10]$, graphs $P_2\uparrow C_5\uparrow \mathrm{S}_m$ have super edge-magic total labeling.



(1) $\mathrm{P}_2\uparrow \mathrm{C}_5\uparrow \mathrm{S}_m$ (2) $\mathrm{P}_2\uparrow \mathrm{C}_6\uparrow \mathrm{S}_m$

Fig.22. $P_2\uparrow C_5\uparrow S_m$ and $P_2\uparrow C_6\uparrow S_m$zz

(4) When $n=6$, the structure of $P_2\uparrow C_6\uparrow \mathrm{S}_m$ is as shown in Figure 22(2).

We know that

$$|V(G)|=|E(G)|=1+6+(m-1)=m+6$$

So, $k\in[2m+15,4m+24]$, when $k=2m+17$, we obtain the labels as follows.

The vertex labels of $P_2\uparrow C_6\uparrow \mathrm{S}_m$ are

$$f(u_i)=\begin{cases}1,\ i=1\\ 2m+9,\ i=2\\ 3,\ i=3\\ 2,\ i=4\\ 4,\ i=5\\ m+4,\ i=6\end{cases}$$

$$f(w_1) = m + 7$$

According to the definition 1, the label of an edge is equal to k minus the sum of the labels of its two vertices. Therefore, some edge labels of $P_2 \uparrow C_6 \uparrow S_m$ are as follows:

$$f(u_1u_2) = k - (2m + 10) = 7$$
$$f(u_2u_3) = k - (2m + 12) = 5$$
$$f(u_3u_4) = k - 5 = 2m + 12$$
$$f(u_4u_5) = k - 6 = 2m + 11$$
$$f(u_5u_6) = k - (m + 8) = m + 9$$
$$f(u_6u_1) = k - (m + 5) = m + 12$$
$$f(w_1w_2) = k - (m + 9) = m + 8$$

According to definition 1, $f(V) \cup f(E)$ should belong to $[1,2m + 12]$.

Then the labels of the rest of the vertices $v_2, v_3,\ldots, v_m$ are the smallest remaining number belonging to $[1,2m + 12]$.

From the above argument, we know that $f(V) \cup f(E) \to [1,2m + 12]$ and $f(V) \cap f(E) = \emptyset$, hence, graphs $P_2 \uparrow C_6 \uparrow S_m$ have edge-magic total labeling.

(5) When $n = 7$, the structure of $P_2 \uparrow C_7 \uparrow S_m$ is as shown in Fig.23.

We know that

$$|V(G)| = |E(G)| = 1 + 7 + (m - 1) = m + 7$$

So, $k \in [2m + 17,4m + 28]$, when $k = 2m + 19$, we get the labels as follows:

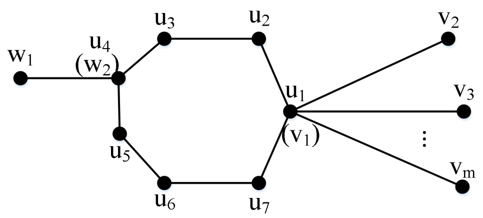


Fig.23. $P_2 \uparrow C_6 \uparrow S_m$

The vertex labels of $P_2 \uparrow C_7 \uparrow S_m$ are

$$f(u_i) = \begin{cases} 1,\ i = 1 \\ 5,\ i = 2 \\ m + 6,\ i = 3 \\ 2,\ i = 4 \\ m + 7,\ i = 5 \\ 3,\ i = 6 \\ m + 3,\ i = 7 \end{cases}$$

$$f(w_1) = \mathrm{m} + 5$$

Then the labels of the rest of the vertices $v_2,v_3,\ldots,v_m$ are the remaining numbers in $[1, m + 7]$.

Since $f(V) \to [1, m + 7]$ and $f(E) \to [\mathrm{m} + 8,2m + 14]$, graphs $P_2 \uparrow C_7 \uparrow S_m$ have super edge-magic total labeling.

IV. CONCLUSION

The algorithm designed in this paper is currently limited to calculating the EMTL and SEMTL of unicyclic graphs with vertices less than or equal to 16. From Table 4, it can be seen that the total number of unicyclic graphs within 16 vertices is 482977; 482714 of them are super edge-magic total labeling graphs, accounting for 99.946% of the total number of graphs.

Some theorems about unicyclic graphs are also deduced from the algorithm's results.

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