A New Class of Generalized Hermite Based Apostol Type Polynomials

Saniya Batra, Prakriti Rai

***Abstract*—In recent years, various mathematicians (such as Ernst, U. Duran) introduced an extension of Apostol Type polynomials of order α. Recently, W. A. Khan introduced a new class of q- Hermite based Apostol type polynomials. Motivated by their research, this article introduces a new class of (p,q)-analogue of Hermite based Apostol type polynomials of order α and investigate its characteristics.In particular, it establishes the generating function, series expression and explicit relation for these polynomials. It also explores the relationship between generalized Bernoulli, Euler and Genocchi polynomials.**

***Index Terms*—(p,q)-Calculus, (p,q)-Hermite based Apostol polynomials, Generating function, Cauchy product.**

I. Introduction

QUANTUM calculus is an old area of research that holds its applications widely in the field of physics, mathematics and engineering sciences. In 2011, Kim [1] derived various identities based on q Bernstein and q-Hermite polynomials. Later in 2014, Ernst [2] introduced q-Apostol type numbers of order α. Very recently, Khan [3] introduced generalized q-Hermite based Apostol type polynomials. Motivated by the fundamental importance of q-calculus in numerous areas, the theory of Post Quantum Calculus, also known as (p,q)-Calculus, was developed. It is the extension of q-Calculus. It should be noted that (p,q)-numbers cannot be derived by swapping q with q/p in q-numbers. However, q-calculus can be derived from (p,q)-calculus by substituting $p=1$.

For $p \neq q$, the (p,q)-numbers are defined as:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}. \tag{1}$$

Also, $[n]_{p,q} = p^{n-1}[n]_{q/p}$.

It should be noted that $[n]_{1,q} = [n]_q$ are called q-numbers. For $n \geq 1$, the (p,q)-factorial is defined by:

$$[n]_{p,q}! = \prod_{m=1}^{n} [m]_{p,q}\,, \tag{2}$$

with initial condition $[0]_{p,q}! = 1$.

For $n \in \mathbb{N}$, the (p,q)-binomial coefficient, is given by:

$$\binom{n}{m}_{p,q} = \frac{[n]_{p,q}!}{[m]_{p,q}![n-m]_{p,q}!}, \quad 0 \leq m \leq n.$$

Consider $p=q=1$, the (p,q)-binomial coefficient reduces to standard binomial coefficient $\binom{n}{k}$.

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The two types of (p,q)-exponential functions (see [4], [5]) are defined by:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \tag{3}$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}, \tag{4}$$

where $0 < \left|\frac{q}{p}\right| < 1$; $|x| < 1$.

The above-mentioned exponential functions satisfy the following relation:

$$e_{p,q}(x)E_{p,q}(x) = 1.$$

For $x \neq 0$, the (p,q)-derivative is given as:

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}\,. \tag{5}$$

The Hermite polynomial is (see [6]) defined as:

$$H_n(x) = \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r n!}{r!(n-2r)!}(2x)^{n-2r}\,.$$

The generating function of these polynomials is given by:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}\,. \tag{6}$$

The q-analogue of Hermite polynomial, known as q-Hermite polynomial (see [1]), is defined as:

$$H_{n,q}(x) = n! \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l 2^{n-2l}[x]_q^{n-2l}}{l!(n-2l)!}\,.$$

The generating function of these polynomials is given by:

$$e^{2t[x]_q - t^2} = \sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{n!}\,.$$

A new q-analogue of Hermite polynomial was introduced by Mahmudov (see [7], [3]) as follows:

$$F_q(x,t) = F_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_{n,q}(x,t)\frac{t^n}{[n]_q!}\,, \tag{7}$$

where $F_q(t) = \sum\limits_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{t^{2n}}{[2n]_q!!}$ and $[2n]_q!! = [2n]_q[2n-2]_q \ldots [2]_q$.

Apostol type polynomials were first introduced by Apostol (see [8]) and investigated by Srivastava (see [9]). Subsequently, Luo (see [10], [11]) established the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α.

Further, Luo et al. (see [12], [13]) derived the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x)$ and generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x)$, both of order α.

Later, in 2014 Ernst (see [2]) introduced the generalized q-Apostol type polynomials of order α using the following generating function:

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha e_q(xt)=\sum_{n=0}^{\infty}F_{n,q}^{\alpha}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, \quad (8)$$

where $\alpha\in\mathbb{N}_0$, $\lambda,a,b\in\mathbb{C}$, $|t|<|\log(-\lambda)|$.

It should be noted that $F_{n,q}^{\alpha}(a,b;\lambda)=F_{n,q}^{\alpha}(0;a,b;\lambda)$ are known as q-Apostol-type numbers of order α.

Consider $\mu=0$ and $\nu=1$ in (8), the generalized q-Apostol-Bernoulli polynomial of order α is defined by the following generating function:

$$\left(\frac{t}{\lambda e_q(t)-1}\right)^\alpha e_q(xt)=\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{\alpha}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, \quad (9)$$

where $\alpha\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, $|t|<|\log(-\lambda)|$.

Take $\mu=1$ and $\nu=0$ in (8), the generalized q-Apostol-Euler polynomial of order α is defined by the following generating function:

$$\left(\frac{2}{\lambda e_q(t)+1}\right)^\alpha e_q(xt)=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{\alpha}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, \quad (10)$$

where $\alpha\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, $|t|<|\log(-\lambda)|$.

Let $\mu=1$ and $\nu=1$ in (8), the generalized q-Apostol-Genocchi polynomial of order α is defined by the following generating function:

$$\left(\frac{2t}{\lambda e_q(t)+1}\right)^\alpha e_q(xt)=\sum_{n=0}^{\infty}\mathcal{G}_{n,q}^{\alpha}(x;a,b;\lambda)\frac{t^n}{[n]_q!}, \quad (11)$$

where $\alpha\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, $|t|<|\log(-\lambda)|$.

Recently, Khan et al. introduced generalized q-Hermite based Apostol type polynomials (see [3]) with the following generating function:

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha F_q(t)e_q(xt) =\sum_{n=0}^{\infty}{}_HF_{n,q}^{\alpha}(x;a,b;\lambda;\mu,\nu)\frac{t^n}{[n]_q!}, \quad (12)$$

where $\alpha\in\mathbb{N}_0$, $\lambda,\mu,\nu,a,b,q,p\in\mathbb{C}$, $0<|q|<|p|<1$ and $|t|<|\log(-\lambda)|$.

Inspired by their research, this paper derives generalized (p,q)-Hermite-based Apostol type polynomials and investigate their properties.

II. (p,q)-Hermite polynomial

This section derives a (p,q)-Hermite polynomials and discusses few exceptional cases.

The generating function for generalized (p,q)-analogue of Hermite polynomial is defined as:

$$F_{p,q}(x,t)=F_{p,q}(t)e_{p,q}(xt) =\sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}, \quad (13)$$

where $0<|q|<|p|<1$, $n\in\mathbb{N}_0$, $q,p\in\mathbb{C}$, and

$$F_{p,q}(t)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}p^{\frac{n(n+1)}{2}}t^{2n}}{[2n]_{p,q}!!}. \quad (14)$$

Remark 1. *Consider $p=1$, equation (13) reduces to the generalized q-polynomial (see [3])*

$$F_q(x,t)=F_q(t)e_q(xt)=\sum_{n=0}^{\infty}H_{n,q}(x)\frac{t^n}{[n]_q!}, \quad (15)$$

where $0<|q|<|p|<1$, $n\in\mathbb{N}_0$, $q,p\in\mathbb{C}$, and

$$F_q(t)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}t^{2n}}{[2n]_{p,q}!!}. \quad (16)$$

Remark 2. *Consider $p=1$ and $q\to1^-$ then*

$$\lim_{q\to1^-}F_q(x,t)=e^{xt-\frac{t^2}{2}}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}.$$

Theorem 1. *The explicit expression of generalized (p,q)-Hermite polynomial is given by:*

$$H_{n,p,q}(x)=\sum_{m=0}^{[\frac{n}{2}]}\frac{\begin{pmatrix}(-1)^n[n]_{p,q}!q^{\frac{m(m-1)}{2}}\\ \times p^{\frac{m(m+1)+(n-2m)(n-2m-1)}{2}}\\ \times x^{n-2m}\end{pmatrix}}{[2m]_{p,q}!!\,[n-2m]_{p,q}!}, \quad (17)$$

where $0<|q|<|p|<1$, $n\in\mathbb{N}_0$, $q,p\in\mathbb{C}$.

Proof: Using (16) and (3) in (13) we get,

$$\sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} =\sum_{m=0}^{\infty}\frac{(-1)^n q^{\frac{m(m-1)}{2}}p^{\frac{m(m+1)}{2}}t^{2m}}{[2m]_{p,q}!!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{(xt)^n}{[n]_{p,q}!}. \quad (18)$$

On using the Cauchy product (18) transforms as follows:

$$\sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} =\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{(-1)^n q^{\frac{m(m-1)}{2}}p^{\frac{m(m+1)+(n-2m)(n-2m-1)}{2}}t^{2m}}{[2m]_{p,q}!!} \times\frac{x^{n-2m}t^{n-2m}}{[n-2m]_{p,q}!}. \quad (19)$$

Now equating the coefficients of t^n in (19), we get the desired result. ∎

III. (p,q)-Apostol Type Polynomials

This section defines the (p,q)-analogue of Apostol type polynomials and investigates certain exceptional cases.

For $q,p \in \mathbb{C}$ and $0 < |q| < |p| < 1$, $\alpha, n \in \mathbb{N}_0$, $a,b \in \mathbb{R}\setminus\{0\}$, $\beta,\mu,\nu \in \mathbb{C}$ the generating function for generalized (p,q)-Apostol type polynomial of order α is defined as follows:

$$\left(\frac{2^{\mu}t^{\nu}}{\beta e_{p,q}(t)+a^{b}}\right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}. \tag{20}$$

The conditions of convergence of generalized (p,q)-Apostol type polynomial are:

(i) If $a^b > 0$ and $\beta = 1$ then $|t| < \pi$; $\beta \neq 1$, then $|t| < |\log(-\beta)|$, $1^{\alpha} = 1$.

(ii) If $a^b < 0$ and $\beta = 1$ then $|t| < 2\pi$; $\beta \neq 1$, then $|t| < |\log(\beta)|$, $1^{\alpha} = 1$.

For $x = 0$, $F^{\alpha}_{n,p,q}(0;a,b;\beta;\mu,\nu) = F^{\alpha}_{n,p,q}(a,b;\beta;\mu,\nu)$ are known as (p,q)-Apostol type numbers.

Remark 3. *Here we investigate few special cases of* $F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)$ *as follows:*

Case 1: Let $a = -1$, $b = 1$, $\mu = 0$ *and* $\nu = 1$ *in* (20)*, we get the generating function of generalized* (p,q)*-Apostol-Bernoulli polynomial of order* α *(see [14]) as follows:*

$$\left(\frac{t}{\beta e_{p,q}(t)-1}\right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} B^{\alpha}_{n,p,q}(x;\beta)\frac{t^n}{[n]_{p,q}!}, \tag{21}$$

where $\beta = 1$*, then* $|t| < 2\pi$*;* $\beta \neq 1$*, then* $|t| < |\log(\beta)|$.

Case 2: Assume $a = 1$, $b = 1$, $\mu = 1$ *and* $\nu = 0$ *in* (20)*, we get the generating function of generalized* (p,q)*-Apostol-Euler polynomial of order* α *(see [14]) as follows:*

$$\left(\frac{2}{\beta e_{p,q}(t)+1}\right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} \mathcal{E}^{\alpha}_{n,p,q}(x;\beta)\frac{t^n}{[n]_{p,q}!}, \tag{22}$$

where $\beta = 1$*, then* $|t| < \pi$*;* $\beta \neq 1$*, then* $|t| < |\log(-\beta)|$.

Case 3: Consider $a = 1$, $b = 1$, $\mu = 1$ *and* $\nu = 1$ *in* (20)*, we establish the generating function of generalized* (p,q)*-Apostol-Genocchi polynomial of order* α *(see [14]) as follows:*

$$\left(\frac{2t}{\beta e_{p,q}(t)+1}\right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} \mathcal{G}^{\alpha}_{n,p,q}(x;\beta)\frac{t^n}{[n]_{p,q}!}, \tag{23}$$

where $\beta = 1$*, then* $|t| < \pi$*;* $\beta \neq 1$*, then* $|t| < |\log(-\beta)|$.

Remark 4. *For* $p = 1$*, the equation* (20) *reduces to generalized q-Apostol type polynomial of order* α*, mentioned in* (8).

Remark 5. *For* $p = 1$ *and* $q \to 1^-$ *in* (20)*, the following cases hold:*

$$\lim_{q\to 1^-} F^{\alpha}_{n,1,q}(x;-1,1;\beta;0,1) = B^{\alpha}_{n}(x;\beta),$$
$$\lim_{q\to 1^-} F^{\alpha}_{n,1,q}(x;1,1;\beta;1,0) = \mathcal{E}^{\alpha}_{n}(x;\beta),$$
$$\lim_{q\to 1^-} F^{\alpha}_{n,1,q}(x;1,1;\beta;1,1) = \mathcal{G}^{\alpha}_{n}(x;\beta),$$

where $B^{\alpha}_{n}(x;\beta)$*,* $\mathcal{E}^{\alpha}_{n}(x;\beta)$ *and* $\mathcal{G}^{\alpha}_{n}t(x;\beta)$ *are defined as the generalization of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order* α *(see [11], [12], [13]).*

IV. (p,q)-Hermite based Apostol Type Polynomials

For $p,q \in \mathbb{C}$, $0 < |q| < |p| < 1$, $\alpha, n \in \mathbb{N}_0$, $a,b \in \mathbb{R}\setminus\{0\}$, $\beta,\mu,\nu \in \mathbb{C}$ the generating function of generalized (p,q)-Hermite based Apostol type polynomial is defined as follows:

$$\left(\frac{2^{\mu}t^{\nu}}{\beta e_{p,q}(t)+a^{b}}\right)^{\alpha} F_{p,q}(t) e_{p,q}(xt) = \sum_{n=0}^{\infty} {}_{H}F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}. \tag{24}$$

The conditions of convergence of generalized (p,q)-Hermite based Apostol type polynomial are:

(iii) If $a^b > 0$ and $\beta = 1$ then $|t| < \pi$; $\beta \neq 1$, then $|t| < |\log(-\beta)|$, $1^{\alpha} = 1$.

(iv) If $a^b < 0$ and $\beta = 1$ then $|t| < 2\pi$; $\beta \neq 1$, then $|t| < |\log(\beta)|$, $1^{\alpha} = 1$.

Remark 6. *For* $x = 0$ *in* (24) *we get,*

$${}_{H}F^{\alpha}_{n,p,q}(a,b;\beta;\mu,\nu) = {}_{H}F^{\alpha}_{n,p,q}(0;a,b;\beta;\mu,\nu), \tag{25}$$

where ${}_{H}F^{\alpha}_{n,p,q}(a,b;\beta;\mu,\nu)$ *are known as* (p,q)*-Hemite based Apostol type numbers of order* α.

Remark 7. *On setting* $\beta = 1$ *in* (24) *we have,*

$${}_{H}F^{\alpha}_{n,p,q}(x;a,b;\mu,\nu) = {}_{H}F^{\alpha}_{n,p,q}(x;a,b;1;\mu,\nu), \tag{26}$$

where ${}_{H}F^{\alpha}_{n,p,q}(x;a,b;\mu,\nu)$ *is known as unified* (p,q)*-Hemite based Apostol type polynomial of order* α.

Remark 8. *If we take* $\alpha = \beta = 1$ *in* (24) *we get,*

$${}_{H}F_{n,p,q}(x;a,b;\mu,\nu) = {}_{H}F^{1}_{n,p,q}(x;a,b;1;\mu,\nu), \tag{27}$$

where ${}_{H}F_{n,p,q}(x;a,b;\mu,\nu)$ *is denoted as unified* (p,q)*-Hemite based Apostol type polynomial.*

Let us now investigate few special cases of (p,q)-Hermite based Apostol type polynomials that are summarized in Table I.

TABLE I
SPECIAL CASES OF (p,q)-HERMITE BASED APOSTOL TYPE POLYNOMIALS

	Assumptions	Generating function	Name of the polynomial
A.	$\mu=0,\ \nu=1,\ a=-1$ and $b=1$	$\left(\frac{t}{\beta e_{p,q}(t)-1}\right)^{\alpha}F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,p,q}^{\alpha}(x;\beta)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{B}_{n,p,q}^{\alpha}(x;\beta)={}_HF_{n,p,q}^{\alpha}(x;-1,1;\beta;0,1)$	(p,q)-Hermite based Apostol type Bernoulli polynomial of order α
	$\mu=0,\ \nu=1,\ a=-1,\ b=1$ and $\beta=1$	$\left(\frac{t}{e_{p,q}(t)-1}\right)^{\alpha}F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,p,q}^{\alpha}(x)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{B}_{n,p,q}^{\alpha}(x)={}_HF_{n,p,q}^{\alpha}(x;-1,1;1;0,1)$	Unified (p,q)-Hermite based Apostol type Bernoulli polynomial of order α
	$\mu=0,\ \nu=1,\ a=-1,\ b=1,\ \alpha=\beta=1$	$\left(\frac{t}{e_{p,q}(t)-1}\right)F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{B}_{n,p,q}(x)={}_HF_{n,p,q}^{1}(x;-1,1;1;0,1)$	Unified (p,q)-Hermite based Apostol type Bernoulli polynomial
B.	$\mu=1,\ \nu=0,\ a=b=1$	$\left(\frac{2}{\beta e_{p,q}(t)+1}\right)^{\alpha}F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,p,q}^{\alpha}(x;\beta)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{E}_{n,p,q}^{\alpha}(x;\beta)={}_HF_{n,p,q}^{\alpha}(x;1,1;\beta;1,0)$	(p,q)-Hermite based Apostol type Euler polynomial of order α
	$\mu=1,\ \nu=0,\ a=b=1$ and $\beta=1$	$\left(\frac{2}{e_{p,q}(t)+1}\right)^{\alpha}F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,p,q}^{\alpha}(x)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{E}_{n,p,q}^{\alpha}(x)={}_HF_{n,p,q}^{\alpha}(x;1,1;1;1,0)$	Unified (p,q)-Hermite based Apostol type Euler polynomial of order α
	$\mu=1,\ \nu=0,\ a=b=1,\ \alpha=\beta=1$	$\left(\frac{2}{e_{p,q}(t)+1}\right)F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{E}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{E}_{n,p,q}(x)={}_HF_{n,p,q}^{1}(x;1,1;1;1,0)$	Unified (p,q)-Hermite based Apostol type Euler polynomial
C.	$\mu=\nu=1,\ a=b=1$	$\left(\frac{2t}{\beta e_{p,q}(t)+1}\right)^{\alpha}F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{G}_{n,p,q}^{\alpha}(x;\beta)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{G}_{n,p,q}^{\alpha}(x;\beta)={}_HF_{n,p,q}^{\alpha}(x;1,1;\beta;1,1)$	(p,q)-Hermite based Apostol type Genocchi polynomial of order α
	$\mu=\nu=1,\ a=b=1$ and $\beta=1$	$\left(\frac{2t}{e_{p,q}(t)+1}\right)^{\alpha}F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{G}_{n,p,q}^{\alpha}(x)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{G}_{n,p,q}^{\alpha}(x)={}_HF_{n,p,q}^{\alpha}(x;1,1;1;1,1)$	Unified (p,q)-Hermite based Apostol type Genocchi polynomial of order α
	$\mu=\nu=1,\ a=b=1,\ \alpha=\beta=1$	$\left(\frac{2t}{e_{p,q}(t)+1}\right)F_{p,q}(t)e_{p,q}(xt)=\sum_{n=0}^{\infty}{}_H\mathcal{G}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}$, where ${}_H\mathcal{G}_{n,p,q}(x)={}_HF_{n,p,q}^{1}(x;1,1;1;1,1)$	Unified (p,q)-Hermite based Apostol type Genocchi polynomial

Theorem 2. *For $q,p\in\mathbb{C}$, $0<|q|<|p|<1$, the explicit expression of generalized (p,q)-Hermite based Apostol type polynomial is given by:*

$$
\begin{aligned}
&{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}_{p,q} p^{\binom{n-k}{2}}\,{}_H F^{\alpha}_{k,p,q}(a,b;\beta;\mu,\nu)x^{n-k}, \quad (28)\\
&{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}_{p,q} F^{\alpha}_{k,p,q}(a,b;\beta;\mu,\nu)H_{n-k,p,q}(x)\,, \quad (29)
\end{aligned}
$$

where $n,\alpha\in\mathbb{N}_0$, $\beta,\mu,\nu,a,b\in\mathbb{C}$.

Proof: Substituting (25) and (3) in (24), we have:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\sum_{k=0}^{\infty}{}_H F^{\alpha}_{k,p,q}(a,b;\beta;\mu,\nu)\frac{t^k}{[k]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{(xt)^n}{[n]_{p,q}!}\,. \quad (30)
\end{aligned}
$$

On using the Cauchy product and equating the coefficients of t^n in (30), we get the desired result mentioned in (28). On using (17) and (20) in (24), we get:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\sum_{k=0}^{\infty}F^{\alpha}_{k,p,q}(a,b;\beta;\mu,\nu)\frac{t^k}{[k]_{p,q}!}\sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}. \quad (31)
\end{aligned}
$$

On using the Cauchy product and equating the coefficients of t^n in (31), we get the desired result mentioned in (29). ■

Theorem 3. *For $q,p\in\mathbb{C}$, $0<|q|<|p|<1$, the following relation holds:*

$$
\begin{aligned}
&{}_H F^{\alpha+\gamma}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}_{p,q}{}_H F^{\alpha}_{k,p,q}(a,b;\beta;\mu,\nu)\\
&\quad\times F^{\gamma}_{n-k,p,q}(x;a,b;\beta;\mu,\nu)\,, \quad (32)
\end{aligned}
$$

where $n,\alpha,\gamma\in\mathbb{N}_0$, $\beta,\mu,\nu,a,b\in\mathbb{C}$.

Proof: In view of (24), (25) and (20), we have:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_H F^{\alpha+\gamma}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\sum_{k=0}^{\infty}{}_H F^{\alpha}_{k,p,q}(a,b;\beta;\mu,\nu)\frac{t^k}{[k]_{p,q}!}\\
&\quad\times\sum_{n=0}^{\infty}F^{\gamma}_{n,p,q}(x;ab;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}. \quad (33)
\end{aligned}
$$

Applying the Cauchy product and equating the coefficients of t^n in (33), we get the desired result. ■

Theorem 4. *For $q,p\in\mathbb{C}$, $0<|q|<|p|<1$, the following differential equation holds:*

$$
\begin{aligned}
&D_{p,q,x}\,{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=[n]_{p,q}\,{}_H F^{\alpha}_{n-1,p,q}(x;a,b;\beta;\mu,\nu), \quad (34)
\end{aligned}
$$

where $n,\alpha\in\mathbb{N}_0$, $\beta,\mu,\nu,a,b\in\mathbb{C}$.

Proof: Differentiating (24) with respect to x, we get:

$$
\begin{aligned}
&D_{p,q,x}\sum_{n=0}^{\infty}{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\left(\frac{2^{\mu}t^{\nu}}{\beta e_{p,q}(t)+a^b}\right)^{\alpha}F_{p,q}(t)D_{p,q,x}(e_{p,q}(xt))\\
&\quad=t\left(\frac{2^{\mu}t^{\nu}}{\beta e_{p,q}(t)+a^b}\right)^{\alpha}F_{p,q}(t)e_{p,q}(pxt). \quad (35)
\end{aligned}
$$

Using Cauchy product and comparing the coefficients of t^n in (35), we get the desired result. ■

Theorem 5. *The following relation hold:*

$$
\begin{aligned}
&{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k q^{\frac{k(k-1)}{2}}p^{\frac{k(k+1)}{2}}F^{\alpha}_{n-2k,p,q}(x;a,b;\beta;\mu,\nu)}{[2k]_{p,q}!!\,[n-2k]_{p,q}!}\,, \quad (36)
\end{aligned}
$$

where $q,p,\beta,\mu,\nu,a,b\in\mathbb{C}$, $0<|q|<|p|<1$ and $n,\alpha\in\mathbb{N}_0$.

Proof: Applying (14) and (20) in (24), we have:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\sum_{n=0}^{\infty}F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}\\
&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k q^{\frac{k(k-1)}{2}}p^{\frac{k(k+1)}{2}}t^{2k}}{[2k]_{p,q}!!}. \quad (37)
\end{aligned}
$$

Using Cauchy product and equating the coefficients of t^n in (37) , we arrive at the desired result (36). ■

V. Relationships between Bernoulli, Euler and Genocchi Polynomials

In this section, we derive explicit relationships between (p,q)-Hermite based Apostol type polynomials with (p,q) Apostol Bernoulli, (p,q) Apostol Euler and (p,q) Apostol Genocchi polynomials.

Theorem 6. *The following identity holds:*

$$
\begin{aligned}
&{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=\frac{{}_H F^{\alpha}_{m,p,q(a,b;\beta;\mu,\nu)}}{[n+1]_{p,q}!}\\
&\quad\times\Bigg[\Bigg(\beta\sum_{m=0}^{n+1}\Bigg(\binom{n+1}{m}_{p,q}\sum_{r=0}^{n+1-m}p^{\binom{r}{2}}\binom{n+1-m}{r}_{p,q}\\
&\quad\times\mathcal{B}_{n+1-m-r,p,q}(x;\beta)\Bigg)\Bigg)\\
&\quad-\left(\sum_{m=0}^{n+1}\binom{n+1}{m}_{p,q}\mathcal{B}_{n+1-m,p,q}(x;\beta)\right)\Bigg] \quad (38)
\end{aligned}
$$

Proof: Multiplying both sides of (24) with $\left(\frac{t}{\beta e_{p,q}(t)-1}\right)$, we have

$$
\sum_{n=0}^{\infty}{}_H F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\frac{t^n}{[n]_{p,q}!}
$$

$$
\begin{aligned}
&= \left(\frac{2^{\mu}t^{\nu}}{\beta e_{p,q}(t)+a^{b}}\right)^{\alpha}\left(\frac{t}{\beta e_{p,q}(t)-1}\right)F_{p,q}(t)\\
&\quad\times\left(\frac{\beta e_{p,q}(t)-1}{t}\right)e_{p,q}(xt)\\
&= \left(\frac{2^{\mu}t^{\nu}}{\beta e_{p,q}(t)+a^{b}}\right)^{\alpha}F_{p,q}(t)\left(\frac{t}{\beta e_{p,q}(t)-1}e_{p,q}(xt)\right)\\
&\quad\times\frac{\beta e_{p,q}(t)}{t}-\frac{1}{t}\left(\frac{2^{\mu}t^{\nu}}{\beta e_{p,q}(t)+a^{b}}\right)^{\alpha}\\
&\quad\times F_{p,q}(t)\left(\frac{t}{\beta e_{p,q}(t)-1}e_{p,q}(xt)\right)\\
&= \frac{1}{t}\Bigg[\Bigg(\beta\sum_{m=0}^{\infty}{}_{H}F^{\alpha}_{m,p,q}(a,b;\beta;\mu,\nu)\frac{t^{m}}{[m]_{p,q}!}\\
&\quad\times\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}(x;\beta)\frac{t^{n}}{[n]_{p,q}!}\sum_{r=0}^{\infty}p^{\binom{r}{2}}\frac{t^{r}}{[r]_{p,q}!}\Bigg)\\
&\quad-\Bigg(\sum_{m=0}^{\infty}{}_{H}F^{\alpha}_{m,p,q}(a,b;\beta;\mu,\nu)\frac{t^{m}}{[m]_{p,q}!}\\
&\quad\times\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}(x;\beta)\frac{t^{n}}{[n]_{p,q}!}\Bigg)\Bigg]\\
&= \frac{1}{t}\Bigg[\Bigg(\beta\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{r=0}^{n-m}\frac{\begin{pmatrix}p^{\binom{r}{2}}\mathcal{B}_{n-m-r,p,q}(x;\beta)\\ \times{}_{H}F^{\alpha}_{m,p,q}(a,b;\beta;\mu,\nu)\end{pmatrix}}{[n-m-r]_{p,q}![m]_{p,q}![r]_{p,q}!}t^{n}\Bigg)\\
&\quad-\Bigg(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\begin{pmatrix}\mathcal{B}_{n-m,p,q}(x;\beta)\\ \times{}_{H}F^{\alpha}_{m,p,q}(a,b;\beta;\mu,\nu)\end{pmatrix}}{[n-m]_{p,q}![m]_{p,q}!}t^{n}\Bigg)\Bigg].
\end{aligned}
\tag{39}
$$

Replacing n with $n+1$ and equating the coefficients of t^n in (39), we get the desired result. ■

Corollary 1. *The following relations hold for (p,q)-Apostol Euler and (p,q)-Apostol Genocchi polynomials.*

$$
\begin{aligned}
&{}_{H}F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=\frac{1}{2}\Bigg[\Bigg(\beta\sum_{m=0}^{n}\binom{n}{m}_{p,q}\sum_{r=0}^{n-m}p^{\binom{r}{2}}\\
&\qquad\times\binom{n-m}{r}_{p,q}\mathcal{E}_{n-m-r,p,q}(x;\beta)\Bigg)\\
&\qquad-\Bigg(\sum_{m=0}^{n}\binom{n}{m}_{p,q}\mathcal{E}_{n-m,p,q}(x;\beta)\Bigg)\Bigg]\\
&\qquad\times{}_{H}F^{\alpha}_{m,p,q}(a,b;\beta;\mu,\nu).
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
&{}_{H}F^{\alpha}_{n,p,q}(x;a,b;\beta;\mu,\nu)\\
&\quad=\frac{1}{2[n+1]_{p,q}!}\\
&\quad\times\Bigg[\beta\sum_{m=0}^{n+1}\Bigg(\binom{n+1}{m}_{p,q}\sum_{r=0}^{n+1-m}\Bigg(p^{\binom{r}{2}}\binom{n+1-m}{r}_{p,q}\\
&\quad\times\mathcal{G}_{n+1-m-r,p,q}(x;\beta)\Bigg)\\
&\quad\times{}_{H}F^{\alpha}_{m,p,q}(a,b;\beta;\mu,\nu)\Bigg)\\
&\quad-\Bigg(\sum_{m=0}^{n+1}\binom{n+1}{m}_{p,q}\mathcal{G}_{n+1-m,p,q}(x;\beta)\\
&\quad\times{}_{H}F^{\alpha}_{m,p,q}(a,b;\beta;\mu,\nu)\Bigg)\Bigg]
\end{aligned}
\tag{41}
$$

Theorem 7. *The following relation holds:*

$$
\begin{aligned}
&{}_{H}\mathcal{G}^{\alpha}_{n,p,q}(x;a,b;\beta)\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{m}_{p,q}\Bigg[\sum_{r=0}^{n-k}\beta p^{\binom{r}{2}}\binom{n-k}{r}_{p,q}\\
&\qquad\times{}_{H}\mathcal{E}_{n-k-r,p,q}(0;\beta)+{}_{H}\mathcal{E}_{n-k,p,q}(0;\beta)\Bigg]\\
&\qquad\times{}_{H}\mathcal{G}^{\alpha}_{k,p,q}(a,b;\beta;\mu,\nu).
\end{aligned}
\tag{42}
$$

Proof: Multiplying both sides of (24) with $\left(\frac{2}{\beta e_{p,q}(t)+1}\right)$, we get:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_{H}\mathcal{G}^{\alpha}_{n,p,q}(x;a,b;\beta)\frac{t^{n}}{[n]_{p,q}!}\\
&\quad=\left(\frac{2t}{\beta e_{p,q}(t)+a^{b}}\right)^{\alpha}\left(\frac{2}{\beta e_{p,q}(t)+1}\right)F_{p,q}(t)\\
&\quad\times\left(\frac{\beta e_{p,q}(t)+1}{2}\right)e_{p,q}(xt)\\
&\quad=\frac{1}{2}\Bigg[\beta\sum_{k=0}^{\infty}{}_{H}\mathcal{G}^{\alpha}_{k,p,q}(x;a,b;\beta)\frac{t^{k}}{[k]_{p,q}!}\\
&\quad\times\sum_{n=0}^{\infty}\mathcal{E}_{n,p,q}(0;\beta)\frac{t^{n}}{[n]_{p,q}!}\sum_{r=0}^{\infty}p^{\binom{r}{2}}\frac{t^{r}}{[r]_{p,q}!}\Bigg]\\
&\quad+\frac{1}{2}\Bigg[\sum_{k=0}^{\infty}{}_{H}\mathcal{G}^{\alpha}_{k,p,q}(x;a,b;\beta)\frac{t^{k}}{[k]_{p,q}!}\\
&\quad\times\sum_{n=0}^{\infty}{}_{H}\mathcal{E}_{n,p,q}(0;\beta)\frac{t^{n}}{[n]_{p,q}!}\Bigg]\\
&\quad=\frac{1}{2}\Bigg[\beta\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\\
&\quad\times\frac{p^{\binom{r}{2}}{}_{H}\mathcal{E}_{n-k-r,p,q}(0;\beta){}_{H}\mathcal{G}^{\alpha}_{k,p,q}(x;a,b;\beta)}{[k]_{p,q}![n-k-r]_{p,q}![r]_{p,q}!}t^{n}\Bigg]\\
&\quad+\frac{1}{2}\Bigg[\sum_{k=0}^{\infty}\sum_{k=0}^{n}\frac{{}_{H}\mathcal{G}^{\alpha}_{k,p,q}(x;a,b;\beta){}_{H}E_{n-k,p,q}(0;\beta)}{[n-k]_{p,q}![k]_{p,q}!}t^{n}\Bigg].
\end{aligned}
\tag{43}
$$

On equating the coefficients of t^n in (43), we get the desired result. ■

Theorem 8. *The following relation holds:*

$$
\begin{aligned}
&{}_{H}\mathcal{E}^{\alpha}_{n,p,q}(x;a,b;\beta)\\
&\quad=\frac{\beta}{(n+1)_{p,q}}\Bigg[\sum_{k=0}^{n}\binom{n+1}{k}_{p,q}\\
&\qquad\times\sum_{s=0}^{k}\binom{k}{s}_{p,q}p^{\binom{k-s}{2}}\frac{\begin{pmatrix}\mathcal{B}_{n-k,p,q}(mx;a,b;\beta)\\ {}_{H}\mathcal{E}^{\alpha}_{s,p,q}(0;a,b;\beta)\end{pmatrix}}{m^{k-s-1}}\Bigg]\\
&\qquad-\frac{a^{b}}{(n+1)_{p,q}}\Bigg[\sum_{s=0}^{\infty}m\mathcal{B}_{n-s,p,q}(mx;a,b;\beta)\\
&\qquad\times{}_{H}\mathcal{E}^{\alpha}_{s,p,q}(0;a,b;\beta)\Bigg]
\end{aligned}
\tag{44}
$$

Proof: Multiplying both sides of (22) with $\left(\frac{\beta e_{p,q}\left(\frac{t}{m}\right)-a^b}{\frac{t}{m}}\right)$, we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\mathcal{E}_{n,p,q}^{\alpha}(x;a,b;\beta)\frac{t^n}{[n]_{p,q}!}\\
&= \left(\frac{2}{\beta e_{p,q}(t)+a^b}\right)^{\alpha} F_{p,q}(t)\left(\frac{\beta e_{p,q}\left(\frac{t}{m}\right)-a^b}{\frac{t}{m}}\right)\\
&\quad\times\left(\frac{\frac{t}{m}}{\beta e_{p,q}\left(\frac{t}{m}\right)-a^b}\right) e_{p,q}\left(mx\frac{t}{m}\right)\\
&= \left[\frac{m\beta}{t}\sum_{s=0}^{\infty} {}_H\mathcal{E}_{s,p,q}^{\alpha}(0;a,b;\beta)\frac{t^s}{[s]_{p,q}!}\right.\\
&\quad\left.\times\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}(mx;a,b;\beta)\frac{t^n}{[n]_{p,q}!}\sum_{k=0}^{\infty}p^{\binom{k}{2}}\frac{t^k}{m^k[k]_{p,q}!}\right]\\
&\quad\times -a^b\frac{m}{t}\left[\sum_{s=0}^{\infty} {}_H\mathcal{E}_{s,p,q}^{\alpha}(0;a,b;\beta)\frac{t^s}{[s]_{p,q}!}\right.\\
&\quad\left.\times\sum_{n=0}^{\infty}\mathcal{B}_{n,p,q}(mx;a,b;\beta)\frac{t^n}{[n]_{p,q}!}\right]\\
&= \beta\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{s=0}^{k}p^{\binom{k-s}{2}}\frac{\begin{pmatrix}\mathcal{B}_{n-k,p,q}(mx;a,b;\beta)\\ {}_H\mathcal{E}_{s,p,q}^{\alpha}(0;a,b;\beta)\end{pmatrix}}{\begin{pmatrix}m^{k-s-1}[k-s]_{p,q}!\\ [s]_{p,q}![n-k]_{p,q}!\end{pmatrix}}t^{n-1}\right]\\
&\quad - m\times a^b\left[\sum_{s=0}^{k}\sum_{s=0}^{\infty}\frac{\begin{pmatrix}\mathcal{B}_{n-s,p,q}(mx;a,b;\beta)\\ {}_H\mathcal{E}_{s,p,q}^{\alpha}(0;a,b;\beta)\end{pmatrix}}{[s]_{p,q}![n-s]_{p,q}!}t^{n-1}\right].
\end{aligned}\tag{45}$$

Replacing n with $n+1$ and then comparing the coefficients of t^n in (45), we get the desired result mentioned in (44). ■

Now we define the (p,q)-Stirling polynomials $S_{2,p,q}(n,v;\beta)$ of the second kind of order v with the following generating function:

$$\sum_{n=0}^{\infty} S_{2,p,q}(n,v;\beta)\frac{t^n}{[n]_{p,q}!} = \frac{\left(\beta e_{p,q}(t)+a^b\right)^{\nu}}{[\nu]_{p,q}!}. \tag{46}$$

Remark 9. *When $q\to p$ and $p=\beta=1$, the above polynomials reduce to standard Stirling numbers of second kind (see [15]):*

$$\sum_{n=0}^{\infty} S(n,v)\frac{t^n}{n!} = \frac{(e^t-1)^{\nu}}{v!}. \tag{47}$$

Theorem 9. *The relationship between generalized (p,q)-Hermite based Bernoulli polynomial and (p,q)-Stirling polynomials is derived as follows:*

$$\begin{aligned}
&{}_H\mathcal{B}_{n-v,p,q}^{\alpha}(x;a,b;\beta)\\
&= \frac{[v]_{p,q}![n-v]_{p,q}!}{[n]_{p,q}!}\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{B}_{k,p,q}^{\alpha-v}(x;a,b;\beta)\\
&\quad\times S_{2,p,q}(n-k,v;\beta).
\end{aligned}\tag{48}$$

Proof: Multiplying both sides of (24) with $\left(\frac{(\beta e_{p,q}(t)+a^b)^v}{[v]_{p,q}!}\right)$, we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,p,q}^{\alpha}(x;a,b;\beta)\frac{t^n}{[n]_{p,q}!}\\
&= \left(\frac{t}{\beta e_{p,q}(t)+a^b}\right)^{\alpha} e_{p,q}(x,t)F_{p,q}(t)\frac{(\beta e_{p,q}(t)+a^b)^v}{[v]_{p,q}!}\\
&\quad\times\frac{[v]_{p,q}!}{(\beta e_{p,q}(t)+a^b)^v}\\
&= \frac{[v]_{p,q}!}{t^v}\sum_{k=0}^{\infty} {}_H\mathcal{B}_{k,p,q}^{\alpha-v}(x;a,b;\beta)\frac{t^k}{[k]_{p,q}!}\\
&\quad\times\sum_{n=0}^{\infty} S_{2,p,q}(n,v;\beta)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}\tag{49}$$

Applying Cauchy product rule in (49), it follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,p,q}^{\alpha}(x;a,b;\beta)\frac{t^{n+v}}{[n]_{p,q}!}\\
&= [v]_{p,q}!\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{B}_{k,p,q}^{\alpha-v}(x;a,b;\beta)\\
&\quad\times S_{2,p,q}(n-k,v;\beta)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}\tag{50}$$

Making the powers identical and equating the coefficients in (50), we get the desired result in (48). ■

VI. Conclusion

The present paper defines (p,q) analogue of Hermite based Apostol type polynomials of order α, multifarious new and exciting relations of extended Hermite-Apostol polynomials with Bernoulli, Euler and Genocchi polynomials were obtained. Moreover, the (p,q) calculus has been efficaciously used in numerous sciences such as physical sciences, field theory, hypergeometric series, lie group, oscillator algebra, differential equations. Hence, approximation functions through polynomials based on basic twin numbers would have a pivotal role. Thus, the (p,q) polynomials derived in the present paper can be investigated and examined in approximation theory.

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