Convergence Rate of the Galërkin Method for Boundary Value Problem for Mixed-Type Operator-Differential Equations

Abdel Baset I. Ahmed

Abstract—**The paper studies the BVP for a mixed-type first-order linear operator-differential equations. We prove that there exists an absolutely continuous-unique solution in a certain Sobolev-type space $\overset{\circ}{W}(H, H_1)$. We provide the Galërkin method for sufficiently smooth selfadjoint positively definite operator $\Phi(t)$ and $\alpha-$ subordinate differentiable operator $S(t)$, $0 \le \alpha < 1$. The investigated method depends on a special basis $\{e_i\}_{i=1}^{\infty}$ of the operator Ψ similar to Φ. Moreover, using the eigenvalues λ_i of the operator Ψ to derive the error estimates of the approximate solutions. Through this work, we investigate how quickly the approximate solutions approach to the exact one.**

Index Terms—**boundary value problems (BVP), Galërkin method, orthoprojector, rate of convergence, mixed-type problem, selfadjoint operator.**

I. Introduction

THE most significant and serious problem that rise in the numerical investigated analysis and its scientific and technological applications is to know at what rate the approximate solutions converge to the unique exact one in various metrics. From nearly eighty years ago, Kantorovich [8] was the cardinal who investigated this type of problems, while Kislov N. V. investigated the inhomogeneous BVPs for differential-operator equations [9]. the Galërkin method is considered to be among the highest crucial approximative methods because it provides Galërkin solution (approximate solutions) in addition to proving the existence of the unique solution under specific conditions [3], [7], [10], [17], [18]. Nowadays, various natural occurrences have been described by IBVPs in Banach spaces for linear parabolic differential equations. For example [10] has considerable attention to the uniform solvability of the perturbed-differential equations using a subordinate operator to the leading perturbed one with some orders. In the Sobolev-type space, we obtain the error of convergence estimates of the ordinary derivative of the Galërkin solution to the exact unique one of the operator-differential equations. The work [15] estimated the method of projection and projection difference to get the estimates of the rate at which the approximate solutions for weakly converge. Furthermore, for the IBVPs of the isomorphic operator-differential equations in a Hilbert-type space, the approximation error and its estimates were provided.

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II. Problem Statement and auxiliary assertions

We consider the BVP in a certain space H_1 as densely embedding in a space H with norm $\|.\| = \|.\|_H$ in which both of H_1 and H are separable Hilbert spaces:

$$\Phi(t)\mathcal{Z}^{'}(t) - S(t)\mathcal{Z}(t) = f(t),\ t \in [0, T], \tag{1}$$

$$\mathcal{Z}(t)|_{t=0} = 0. \tag{2}$$

Where $\mathcal{Z}(t)$ and $f(t)-$ the unknown and the given functions, $\Phi(t)$ and $S(t)-$ acting in H linear operators. The operator $\Phi(t)$ is considered to be an independent of $t-$ selfadjoint in H, $((\Phi(t)x, x) > 0\,\forall x \in H,\ x \neq 0)$ in which $Dom(\Phi(t)) = H_1$, $(\Phi(t) = \Phi^*(t) \ge \lambda_1 E, \lambda_1 > 0)$, λ_1- is the spectrum lower bound $(\lambda_1 \in \sigma(\Phi))$ [1], [2], [6]. All derivatives here are perceived according to distributions theory.

from now on, concerning the operators $\Phi(t)$ and $S(t)$, we assume that they'll fulfill the following necessary assumptions:

(i) $\Phi(t)$ and $S(t)-$ differentiable operators and the derivatives $\Phi^{'}(t)$, $S^{'}(t) : H_1 \to H$ are strongly continuous on the interval $[0, T]$.

(ii) $S(t)$ is $\alpha-$ subordinate operator to $\Phi(0)$ where, $Dom(S(t) \supset Dom(\Phi(t))$ [12]. For any function $v \in H_1$ there's a $k_1 \ge 0$ constant positive value such that,

$$\|S(t)v\| \le k_1\|\Phi(0)v\|^{\alpha}\|v\|^{1-\alpha},\quad 0 \le \alpha < 1. \tag{3}$$

(iii) the operator Ψ is similar to $\Phi(0)$ [13] i.e., Ψ is considered to be selfadjoint, $((\Psi(t)x, x) > 0\,\forall x \in H,\ x \neq 0)$ with the domain $Dom(\Psi) = Dom(\Phi(0))$. Hence, Φ^{-1}, Ψ^{-1} are completely continuous. For the angel between $\Phi(t)$ and Ψ to be acute in Hilbert-type space H [16], i.e., $\forall v \in H_1$ we have:

$$\big(\Phi(t)v,\ \Psi v\big) \ge k_2\|\Phi(0)v\|\|\Psi v\|,\ v \in H_1, \tag{4}$$

where $0 < k_2 \le 1$ does not depend on t or v.

Let the function $f(t)$ be in the space $\mathscr{B}_2(0, T; H)$ over the interval $[0, T]$, where

$$\|f(t)\|_{\mathscr{B}_2(0,T;H)} = \left(\int_0^T \|f(t)\|_H^2 dt\right)^{\frac{1}{2}} < \infty.$$

Further, we consider the function $\mathcal{Z}(t)$ with values in Hilbert space H_1 having a continuous derivative in the space H. Assume that $\mathcal{Z}(t)$ and $\Phi(t)$ have a continuous derivative$\dfrac{d\mathcal{Z}(t)}{dt}$, $\dfrac{d\Phi(t)}{dt}$ respectively in H.

Moreover,

for all $\mathcal{Z}, v \in H_1$ [11] we get the following property

$$\left(\frac{d\Phi(t)}{dt}\mathcal{Z}, v\right) = \left(\mathcal{Z}, \frac{d\Phi(t)}{dt}v\right),\ \ t \in (0, t).$$

Such that

$$W_2^1(H,H_1) = \left\{ \mathcal{Z}(t) \in \mathscr{B}_2(0,T;H_1) : \frac{d}{dt}\Big(\Phi(t)\mathcal{Z}(t)\Big) \in \mathscr{B}_2(0,T;H) \right\},$$

$$\|\mathcal{Z}\|_{W_2^1(H,H_1)} = \left(\|\mathcal{Z}(t)\|^2_{\mathscr{B}_2(0,T;H_1)} + \left\| \frac{d}{dt}\Big(\Phi(t)\mathcal{Z}(t)\Big) \right\|^2_{\mathscr{B}_2(0,T;H)} \right)^{\frac{1}{2}}.$$

Consider the Sobolev space

$$\overset{\circ}{W}(H,H_1) = \left\{ \mathcal{Z}(t) : \mathcal{Z}(t) \in W_2^1(H,H_1),\ \mathcal{Z}(t)_{t=0} = 0 \right\}.$$

Then the function $\mathcal{Z}(t) \in \overset{\circ}{W}(H,H_1)$ which fulfills (1) is called the strong solution and problem (1), (2) is called strongly solvable.

According to the assumption (i) and (ii) the operators $\Phi(t)\Phi^{-1}(0)$ and $\Phi(0)\Phi^{-1}(t)$ acting in H are norm bounded (uniform bounded) [9], i.e.,

$$\begin{aligned} &\|\Phi(t)\Phi^{-1}(0)\|_{H\to H} \le m_1, \\ &\|\Phi(0)\Phi^{-1}(t)\|_{H\to H} \le m_2, \end{aligned} \tag{5}$$

where $m_1, m_2 > 0$ don't depend on t.

Let $\{e_i\}_{i=1}^{\infty}$ be a "basis" of eigenvectors of Ψ, which represents a complete orthonormal system and let $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues of operator Ψ, in which $\Psi e_i = \lambda_i e_i$,

$$0 < \lambda_1 \le \ldots \le \lambda_n \le \ldots, \quad \lambda_n \to +\infty \quad \text{as} \quad n \to +\infty.$$

The approximate Galërkin solutions for problem (1), (2) is identified as follows:

$$\mathcal{Z}_n(t) = \sum_{i=1}^{n} \Theta_i(t) e_i,$$

where $\Theta_i(t),\ i = \overline{1,n}$ is an unknown functions denote the exact solution of the following zero-initial condition problem for a system of operator-differential equations:

$$\begin{aligned} &\Phi(t)\Theta_j^{'}(t) - \sum_{i=1}^{n} \Theta_j(t)\big(S(t)e_i, e_j\big) = \big(f(t), e_j\big)_H, \\ &\Theta_j(t)_{t=0} = 0,\ j = \overline{1,n}. \end{aligned} \tag{6}$$

Let R_n be the unique orthogonal projection in H onto H^n, which defined as the span $(e_1, e_2, \ldots, e_n)$. Then problem (6) can be expressed in the following equivalent form:

$$\Phi(t)\mathcal{Z}_n^{'}(t) - R_n S(t)\mathcal{Z}_n(t) = R_n f(t), \quad \mathcal{Z}_n(t)|_{t=0} = 0, \tag{7}$$

where $\mathcal{Z}_n(t)-$ the Galërkin solution (approximate solutions) of problem (1), (2) [12].

The local BVP for Eq. (1) using the properties of the corresponding spectral problem were studied in [5]. While the non-local BVP with constant coefficients were studied in [4].

III. Main results

First we look over the questions of the solvability of problem (1), (2) in H. The strong solution $\mathcal{Z}(t) \in \overset{\circ}{W}(H,H_1)$ of problem (1), (2) is absolutely continuous for almost all $t \in [0,T]$ and fulfills the Eq. (1) and the assumption (2).

In what follows, $c > 0$ denotes different constants that are independent of t, n.

In this section, we will consider the question of the unique solvability of problrm (1), (2).

Theorem 1. *Assume that $f(t) \in \mathscr{B}_2(0,T;H)$ and the operators $\Phi(t)$ and $S(t)$ fulfill the assumptions $(i)-(iii)$, then problem (1), (2) is solvable and there exists uniquely a strong solution $\mathcal{Z}(t) \in \overset{\circ}{W}(H,H_1)$.*

Proof: Consider the following problem of ordinary differential equation

$$\Phi(t)\frac{dv}{dt} = \varrho(t), \quad v(t)|_{t=0} = 0, \tag{8}$$

if $\varrho(t) \in \mathscr{B}_2(0,T;H)$, then problem (8) is solvable and possesses a unique solution $v \in \overset{\circ}{W}(H,H_1)$ and

$$\int_o^T \left\| \Phi(t)\frac{dv(t)}{dt} \right\|^2 dt \le c_1 \int_0^t \|\varrho(t)\|^2 dt.$$

holds true.

Let the operator $\left(\Phi(t)\frac{d}{dt}\right)^{-1} : \mathscr{B}_2(0,T;H) \to \overset{\circ}{W}(H,H_1)$ be a homeomorphism.

Consequently, in $\mathscr{B}_2(0,T;H)$, we will provide the following estimate

$$\left\| \left(\Phi(t)\frac{d}{dt}\right)^{-1} \right\|_{\mathscr{B}_2(0,T;H)\to\mathscr{B}_2(0,T;H)} \le c_2. \tag{9}$$

As the operator $\Phi(0)\Phi^{-1}(t)$ in $\mathscr{B}_2(0,T;H)$ is actually uniformly bounded and then from the Heinz's inequality [10], the operator $\Phi^{\frac{1}{2}}(0)\Phi^{-\frac{1}{2}}(t)$ is also uniformly bounded. Therefore

$$\Big(\Phi(t)v, v\Big)_H \ge \beta_1 \|\Phi^{\frac{1}{2}}(0)v\|^2, \quad \beta_1 > 0. \tag{10}$$

Multiply equation (8) scalarly by $v(t)$ in H and integrate with respect to the time t from 0 to $s,\ s \le T$. Then using (10), we obtain

$$\frac{\beta_1}{2}\|\Phi^{\frac{1}{2}}(0)v(s)\|^2 \le \|\varrho(t)\|_{\mathscr{B}_2(0,T;H)} \left(\int_0^s \|v(t)\|^2 \right)^{1/2},$$

$$\max_{0\le s\le T} \|\Phi^{\frac{1}{2}}(0)v(s)\| \le c_3 \|\varrho(t)\|_{\mathscr{B}_2(0,T;H)}.$$

Hence,

$$\max_{0\le s\le T} \|\Phi^{\frac{1}{2}}(0)\left(\Phi(t)\frac{d}{dt}\right)^{-1} \varrho(t)\| \le c_3 \|\varrho(t)\|_{\mathscr{B}_2(0,T;H)}. \tag{11}$$

We can simply take

$$\Phi(t)\frac{d\mathcal{Z}}{dt}(t) \equiv \omega(t), \quad \mathcal{Z}(t)|_{t=0} = 0.$$

Then, in the Hilbert space $\mathscr{B}_2(0,T;H)$, problem (1), (2) is identical to

$$\omega(t) - S(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1}\omega(t) = f(t). \tag{12}$$

On the other side, we'll investigate the operator $S(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1}$ as follows:

Using the relation (3) and Hölder's inequality, for any $\mathcal{Y} \in H_1$ we get

$$\left\| S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}\mathcal{Y}\right\|_{\mathscr{B}_2(0,T;H)} \le k_1 \left\|\Phi(0)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}\mathcal{Y}\right\|^{\alpha}_{\mathscr{B}_2(0,T;H)} \times \left\|\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}\mathcal{Y}\right\|^{1-\alpha}_{\mathscr{B}_2(0,T;H)}. \quad (13)$$

As $\Phi(0)\Phi^{-1}(t)$ in $\mathscr{B}_2(0,T;H)(0,T;H)$ is continuous linear operator- uniformly bounded- and from (9), (13) we get

$$\left\| S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}\mathcal{Y}\right\|_{\mathscr{B}_2(0,T;H)} \le c_4 \left\|\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}\mathcal{Y}\right\|^{1-\alpha}_{\mathscr{B}_2(0,T;H)} \|\mathcal{Y}\|^{\alpha}_{\mathscr{B}_2(0,T;H))}. \quad (14)$$

Hence ($H_1 \subset\subset H$), other words from the compact embedding of H_1 into H, and as stated in the lemma of compactness [8], the Sobolev space $W_2^1(H,H_1)$ is compactly embedded into $\mathscr{B}_2(0,T;H)$. Further, As the operator $\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}$ is compact and from (14) yields that the operator $S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}$ in $\mathscr{B}_2(0,T;H)$ is also compact.

We utilize the principle of the classical Fredholm alternative [14] to verify the resolvability of equation (12). the successive Cauchy problem is identical to the homogeneous equation (12):

$$\Phi(t)\frac{d\mathcal{Z}(t)}{dt} - S(t)\mathcal{Z}(t) = 0, \quad \mathcal{Z}(t)|_{t=0} = 0. \quad (15)$$

Let $\mathcal{Z}(t)$ be a solution of problem (15) and use (3) to get

$$\int_0^s \left\|\Phi(t)\frac{d\mathcal{Z}(t)}{dt}\right\|^2 dt \le k_2^2 \int_0^s \|\Phi(0)\mathcal{Z}(t)\|^{2\alpha}\|\mathcal{Z}(t)\|^{(2-2\alpha)}dt \le k_2^2\|\Phi(0)\Phi^{-1}(t)\|^{2\alpha}_{H\to H} \times \int_0^s \|\Phi(t)\mathcal{Z}(t)\|^{2\alpha}\|\mathcal{Z}(t)\|^{(2-2\alpha)}dt.$$

For any $s \in [0,T]$, follows

$$\int_0^s \left\|\Phi(t)\frac{d\mathcal{Z}(t)}{dt}\right\|^2 dt \le c_5 \int_0^s \|\Phi(t)\mathcal{Z}(t)\|^{2\alpha}\|\mathcal{Z}(t)\|^{(2-2\alpha)}dt.$$

Applying the Hölder's inequality to the preceding inequality

$$\int_0^s \left\|\Phi(t)\frac{d\mathcal{Z}(t)}{dt}\right\|^2 dt \le c_5 \left(\int_0^s \|\Phi(t)\mathcal{Z}(t)\|^2\right)^{\alpha} \left(\int_0^s \|\mathcal{Z}(t)\|^2 dt\right)^{(1-\alpha)}.$$

Further,
applying the Young inequality to get:

$$\int_0^s \left\|\Phi(t)\frac{d\mathcal{Z}(t)}{dt}\right\|^2 dt \le c_5\left(\varepsilon\int_0^s \|\Phi(t)\mathcal{Z}\|^2 + \Big(\frac{\alpha}{\varepsilon}\Big)^{\frac{\alpha}{1-\alpha}}(1-\alpha)\int_0^s \|\mathcal{Z}(t)\|^2 dt\right).$$

Chosen $\varepsilon = \frac{1}{2c_5}$, we get

$$\int_0^s \left\|\Phi(t)\frac{d\mathcal{Z}(t)}{dt}\right\|^2 dt \le c_6 \int_0^s \|\mathcal{Z}(t)\|^2 dt. \quad (16)$$

Multiply equation (15) scalarly by $\mathcal{Z}(t)$ and integrate with respect to the time t from 0 to s, $s \le T$. Then, by utilizing the characteristics of $\Phi(t)$ and $S(t)$, It would follow that

$$\frac{\beta_1}{2}\|\Phi^{\frac{1}{2}}(0)\mathcal{Z}(s)\|^2 \le \int_0^s \|\mathcal{Z}(t)\|\|S(t)\mathcal{Z}(t)\|dt \le c_7\int_0^s \|\Phi(t)\mathcal{Z}(t)\|^{\alpha}\|\mathcal{Z}(t)\|^{(2-\alpha)}dt \le c_7\left(\int_0^s \|\Phi(t)\mathcal{Z}(t)\|^2 dt\right)^{\frac{\alpha}{2}}\left(\int_0^s \|\mathcal{Z}(t)\|^2 dt\right)^{\frac{2-\alpha}{2}}.$$

Use (16) to get

$$\|\mathcal{Z}(s)\|^2 \le c_8 \int_0^s \|\mathcal{Z}(t)\|^2 dt.$$

From the Bellman–Gronwall result [3], we come to the fact that the preceding inequality is achievable only if $\mathcal{Z}(t)$ is equivalent to zero.

Thus, from the (Fredholm) Alternative, problem (1), (2) is solvable and has the unique solution $\mathcal{Z}(t) \in \overset{\circ}{W}(H,H_1)$. ■

We must utilize the following required lemmas for establishing the main results.

Lemma 2. *Let Φ_0 and D be bounded linear operators mapping H_1 into H where Φ_0 has bounded inverse Φ_0^{-1} and $|D| < 1/|\Phi_0^{-1}|$. Then, $\Phi = \Phi_0 + D$ that mapping H_1 into H invertible and has bounded inverse operator [14].*

Lemma 3. *Let $\Phi(t)$ and $S(t)$ fulfill assumptions $(i)-(iii)$. Then, $\exists N_0 > 0 : \forall n > N_0$, the operator $\Phi(t)\frac{d}{dt} - R_n S(t)$ has a bounded inverse operator $\big(\Phi(t)\frac{d}{dt} - R_n S(t)\big)^{-1}$ mapping the space $\mathscr{B}_2(0,T;H^n)$ into the space $\overset{\circ}{W}(H,H_1)$ and holds the inequality*

$$\left\|\Big(\Phi(t)\frac{d}{dt} - R_n S(t)\Big)^{-1}\right\|_{\mathscr{B}_2(0,T;H)} \le c_9. \quad (17)$$

Proof: It's clearly, Theorem 1 provided that the inverse operator

$$\Big(\Phi(t)\frac{d}{dt} - S(t)\Big)^{-1} : \mathscr{B}_2(0,T;H) \to W_2^1(H,H_1)$$

exists.

Then we investigate the invertibility of the operator

$$S(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1}R_n - I = -I + S(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1} + S(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1}(R_n - I).$$

At every $0 \le t \le T$, the linear operator

$$I - S(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1} : \mathscr{B}_2(0,T;H) \to \mathscr{B}_2(0,T;H)$$

is invertible and possesses a bounded inverse operator

$$\left(I + S(t)\left(\Phi(t)\frac{d}{dt} + S(t)\right)^{-1}\right).$$

Assume that the following problem:

$$\Phi(t)\frac{dv(t)}{dt} = (I - R_n)\Sigma(t),$$
$$v(t)|_{t=0} = 0$$

possesses a unique solution $v(t)$ in H_1.
Then,
multiplying the equation $\Phi(t)\frac{dv(t)}{dt} = (I - R_n)\Sigma(t)$ scalarly by $v(t)$ and integrating with respect to the time t from 0 to $s,\ s \le T$ and then using (10), we obtain

$$\frac{\beta_1}{2}\|\Phi^{\frac{1}{2}}(0)v(s)\|^2 \le \int_0^s |((I - R_n)\Sigma(t), v(t))_H|\, dt$$
$$\le \int_0^s \left\|\Phi^{-\frac{1}{2}}(0)(I - R_n)\Sigma(t)\right\| \left\|\Phi^{\frac{1}{2}}(0)v(t)\right\| dt.$$

Estimating the right-hand side of this relation with the help of the Cauchy inequality

$$\beta_1\|\Phi^{\frac{1}{2}}(0)v(s)\|^2 \le \frac{1}{\beta_1}\int_0^T \left\|\Phi^{-\frac{1}{2}}(0)(I - R_n)\Sigma(t)\right\|^2 dt.$$

Clearly, cause the operator Ψ is similar to $\Phi(0)$, we have

$$\beta_1\|\Phi^{\frac{1}{2}}(0)v(s)\|^2$$
$$\le \frac{1}{\beta_1}\left\|\Phi^{-\frac{1}{2}}(0)\Psi^{\frac{1}{2}}\right\|^2_{H\to H}\int_0^T \left\|\Psi^{-\frac{1}{2}}(I - R_n)\Sigma(t)\right\|^2 dt$$
$$\le \frac{1}{\beta_1}\left\|\Phi^{-\frac{1}{2}}(0)\Psi^{\frac{1}{2}}\right\|^2_{H\to H}\|\Sigma(t)\|^2_{\mathscr{B}_2(0,T;H)}\lambda^{-1}_{n+1}.$$

Consequently, we get that

$$\max_{0\le t\le T}\left\|\left(\Phi(t)\frac{d}{dt}\right)^{-1}(I - R_n)\Sigma(t)\right\| \le c_{10}\lambda^{-\frac{1}{2}}_{n+1}|\Sigma(t)\|_{\mathscr{B}_2(0,T;H)}. \tag{18}$$

Putting $\mathcal{Y} = (I - R_n)\Sigma(t)$ in (14), we obtain

$$\left\|S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}(I - R_n)\Sigma(t)\right\|_{\mathscr{B}_2(0,T;H)}$$
$$\le c_{11}\left\|\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}(I - R_n)\Sigma(t)\right\|^{1-\alpha}_{\mathscr{B}_2(0,T;H)}$$
$$\times \|\Sigma(t)(t)\|^{\alpha}_{\mathscr{B}_2(0,T;H)}.$$

Using (18), we obtain

$$\left\|S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}(I - R_n)\right\|_{\mathscr{B}_2(0,T;H)\to\mathscr{B}_2(0,T;H)} \le c_{12}\lambda^{\frac{\alpha-1}{2}}_{n+1}. \tag{19}$$

As the eigenvalues $\lambda_n \to +\infty$ as $n \to +\infty$ and based on (19), then $\exists$ a number $N_1 : \forall n > N_1$, we have

$$\left\|\left(I - S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}R_n\right)^{-1}\right\|_{\mathscr{B}_2(0,T;H)\to\mathscr{B}_2(0,T;H)}$$
$$\times\left\|S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}(I - R_n)\right\|_{\mathscr{B}_2(0,T;H)\to\mathscr{B}_2(0,T;H)}$$
$$\le c_{13}\lambda^{\frac{\alpha-1}{2}}_{N_1+1} \equiv \tau_1,$$

where the positive constant $\tau_1 < 1$.
Then, $\forall n > N_1$, as stated by Lemma (2), the operator

$$I - S(t)\Big(R_n\Phi(t)\frac{d}{dt}\Big)^{-1}R_n$$

has an inverse operator and

$$\left\|\left(I - S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}R_n\right)^{-1}\right\|_{\mathscr{B}_2(0,T;H)\to\mathscr{B}_2(0,T;H)} \le c_{14}. \tag{20}$$

It is simple to verify the relation

$$\left(I - R_nS(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}\right)^{-1} = I + R_n\left(I - S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}R_n\right)^{-1} \times S(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}. \tag{21}$$

Because for $n > N_1$ the operator

$$R_nS(t) + \Phi(t)\frac{d}{dt} = \left(I - R_nS(t)\Big(\Phi(t)\frac{d}{dt}\Big)^{-1}\right)\Big(\Phi(t)\frac{d}{dt}\Big),$$

for $n > N_1$ and from (20), (21) the operator $\big(\Phi(t)\frac{d}{dt} - R_nS(t)\big)$ has an inverse and inequality (17) holds true. ∎

In this section, we will consider the question of the rate of convergence of Galërkin solution (approximate solutions), established by the Galërkin method, to the exact solution.

Theorem 4. *Let $f(t) \in \mathscr{B}_2(0,T;H)$ and $\Phi(t), S(t)$ fulfill assumptions $(i), (ii)$, and (iii). Then, $\exists N_0 > 0 : \forall n > N_0$, problem (7) has a unique Galërkin solution $\mathcal{Z}_n(t)$ and*

$$\max_{t\in[0,T]}\|\mathcal{Z}_n(t) - \mathcal{Z}(t)\| \le c_{15}\lambda^{\frac{\alpha-1}{2}}_{n+1} \tag{22}$$

holds.

Proof: From Lemma (3) follows that problem (7) is solvable and has the solution $\mathcal{Z}_n(t) \in \overset{\circ}{W}(H, H_1)$. Therefore, from (17) we find out the approximate solutions $\mathcal{Z}_n(t)$ with regard to the number n are uniformly bounded i.e.,

$$\|\mathcal{Z}_n(t)\|_{\overset{\circ}{W}(H,H_1)} \le c_{10}\|f(t)\|_{\mathscr{B}_2(0,T;H)}. \tag{23}$$

For $n > N_0$ yields

$$\mathcal{Z}_n(t) - \mathcal{Z}(t) = \left(\Phi(t)\frac{d}{dt} - R_nS(t)\right)^{-1}(R_n - I)\left(\Phi(t)\frac{d\mathcal{Z}(t)}{dt}\right). \tag{24}$$

As the inverse operator

$$\left(\Phi(t)\frac{d}{dt} - R_nS(t)\right)^{-1} = \left(\Phi(t)\frac{d}{dt}\right)^{-1} - \left(\Phi(t)\frac{d}{dt} - R_nS(t)\right)^{-1}R_nS(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1}.$$

Eq. (24) could be expressed as follows:

$$\mathcal{Z}_n(t)-\mathcal{Z}(t)=\left(\Phi(t)\frac{d}{dt}\right)^{-1}(R_n-I)\left(\Phi(t)\frac{\mathcal{Z}(t)}{dt}\right)$$
$$-\left(\Phi(t)\frac{d}{dt}-R_nS(t)\right)^{-1}R_nS(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1}$$
$$\times(R_n-I)\left(\Phi(t)\frac{\mathcal{Z}(t)}{dt}\right). \tag{25}$$

Hence, from the continuous embedding between the space $W_2^1(H,H_1)$ and the space of continuous functions $C(0,T;H)$, we get

$$\max_{0\le t\le T}\|v\|\le c_{16}\|v\|_{W_2^1(H,H_1)}$$

holds true for all element v in the Sobolev space $W_2^1(H,H_1)$. Using inequalities (17) and (18) to estimate inequality (25) as follows:

$$\max_{0\le t\le T}\|\mathcal{Z}_n(t)-\mathcal{Z}(t)\|\le c_{10}\lambda_{n+1}^{-\frac{1}{2}}\|\Phi(t)\mathcal{Z}'(t)\|_{\mathscr{B}_2(0,T;H)}$$
$$+c_{17}\left\|S(t)\left(\Phi(t)\frac{d}{dt}\right)^{-1}(R_n-I)(\Phi(t)\mathcal{Z}'(t))\right\|_{\mathscr{B}_2(0,T;H)}.$$

Consequently, using inequalities (3) and (23), follows estimate (22).
Theorem is proved. ■

Theorem 5. *Let $f(t)\in C^1(0,T;H)$, $f(t)|_{t=0}=0$, the operator Φ is independent on the variable t, $S'(t)$ is $\gamma-$ subordinate to Φ, $0\le\gamma<1$, and Φ, $S(t)$ fulfill the assumptions (i), (ii), (iii). Then*

$$\max_{t\in[0,T]}\|\mathcal{Z}_n'(t)-\mathcal{Z}'(t)\|\le c_{18}\lambda_{n+1}^{\frac{(\alpha-1)(1-\gamma)}{2}}, \tag{26}$$

where $\mathcal{Z}(t)$ and $\mathcal{Z}_n(t)$ are the solutions of problem (1), (2) and problem (7) respectively.

Proof: We return back to the Cauchy problem (6): If the derivative $f'(t)$ is continuous and $\{\Theta_j(t)\}_{j=1}^n\in C^2(0,T;H)$ is the solution of problem (6), then we can differentiate Eq. (7).
Moreover, put $\mathcal{Z}_n'(t)=v_n(t)$ and by using $f(0)=0$, we get

$$\mathcal{Z}(t)v_n'(t)-R_nS(t)v_n(t)=R_nf'(t)+R_nS'(t)\mathcal{Z}_n(t),$$
$$v_n(t)|_{t=0}=0. \tag{27}$$

Consider the Cauchy problem of ordinary differential equation at $t=0$:

$$\Phi v'(t)-S(t)v(t)=f'(t)+S'(t)\mathcal{Z}(t),\quad v(0)=0. \tag{28}$$

From Theorem 1, problem (28) is solvable and has a solution $v(t)\in\overset{\circ}{W}(H,H_1)$ because $\left(f'(t)+S'(t)\mathcal{Z}(t)\right)\in\mathscr{B}_2(0,T;H)$. Then

$$\left(\Phi\frac{d}{dt}-R_nS(t)\right)\left(v_n(t)-v(t)\right)$$
$$=(R_n-I)\left[f'(t)+S'(t)\mathcal{Z}_n(t)+S(t)v(t)\right]$$
$$-S'(t)\left(\mathcal{Z}(t)-\mathcal{Z}_n(t)\right).$$

After that, from (17) we obtain that the approximate solutions $v_n(t)$ with respect to n are uniformly bounded, i.e.,

$$\|v_n(t)\|_{\overset{\circ}{W}(H,H_1)}\le c_{19}. \tag{29}$$

By analogy with the deducing of (22), using (23), and (29), we obtain

$$\max_{0\le t\le T}\|v_n(t)-v(t)\|$$
$$\le c_{20}\lambda_{n+1}^{\frac{(\alpha-1)}{2}}+c_{21}\|S'(t)(\mathcal{Z}(t)-\mathcal{Z}_n(t))\|_{\mathscr{B}_2(0,T;H)}.$$

Hence, $\left(\dfrac{dS(t)}{dt}\right)$ is subordinate to Φ with the order γ. Therefore,

$$\max_{0\le t\le T}\|v_n(t)-v(t)\|\le c_{20}\lambda_{n+1}^{\frac{(\alpha-1)}{2}}$$
$$+c_{22}\|\Phi(\mathcal{Z}(t)-\mathcal{Z}_n(t))\|^{\gamma}_{\mathscr{B}_2(0,T;H)}\|\mathcal{Z}(t)-\mathcal{Z}_n(t)\|^{1-\gamma}_{\mathscr{B}_2(0,T;H)}.$$

Using (22) and (23), we obtain

$$\max_{0\le t\le T}\|v_n(t)-v(t)\|\le c_{20}\lambda_{n+1}^{\frac{(\alpha-1)}{2}}+c_{23}\lambda_{n+1}^{\frac{(\alpha-1)(1-\gamma)}{2}}.$$

Hence,

$$\max_{0\le t\le T}\|v_n(t)-v(t)\|\le c_{24}\lambda_{n+1}^{\frac{(\alpha-1)(1-\gamma)}{2}}.$$

Theorem is proved. ■

IV. Applications of initial-boundary value problems (IBVP)

Finally, we apply the Galërkin method for IBVP.
In a rectangular $Q=[0,1]\times[0,T]$, we will study the next problem

$$\frac{\partial}{\partial x}\left(\chi(x,t)\frac{\partial}{\partial x}\right)\frac{\partial\mathcal{Z}(x,t)}{\partial t}$$
$$-\zeta_1(x,t)\frac{\partial\mathcal{Z}(x,t)}{\partial x}-\zeta_0(x,t)\mathcal{Z}(x,t)=f(x,t),\ (x,t)\in Q \tag{30}$$

$$\mathcal{Z}(x,t)|_{x=0}=\mathcal{Z}(x,t)|_{x=1}=0,\ \ t\in[0,T], \tag{31}$$

$$\mathcal{Z}(x,t)|_{t=0}=0,\ \ 0\le x\le1. \tag{32}$$

Suppose that all of the functions:

$$\chi(x,t),\quad\frac{\partial^{i+1}}{\partial x^i\partial t}\chi(x,t),\quad\zeta_0(x,t),\quad\zeta_1(x,t),$$
$$\frac{\partial}{\partial t}\zeta_0(x,t),\quad\frac{\partial}{\partial t}\zeta_1(x,t)$$

be in the space $C(Q)$ and $\chi(x,t)\ge\chi_0>0$.

In this section we redefine some of the following spaces: Assign $H=\mathscr{B}_2(0,1),H_1=W_2^2(0,1)\cap\overset{\circ}{W}(0,1)$, where $W_2^k(0,1)$ is a Sobolev space and

$$\overset{\circ}{W}(0,1)=\{v(x):v(x)\in W_2^2(0,1),v(0)=v(1)=0\}.$$

Because the partial derivative $\frac{\partial}{\partial x}\chi(x,t)\in C(\overline{Q})$, there is such a constant $\chi_1>0$ in which, $\left|\frac{\partial}{\partial x}\chi(x,t)\right|\le\chi_1$.
Assume that

$$\Phi(t)=\frac{\partial}{\partial x}\left(\chi(x,t)\frac{\partial}{\partial x}\right)-\Upsilon I$$

and

$$S(t) = \zeta_1(x,t)\frac{\partial \mathcal{Z}}{\partial x} + (\zeta_0(x,t) - \Upsilon) I,$$

where the constant $\Upsilon < \frac{a_1^2}{\chi_0}$. These operators are defined on H_1 and $D(S(t)) \supset D(\Phi(t))$.

For this problem we put $\Psi = \frac{d^2}{dx^2}$. It is obvious that the assumptions (i), (ii) and (iii) are fully achieved.

Moreover, It is not difficult to check that $S(t)$ is $\alpha-$ subordinate to $\Phi(0)$, $\alpha = \frac{1}{2}$.

Next, we'll demonstrate how "the acute-angle inequality (4)" is fulfilled by the operators $\Phi(t)$ and Ψ.

For each element $v \in H_1$, which fulfills the inequality

$$\begin{aligned}
&(\Phi(t)v, \Psi v)_{\mathscr{B}_2(0,1)} \\
&\geq \chi_0 \int_0^1 \left(\frac{d^2 v}{dx^2}\right)^2 dx - \Upsilon \int_0^1 \left(\frac{dv}{dx}\right)^2 dx \\
&+ \chi_1 \int_0^1 \left|\frac{dv}{dx}\right| \left|\frac{d^2 v}{d^2 x}\right| dx \\
&\geq \chi_0 \int_0^1 \left(\frac{d^2 v}{dx^2}\right)^2 dx + \frac{\chi_1 \varepsilon}{2} \int_0^1 \left(\frac{d^2 v}{dx^2}\right)^2 dx \\
&+ \frac{\chi_1}{2\varepsilon} \int_0^1 \left(\frac{dv}{dx}\right)^2 dx - \Upsilon \int_0^1 \left(\frac{dv}{dx}\right)^2 dx.
\end{aligned}$$

Choosing $\varepsilon = \frac{\chi_0}{2\chi_1}$, we get

$$\begin{aligned}
&(\Phi(t)v, \Psi v)_{\mathscr{B}_2(0,1)} \\
&\geq \left(\frac{\chi_1^2}{\chi_0} - \Upsilon\right) \int_0^1 \left(\frac{dv}{dx}\right)^2 dx \geq c_{25} \|v\|^2_{W_2^2(0,1)},
\end{aligned}$$

which achieves the acute-angle inequality.

Further, for Ψ we have $\Psi e_s(x) = \lambda_s e_s(x)$, $s = 1, 2, \ldots$ where, $e_s(x) = 2^{-1/2} \sin s\pi x$ is a complete orthonormalized system of eigenvectors, $\lambda_s = (s\pi)^2$, $s = 1, 2, \ldots$ is the corresponding eigenvalues.

All of Theorem 1's and Theorem 4's hypotheses are clearly fulfilled.

Thus, for the Galërkin solution of problem (30) - (32), the estimate

$$\max_{0 \leq t \leq T} \|\mathcal{Z}_n(x,t) - \mathcal{Z}(x,t)\|_{\mathscr{B}_2(0,1)} \leq c_{26} n^{-\frac{1}{2}}$$

holds.

V. Conclusion

We proved that there exists uniquely a strong solution of the mixed-type problem with boundary conditions for first-order differential and approximate equations. The convergence rate of approximate solutions generated by using the Galërkin method is estimated asymptotically. We utilized the new outcomes of this work to provide an applied example of the IBVP for mixed-type operator-differential equations.

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