Merging The A- and Q-spectrum of Three Kinds of Graph Product

Mengyue Yuan, Fei Wen, Ranran Wang

***Abstract*—For a graph G, let $A(G)$ and $D(G)$ be the adjacency matrix and the degree matrix of G, respectively. Nikiforov defined the matrix $A_\alpha(G)$ as**

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

where $\alpha \in [0,1]$. In this paper, we first give the A_α-spectral radius of semistrong product of arbitrary graph G and a regular graph H. Then we respectively present the A_α-spectra of semistrong product $G \bullet H$ and special product $G \oplus H$ when G and H are two regular graphs. Moreover, we give the A_α-spectral radius of wreath product $G\rho H$.

***Index Terms*—semistrong product, special product, wreath product, A_α-spectral radius, A_α-spectral spectrum.**

I. Introduction

THROUGHOUT this paper, all graphs considered are simple finite undirected graph. Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $A(G)$ and $D(G)$ denote the adjacency matrix and the diagonal matrix of G, respectively. For any real $\alpha \in [0,1]$, Nikiforov [1] defined the matrix $A_\alpha(G)$ as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G).$$

It is easy to see that $A_0(G) = A(G)$, $A_1(G) = D(G)$ and $2A_{1/2}(G) = Q(G)$ if $\alpha = 0, 1, 1/2$, and $L(G) = \frac{A_\alpha - A_\beta}{\alpha-\beta}$ if $\alpha \neq \beta$ for any $\alpha, \beta \in [0,1]$, where $Q(G)$ and $L(G)$ are *signless Laplacian matrix* and *Laplacian matrix* of G, respectively. Let $M(G) = M$ be a graph matrix with respect to $A(G)$ of G. We denote the eigenvalues of M by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$, where $\lambda_1(M)$ is called the *spectral radius*. The collection of eigenvalues of M together with multiplicities are called M-spectrum, denoted by $Spec(M)$ for short. Let $\Phi_M(x) = \det(xI_n - M)$ denote the *characteristic polynomial* of M, where I_n is the identity matrix. Clearly, the graph matrix M is respectively called the *adjacency matrix* and A_α*-matrix* if M equals $A(G)$ and $A_\alpha(G)$, etc.

We use $J_{m\times n}$ to denote the $m \times n$ matrix with each entry one, and write $\vec{j}_m$ for the column m-vector of ones. Further, a graph G is *k-regular* if $d(v) = k$ for all $v \in V(G)$. For other graph theoretic notations and terminologies not defined here, we refer the readers to [2].

The matrix $A_\alpha(G)$ was first defined by Nikiforov (2017) in [1], it has attracted the attention of several scholars in the field of spectral graph. Clearly, $A_\alpha(G)$ is the linear combinations of $A(G)$ and $D(G)$, it was claimed that the matrices $A_\alpha(G)$ can underpin a unified theory of $A(G)$ and $Q(G)$ in [1], [3]. In recent years, a few properties on $A_\alpha(G)$ have been investigated. Nikiforov and Rojo in [4] studied the positive semi-definiteness of matrix $A_\alpha(G)$. Lin et al. [5] considered the A_α-spectral radius of graphs, and then, Wang et al. [7] characterized the connected graphs whose A_α-spectral radius is at most 2. For more properties of $A_\alpha(G)$, we refer the readers to [8]–[11].

In [12], Jaradat introduced seven basic graph products such as *cartesian product*, *lexicographic product*, *directed product*, *strong product*, *semi-strong product*, *special product* and *wreath product*. More recently, Li and Wang [6] respectively presented A_α-spectrum of *cartesian product*, *lexicographic product*, *directed product* and *strong product* of graphs.

Motivated above, in this paper, we continue to study the A_α-spectrum of the remaining three graph products, which are *semi-strong product*, *special product* and *wreath product*. We first give the A_α-spectral radius of semistrong product of arbitrary graph G and a regular graph H. Then we respectively present the A_α-spectra of semistrong product $G \bullet H$ and special product $G \oplus H$ when G and H are two regular graphs. Moreover, we give the A_α-spectral radius of wreath product $G\rho H$.

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II. Preliminaries

In this section, we introduce some definitions and lemmas which will be useful for the proof of main results.

Definition II.1 ([12]). *Given graphs G and H with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Their semistrong product $G \bullet H$ is graph with vertex set $V(G) \times V(H)$, where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent when $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$ (shown in Fig.1 for example).*

Definition II.2 ([12]). *Given graphs G and H with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Their special product $G \oplus H$ is graph with vertex set $V(G) \times V(H)$, where (u_1, v_1) and (u_2, v_2) are adjacent when $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$.*

Remark II.1. *Under the definition of special product, when $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$, there will be multiple edges between (u_1, v_1) and (u_2, v_2). In order to avoid multiple edges, we only add one edge between (u_1, v_1) and (u_2, v_2), other places remain unchange (see Fig.2 for instance).*

Definition II.3 ([12]). *Given graphs G and H with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Their wreath product $G\rho H$ is graph with vertex set*

$V(G) \times V(H)$, and edge set $E(G\rho H) = \{(u_1, v_1)(u_2, v_2) |\, u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and there is $\pi \in Aut(H)$ such that $\pi(v_1) = v_2\}$, where $Aut(H)$ is the automorphism group of H.

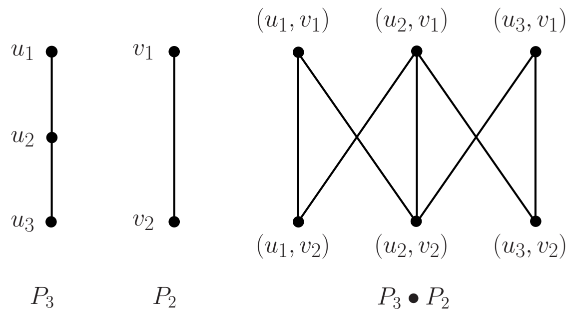

Fig. 1. semistrong product $P_3 \bullet P_2$

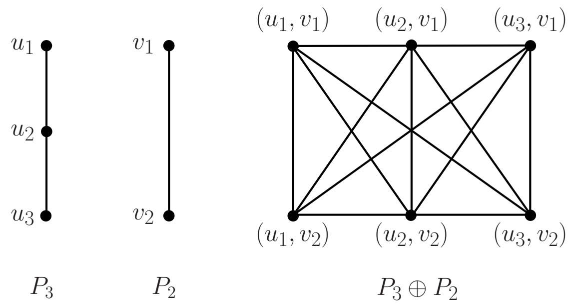

Fig. 2. special product $P_3 \oplus P_2$

Remark II.2. *Under the definition of wreath product, if graph H has more than one permutations, then $G\rho H$ will produce multiple edges. Therefore, we only consider one of all the permutations of H to avoid this case (shown in Fig.3).*

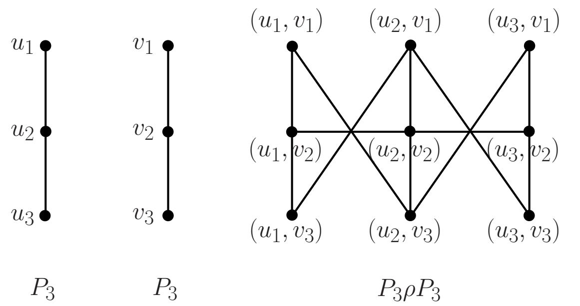

Fig. 3. wreath product $P_3\rho P_3$

Lemma II.1 ([6]). *Let H be a p-regular graph with order n. If $p \geq \lambda_2(H) \geq \cdots \geq \lambda_m(H)$ are the adjacency eigenvalues of H, then*

$$Spec(A_\alpha(H)) = \{p, \alpha p + (1-\alpha)\lambda_2(H), \cdots, \alpha p + (1-\alpha)\lambda_m(H)\}.$$

Furthermore, if $Y = [\vec{j}_m\ \vec{Y}_2\ \cdots\ \vec{Y}_m]$ is an orthogonal matrix whose columns $\vec{j}_m, \vec{Y}_2, \cdots, \vec{Y}_m$ are eigenvectors corresponding to the eigenvalues $p, \lambda_2(H), \cdots, \lambda_m(H)$, respectively, then Y is also an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalues $p, \alpha p + (1-\alpha)\lambda_2(H), \cdots, \alpha p + (1-\alpha)\lambda_m(H)$ of $A_\alpha(H)$, respectively.

III. Main results

Theorem III.1. *Let G be a connected graph with order n, and H be a r-regular graph with order m. Then the A_α-spectral radius of $G \bullet H$ is*

$$\lambda_1(A_\alpha(G \bullet H)) = r + r\lambda_1(A_\alpha(G)).$$

Proof: Let G be a connected graph with $V(G) = \{u_1, u_2, \cdots, u_n\}$, and H be a r-regular graph with $V(H) = \{v_1, v_2, \cdots, v_m\}$. Firstly, we give a partition of the vertex set of $G \bullet H$ and denote by $V(G \bullet H) = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i = \{(u_i, v_j)|j = 1, 2, \cdots, m\}$. For the adjacency matrix $A(G \bullet H)$, it is obvious that

$$A(G \bullet H) = I_n \otimes A(H) + A(G) \otimes A(H).$$

Meanwhile, one can find that

$$d_{G\bullet H}(u_i, v_j) = d_H(v_j) + d_G(u_i) \times d_H(v_j),$$

and further,

$$D(G \bullet H) = I_n \otimes D(H) + D(G) \otimes D(H).$$

Thus we have

$$\begin{aligned} A_\alpha(G \bullet H) &= \alpha D(G \bullet H) + (1-\alpha)A(G \bullet H) \\ &= \alpha(I_n \otimes D(H) + D(G) \otimes D(H)) \\ &\quad +(1-\alpha)(I_n \otimes A(H) + A(G) \otimes A(H)) \\ &= \alpha I_n \otimes D(H) + \alpha D(G) \otimes D(H) \\ &\quad +(1-\alpha)I_n \otimes A(H) + (1-\alpha)A(G) \otimes A(H) \\ &= I_n \otimes A_\alpha(H) + \alpha D(G) \otimes D(H) \\ &\quad +(1-\alpha)A(G) \otimes A(H). \end{aligned}$$

Note that for r-regular graph H of order m, $\vec{j}_m$ is an eigenvector of H corresponding to the spectral radius r. Let $\vec{X}_1 = (x_1, x_2, \cdots, x_n)^T$ be the *Perron vector* of $A_\alpha(G)$, i.e., $x_i > 0$ for each i. It follows that

$$\begin{aligned} &A_\alpha(G \bullet H)(\vec{X}_1 \otimes \vec{j}_m) \\ &= (I_n \otimes A_\alpha(H) + \alpha D(G) \otimes D(H) \\ &\quad +(1-\alpha)A(G) \otimes A(H))(\vec{X}_1 \otimes \vec{j}_m) \\ &= \vec{X}_1 \otimes A_\alpha(H)\vec{j}_m + \alpha D(G)\vec{X}_1 \otimes D(H)\vec{j}_m \\ &\quad +(1-\alpha)A(G)\vec{X}_1 \otimes A(H)\vec{j}_m \\ &= r\vec{X}_1 \otimes \vec{j}_m + \alpha r D(G)\vec{X}_1 \otimes \vec{j}_m + r(1-\alpha)A(G)\vec{X}_1 \otimes \vec{j}_m \\ &= r\vec{X}_1 \otimes \vec{j}_m + rA_\alpha(G)\vec{X}_1 \otimes \vec{j}_m \\ &= (r + r\lambda_1(A_\alpha(G)))(\vec{X}_1 \otimes \vec{j}_m). \end{aligned}$$

Hence, the proof completes. ■

Theorem III.2. *Let G be a p-regular graph with order n and H be a q-regular graph with order m . If $p \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ are adjacency eigenvalues of G and $q \geq \mu_2(H) \geq \cdots \geq \mu_m(H)$ are adjacency eigenvalues of H, then the A_α-spectrum of $G \bullet H$ is*

$$\begin{aligned} &Spec(A_\alpha(G \bullet H)) \\ &= \{pq + q\} \cup \textstyle\bigcup_{i=2}^{n}\{q(1 + \alpha p + (1-\alpha)\lambda_i(G))\} \\ &\cup \textstyle\bigcup_{j=2}^{m}\{(1+p)(\alpha q + (1-\alpha)\mu_j(H))\} \\ &\cup \textstyle\bigcup_{i=2}^{n}\bigcup_{j=2}^{m}\{\alpha q(1+p) + (1-\alpha)\mu_j(H)(\lambda_i(G)+1)\}. \end{aligned}$$

Proof: Let G be a p-regular graph with $V(G) = \{u_1, u_2, \cdots, u_n\}$ and H be a q-regular graph with $V(H) = \{v_1, v_2, \cdots, v_m\}$. Given the vertex partition $V(G \bullet H) = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i = \{(u_i, v_j)|j = 1, 2, \cdots, m\}$, it follows from Theorem III.1 that

$$A_\alpha(G\bullet H) = I_n \otimes A_\alpha(H) + \alpha D(G) \otimes D(H) + (1-\alpha)A(G) \otimes A(H).$$

Let $X = [\vec{j}_n\ \vec{X}_2\ \cdots\ \vec{X}_n]$ be an orthogonal matrix whose columns $\vec{j}_n, \vec{X}_2, \cdots, \vec{X}_n$ are eigenvectors corresponding to the eigenvalues $p, \lambda_2(G), \cdots, \lambda_n(G)$, respectively. Let $Y = [\vec{j}_n\ \vec{Y}_2\ \cdots\ \vec{Y}_m]$ be an orthogonal matrix whose columns $\vec{j}_m, \vec{Y}_2, \cdots, \vec{Y}_m$ are eigenvectors corresponding to the eigenvalues $q, \mu_2(H), \cdots, \mu_m(H)$, respectively. It follows from Lemma II.1 that $\vec{j}_n, \vec{X}_2, \cdots, \vec{X}_n$ are eigenvectors corresponding to the eigenvalues $p, \alpha p + (1-\alpha)\lambda_2(G), \cdots, \alpha p + (1-\alpha)\lambda_n(G)$. Then by Theorem III.1, one can see that $\lambda_1(A_\alpha(G \bullet H)) = pq + q$ since $\vec{j}_n \otimes \vec{j}_m$ is the Perron vector of $A_\alpha(G \bullet H)$ corresponding to $pq + q$.

We then prove that $(1+p)(\alpha q + (1-\alpha)\mu_j(H))$ is an eigenvalue of $A_\alpha(G \bullet H)$, where $j = 2, 3, \cdots, m$. For the vector $\vec{j}_n \otimes \vec{Y}_j$, we have

$$
\begin{aligned}
& A_\alpha(G \bullet H)(\vec{j}_n \otimes \vec{Y}_j) \\
&= (I_n \otimes A_\alpha(H) + \alpha D(G) \otimes D(H) \\
&\quad +(1-\alpha)A(G) \otimes A(H))(\vec{j}_n \otimes \vec{Y}_j) \\
&= \vec{j}_n \otimes A_\alpha(H)\vec{Y}_j + \alpha D(G)\vec{j}_n \otimes D(H)\vec{Y}_j \\
&\quad +(1-\alpha)A(G)\vec{j}_n \otimes A(H)\vec{Y}_j \\
&= (\alpha q + (1-\alpha)\mu_j(H))\vec{j}_n \otimes \vec{Y}_j + \alpha pq\vec{j}_n \otimes \vec{Y}_j \\
&\quad +(1-\alpha)p\mu_j(H)\vec{j}_n \otimes \vec{Y}_j \\
&= ((1+p)(\alpha q + (1-\alpha)\mu_j(H)))(\vec{j}_n \otimes \vec{Y}_j).
\end{aligned}
$$

Thus, $\vec{j}_n \otimes \vec{Y}_j$ is an eigenvector of $A_\alpha(G \bullet H)$ corresponding to $(1+p)(\alpha q + (1-\alpha)\mu_j(H))$.

Next we will show that $q(1 + \alpha p + (1-\alpha)\lambda_i(G))$ is an eigenvalue of $A_\alpha(G \bullet H)$, where $i = 2, 3, \cdots, n$. For the vector $\vec{X}_i \otimes \vec{j}_m$ we have

$$
\begin{aligned}
& A_\alpha(G \bullet H)(\vec{X}_i \otimes \vec{j}_m) \\
&= (I_n \otimes A_\alpha(H) + \alpha D(G) \otimes D(H) \\
&\quad +(1-\alpha)A(G) \otimes A(H))(\vec{X}_i \otimes \vec{j}_m) \\
&= \vec{X}_i \otimes A_\alpha(H)\vec{j}_m + \alpha D(G)\vec{X}_i \otimes D(H)\vec{j}_m \\
&\quad +(1-\alpha)A(G)\vec{X}_i \otimes A(H)\vec{j}_m \\
&= q\vec{X}_i \otimes \vec{j}_m + \alpha pq\vec{X}_i \otimes \vec{j}_m + (1-\alpha)q\lambda_i(G)\vec{X}_i \otimes \vec{j}_m \\
&= (q + \alpha pq + (1-\alpha)q\lambda_i(G))(\vec{X}_i \otimes \vec{j}_m) \\
&= (q(1 + \alpha p + (1-\alpha)\lambda_i(G)))(\vec{X}_i \otimes \vec{j}_m).
\end{aligned}
$$

Hence, $\vec{X}_i \otimes \vec{j}_m$ is an eigenvector of $A_\alpha(G \bullet H)$ corresponding to $q(1 + \alpha p + (1-\alpha)\lambda_i(G))$ for $i = 2, 3, \cdots, n$.

At last, we will verify that $\alpha q(1+p) + (1-\alpha)\mu_j(H)(\lambda_i(G) + 1)$ are A_α-eigenvalues of $G \bullet H$ for $i = 2, 3, \cdots, n$ and $j = 2, 3, \cdots, m$.

$$
\begin{aligned}
& A_\alpha(G \bullet H)(\vec{X}_i \otimes \vec{Y}_j) \\
&= (I_n \otimes A_\alpha(H) + \alpha D(G) \otimes D(H) \\
&\quad +(1-\alpha)A(G) \otimes A(H))(\vec{X}_i \otimes \vec{Y}_j) \\
&= \vec{X}_i \otimes A_\alpha(H)\vec{Y}_j + \alpha D(G)\vec{X}_i \otimes D(H)\vec{Y}_j \\
&\quad +(1-\alpha)A(G)\vec{X}_i \otimes A(H)\vec{Y}_j \\
&= (\alpha q + (1-\alpha)\mu_j(H))\vec{X}_i \otimes \vec{Y}_j + \alpha pq\vec{X}_i \otimes \vec{Y}_j \\
&\quad +(1-\alpha)\lambda_i(G)\mu_j(H)\vec{X}_i \otimes \vec{Y}_j \\
&= (\alpha q(1+p) + (1-\alpha)\mu_j(H)(\lambda_i(G)+1))(\vec{X}_i \otimes \vec{Y}_j).
\end{aligned}
$$

Therefore, $\vec{X}_i \otimes \vec{Y}_j$ is an eigenvector of $A_\alpha(G \bullet H)$ corresponding to $\alpha q(1+p) + (1-\alpha)\mu_j(H)(\lambda_i(G)+1)$.

Note that $\vec{X}_i$ and $\vec{Y}_j$ are orthogonal to all one column vector, it is easy to prove that $\vec{X}_i \otimes \vec{Y}_j$, $\vec{X}_i \otimes \vec{j}_m$, $\vec{j}_n \otimes \vec{Y}_j$ and $\vec{j}_n \otimes \vec{j}_m$ are orthogonal.

To sum up, the proof is completed. ■

Corollary III.1. *If G is a p-regular graph and H is q-regular graph, then the A_α-spectral radius of $G \bullet H$ is $q(p+1)$.*

From Theorem III.2, one can easily obtained the A_α-spectrum of the semistrong product of two regular graphs.

Example III.1. *Let $G = C_5$ and $H = C_3$. The semistrong product $C_5 \bullet C_3$ is shown in Fig.4. By simple computation, one can get $Spec(A(C_5)) = \{2, 0.618^2, -1.618^2\}$ and $Spec(A(C_3)) = \{2, -1^2\}$. It follows from Theorem III.2 that*

$$
\begin{aligned}
Spec(A_\alpha(C_5 \bullet C_3)) = \{&6, (9\alpha-3)^2, (2.764\alpha+3.236)^2, (7.236\alpha- \\
&1.236)^2, (7.618\alpha-1.618)^4, (5.382\alpha+0.618)^4\}.
\end{aligned}
$$

On the other hand, by Matlab 7.0 we get $Spec(A(C_5 \bullet C_3)) = \{6, 3.236^2, 0.618^4, -1.236^2, -1.618^4, -3^2\}$. Furthermore, from Corollary III.2 we know that $C_5 \bullet C_3$ is 6-regular graph. According to the definition $A_\alpha(C_5 \bullet C_3) = \alpha D(C_5 \bullet C_3) + (1-\alpha)A(C_5 \bullet C_3)$, we can also calculate the A_α-spectrum of $C_5 \bullet C_3$, which is the same as the above calculation.

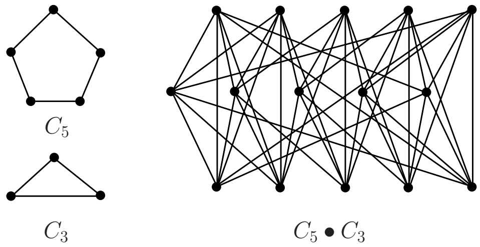

Fig. 4. C_5, C_3 and semistrong product $C_5 \bullet C_3$

Besides, one can use Theorem III.2 to construct infinitely many pairs of A_α-cospectral graphs.

Corollary III.2. *Let H be a regular graph. If G_1 and G_2 are A-cospectral regular graphs but not isomorphic with each other, then $G_1 \bullet H$ and $G_2 \bullet H$ are A_α-cospectral.*

Example III.2. *Let G_1 and G_2 be two 4-regular graphs (shown in Fig.6). Then by Matlab 7.0 we get $Spec(A(G_1)) = Spec(A(G_2)) = \{4, 2.2361, 1.5616, 1, -1^4, -2.2361, -2.5616\}$ and $Spec(A(P_2)) = \{1, -1\}$. It is easy to find that G_1 and G_2 are A-cospectral but not isomorphic with each other. It follows from Theorem III.2 that $Spec(A_\alpha(G_1 \bullet H)) = Spec(A_\alpha(G_2 \bullet H)) = \{5, 1.7639\alpha + 3.2361, 2.4384\alpha + 2.5616, 3\alpha+2, 3.4384\alpha+1.5616, 3.7639\alpha+1.2361, 5\alpha^8, 6.2361\alpha-1.2361, 6.5616\alpha - 1.5616, 7\alpha - 2, 7.5616\alpha - 2.5616, 8.2361\alpha - 3.2361, 10\alpha-5\}$, which means $G_1 \bullet P_2$ and $G_2 \bullet P_2$ are A_α-cospectral.*

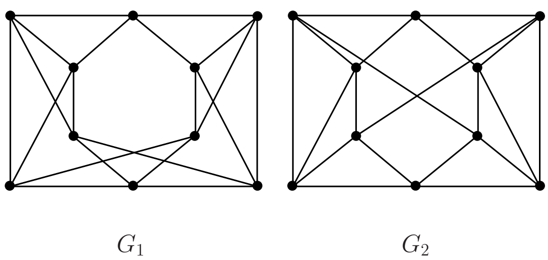

Fig. 5. G_1 and G_2

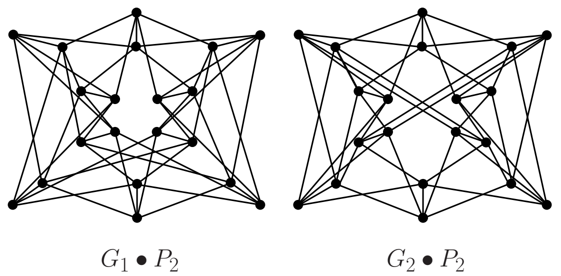

Fig. 6. $G_1 \bullet P_2$ and $G_2 \bullet P_2$

Theorem III.3. *Let G be a p-regular graph with order n and H be a q-regular graph with order m. If $p \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ are adjacency eigenvalues of G and $q \geq \mu_2(H) \geq \cdots \geq \mu_m(H)$ are adjacency eigenvalues of H, then under the condition of Remark II.1, we have*

$$\begin{aligned}&Spec(A_\alpha(G \oplus H))\\&= \bigcup_{i=2}^{n}\bigcup_{j=2}^{m}\{\alpha(mp+nq-pq)-(1-\alpha)\lambda_i(G)\mu_j(H)\}\\&\cup\bigcup_{i=2}^{n}\{\alpha(mp+nq-pq)+(1-\alpha)(m-q)\lambda_i(G)\}\\&\cup\bigcup_{j=2}^{m}\{\alpha(mp+nq-pq)+(1-\alpha)(n-p)\mu_j(G)\}\\&\cup\{mp+nq-pq\}.\end{aligned}$$

Proof: Let G be a p-regular graph with $V(G) = \{u_1, u_2, \cdots, u_n\}$ and H be a q-regular graph with $V(H) = \{v_1, v_2, \cdots, v_m\}$. Under the condition of Remark II.1, by the partition $V(G \bullet H) = V_1 \cup V_2 \cup \cdots \cup V_n$ one can get

$$A(G \oplus H) = A(G) \otimes J_{m\times m} + J_{n\times n} \otimes A(H) - A(G) \otimes A(H)$$

where $V_i = \{(u_i, v_j)|j = 1, 2, \cdots, m\}$. It is easy to see that

$$d_{G\oplus H}(u_i,v_j)=d_G(u_i)\cdot|V(H)|+d_H(v_j)\cdot|V(G)|-d_G(u_i)\cdot d_H(v_j),$$

which leads to

$$D(G \oplus H) = mD(G) \otimes I_m + nI_n \otimes D(H) - D(G) \otimes D(H).$$

It therefore follows that

$$\begin{aligned}A_\alpha(G \oplus H) &= \alpha D(G \oplus H)+(1-\alpha)A(G \oplus H)\\&= \alpha mD(G) \otimes I_m + \alpha nI_n \otimes D(H)\\&\quad -\alpha D(G) \otimes D(H) + (1-\alpha)A(G) \otimes J_{m\times m}\\&\quad +(1-\alpha)J_{n\times n}\otimes A(H)-(1-\alpha)A(G) \otimes A(H).\end{aligned}$$

Let $X = [\vec{j}_n\ \vec{X}_2\ \cdots\ \vec{X}_n]$ be an orthogonal matrix whose columns $\vec{j}_n, \vec{X}_2, \cdots, \vec{X}_n$ are eigenvectors corresponding to the eigenvalues $p, \lambda_2(G), \cdots, \lambda_n(G)$, respectively. Let $Y = [\vec{j}_n\ \vec{Y}_2\ \cdots\ \vec{Y}_m]$ be an orthogonal matrix whose columns $\vec{j}_m, \vec{Y}_2, \cdots, \vec{Y}_m$ are eigenvectors corresponding to the eigenvalues $q, \mu_2(H), \cdots, \mu_m(H)$, respectively.

Using the same argument as the A_α-spectrum of $G \bullet H$, we have

$$\begin{aligned}&A_\alpha(G \oplus H)(\vec{j}_n \otimes \vec{j}_m)\\&= \alpha mD(G)\vec{j}_n \otimes \vec{j}_m+\alpha n\vec{j}_n \otimes D(H)\vec{j}_m-\alpha D(G)\vec{j}_n\otimes D(H)\vec{j}_m\\&\quad +(1-\alpha)A(G)\vec{j}_n\otimes J_{m\times m}\vec{j}_m+(1-\alpha)J_{n\times n}\vec{j}_n\otimes A(H)\vec{j}_m\\&\quad -(1-\alpha)A(G)\vec{j}_n\otimes A(H)\vec{j}_m\\&= \alpha mp\vec{j}_n\otimes\vec{j}_m+\alpha nq\vec{j}_n \otimes \vec{j}_m - \alpha pq\vec{j}_n\otimes\vec{j}_m\\&\quad +(1-\alpha)mp\vec{j}_n\otimes\vec{j}_m+(1-\alpha)nq\vec{j}_n\otimes\vec{j}_m - (1-\alpha)pq\vec{j}_n \otimes \vec{j}_m\\&= (\alpha mp+\alpha nq-\alpha pq+(1-\alpha)mp+(1-\alpha)nq\\&\quad -(1-\alpha)pq)(\vec{j}_n\otimes\vec{j}_m)\\&= (mp + nq - pq)(\vec{j}_n\otimes\vec{j}_m)\end{aligned}$$

Therefore, the spectral radius of $A_\alpha(G \oplus H)$ is $mp+nq-pq$. Besides,

$$\begin{aligned}&A_\alpha(G \oplus H)(\vec{X}_i \otimes \vec{Y}_j)\\&= \alpha mD(G)\vec{X}_i \otimes \vec{Y}_j+\alpha n\vec{X}_i\otimes D(H)\vec{Y}_j-\alpha D(G)\vec{X}_i\otimes D(H)\vec{Y}_j\\&\quad +(1-\alpha)A(G)\vec{X}_i \otimes J_{m\times m}\vec{Y}_j + (1-\alpha)J_{n\times n}\vec{X}_i \otimes A(H)\vec{Y}_j\\&\quad -(1-\alpha)A(G)\vec{X}_i \otimes A(H)\vec{Y}_j\\&= \alpha mp\vec{X}_i \otimes \vec{Y}_j + \alpha nq\vec{X}_i \otimes \vec{Y}_j - \alpha pq\vec{X}_i \otimes \vec{Y}_j + 0 + 0\\&\quad -(1-\alpha)\lambda_i(G)\mu_j(H)\vec{X}_i \otimes \vec{Y}_j\\&= (\alpha(mp + nq - pq) - (1-\alpha)\lambda_i(G)\mu_j(H))(\vec{X}_i \otimes \vec{Y}_j).\end{aligned}$$

Hence, $\vec{X}_i\otimes\vec{Y}_j$ is an eigenvector of $A_\alpha(G\oplus H)$ corresponding to $\alpha(mp+nq-pq)-(1-\alpha)\lambda_i(G)\mu_j(H)$, where $i = 2, 3, \cdots, n$ and $j = 2, 3, \cdots, m$.

Moreover,

$$\begin{aligned}&A_\alpha(G \oplus H)(\vec{X}_i \otimes \vec{j}_m)\\&= \alpha mD(G)\vec{X}_i \otimes \vec{j}_m + \alpha n\vec{X}_i \otimes D(H)\vec{j}_m\\&\quad -\alpha D(G)\vec{X}_i \otimes D(H)\vec{j}_m + (1-\alpha)A(G)\vec{X}_i \otimes J_{m\times m}\vec{j}_m\\&\quad +(1-\alpha)J_{n\times n}\vec{X}_i \otimes A(H)\vec{j}_m - (1-\alpha)A(G)\vec{X}_i \otimes A(H)\vec{j}_m\\&= \alpha mp\vec{X}_i \otimes \vec{j}_m + \alpha nq\vec{X}_i \otimes \vec{j}_m - \alpha pq\vec{X}_i \otimes \vec{j}_m\\&\quad +(1-\alpha)m\lambda_i(G)\vec{X}_i \otimes \vec{j}_m + 0 - (1-\alpha)q\lambda_i(G)\vec{X}_i \otimes \vec{j}_m\\&= (\alpha(mp + nq - pq) + (1-\alpha)(m-q)\lambda_i(G))(\vec{X}_i \otimes \vec{j}_m)\end{aligned}$$

Thus, $\vec{X}_i\otimes\vec{j}_m$ is an eigenvector of $A_\alpha(G\oplus H)$ corresponding to $\alpha(mp+nq-pq)-(1-\alpha)(m-q)\lambda_i(G)$. By the same way, for $j = 2, 3, \cdots, m$, we can obtain that $\vec{j}_n \otimes \vec{Y}_j$is an eigenvector of $A_\alpha(G\oplus H)$ corresponding to $\alpha(mp+nq-pq)-(1-\alpha)(n-p)\mu_j(H)$.

It is not hard to see that $\vec{X}_i \otimes \vec{Y}_j$, $\vec{X}_i \otimes \vec{j}_m$, $\vec{j}_n \otimes \vec{Y}_j$ and $\vec{j}_n \otimes \vec{j}_m$ are orthogonal.

Thus, the proof completes.

■

Corollary III.3. *Let G be a p-regular graph and H be a q-regular graph. Then, under Remark II.1, $G \oplus H$ is a $(mp+nq-pq)$-regular graph.*

In accordance with Theorem III.3, we give an example to find the A_α-spectrum of the special product of two regular graphs.

Example III.3. *Let $G = C_4$ and $H = K_2$. The special product $C_4 \oplus K_2$ is shown in Fig.7. By direct calculation, we have $Spec(A(C_4)) = \{2, 0^2, -2\}$ and $Spec(A(K_2)) = \{1, -1\}$. Thus, it follows from Theorem III.3 that*

$$Spec(A_\alpha(C_4 \oplus K_2)) = \{6, 6\alpha^4, 8\alpha - 2^3\}.$$

On the other hand, by Matlab 7.0 one can obtain $Spec(A(C_4 \oplus K_2)) = \{6, 0^4, -2^3\}$, and then according to $A_\alpha(C_4 \oplus K_2) = \alpha D(C_4 \oplus K_2) + (1-\alpha)A(C_4 \oplus K_2)$, we can directly calculate the A_α-spectrum of $C_4 \oplus K_2$, the result is the same as above.

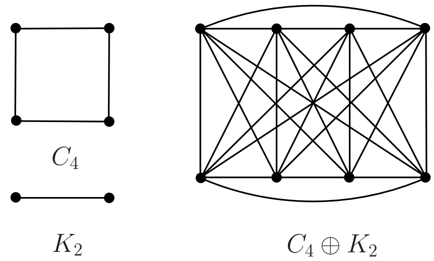

Fig. 7. C_4, K_2 and special product $C_4 \oplus K_2$

In addition, we use the Theorem III.3 to construct families of A_α-cospectral graphs.

Corollary III.4. *Let H be a regular graph and let G_1 and G_2 be A-cospectral regular graphs but not isomorphic with each other, under the condition of Remark II.1, $G_1 \oplus H$ and $G_2 \oplus H$ are A_α-cospectral.*

Theorem III.4. *Let G be a p-regular graph with order n and H be a q-regular graph with order m. If $Spec(A(G)) = \{p, \lambda_2(G), \cdots, \lambda_n(G)\}$ and $Spec(A(H)) = \{q, \mu_2(H), \cdots, \mu_m(H)\}$, then under the Remark II.2, we have*

$$\lambda_1(A_\alpha(G\rho H)) = p + q.$$

Proof: Let G be a p-regular graph with $V(G) = \{u_1, u_2, \cdots, u_n\}$ and H be a q-regular graph with $V(H) = \{v_1, v_2, \cdots, v_m\}$. A permutation π of $V(H)$ can be represented by a permutation matrix $P = (p_{ij})$, where $p_{ij} = 1$ if $v_i = \pi(v_j)$, and $p_{ij} = 0$ otherwise. Under the condition of Remark II.2 and vertex partition $V(G \bullet H) = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i = \{(u_i, v_j) | j = 1, 2, \cdots, m\}$, it is obvious that

$$A(G\rho H) = \begin{pmatrix} A(H) & a_{12}P & \cdots & a_{1n}P \\ a_{21}P & A(H) & \cdots & a_{2n}P \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}P & a_{n2}P & \cdots & A(H) \end{pmatrix} = I_n \otimes A(H) + A(G) \otimes P$$

and

$$D(G\rho H) = I_n \otimes D(H) + D(G) \otimes I_m.$$

Then

$$\begin{aligned} A_\alpha(G\rho H) &= \alpha(I_n \otimes D(H) + D(G) \otimes I_m) \\ &\quad + (1-\alpha)(I_n \otimes A(H) + A(G) \otimes P) \\ &= I_n \otimes A_\alpha(H) + \alpha D(G) \otimes I_m + (1-\alpha)A(G) \otimes P. \end{aligned}$$

It is well known that $\vec{j}_n$ and $\vec{j}_m$ are the Perron vectors of regular graphs G and H corresponding to eigenvalues p and q, respectively. Thus, one can get

$$\begin{aligned} & A_\alpha(G\rho H)(\vec{j}_n \otimes \vec{j}_m) \\ &= (I_n \otimes A_\alpha(H) + \alpha D(G) \otimes I_m + (1-\alpha)A(G) \otimes P)(\vec{j}_n \otimes \vec{j}_m) \\ &= \vec{j}_n \otimes A_\alpha(H)\vec{j}_m + \alpha D(G)\vec{j}_n \otimes \vec{j}_m + (1-\alpha)A(G)\vec{j}_n \otimes P\vec{j}_m \\ &= q\vec{j}_n \otimes \vec{j}_m + \alpha p\vec{j}_n \otimes \vec{j}_m + (1-\alpha)p\vec{j}_n \otimes \vec{j}_m \\ &= (q + \alpha p + (1-\alpha)p)(\vec{j}_n \otimes \vec{j}_m) \\ &= (p+q)(\vec{j}_n \otimes \vec{j}_m) \end{aligned}$$

It is completed the proof. ∎

Corollary III.5. *Let G be a p-regular graph and H be a q-regular graph. Then, under the condition of Remark II.2, $G\rho H$ is a (p+q)-regular graph.*

According to the above Corollary III.5, one can give the A_α-spectral radius of two regular graphs.

Example III.4. *Let $G = C_4$ and $H = C_3$, their wreath product $C_4 \rho C_3$ is shown in Fig.8. According to Corollary III.5, one can easily obtain $\lambda_1(A_\alpha(C_4 \rho C_3)) = 4$.*

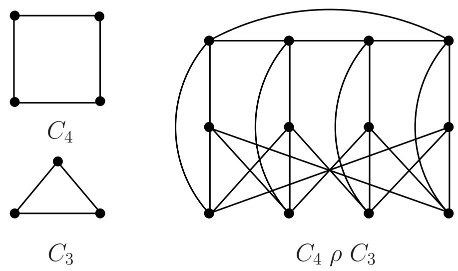

Fig. 8. C_4, C_3 and wreath product $C_4 \oplus C_3$

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