Fractional-order Quadratic Time-varying Parameters Discrete Grey Model FQDGM (1, 1) and Its Application

Xiaogao Yang, Deqiong Ding*, Youxin Luo

***Abstract*— Several objects in scientific research and engineering applications present fractional-order time-varying characteristics. When developing mathematical models of considering fractional-order time-varying, the dynamic analysis results are often inconsistent with the actual statistical data due to inaccurate description of the analysis model. To solve this problem, the present paper proposes a discrete grey model of fractional-order quadratic time-varying parameters based on grey fractional-order characteristics and integral theory. By taking the minimum mean relative error and the minimum mean square error as the objective function, two optimization models, namely FQDGM-I and FQDGM-II, are constructed to perform parameter estimation. A comparative analysis based on examples shows that the background value reconstructed using the proposed model is more accurate. The fitting accuracy is greatly improved compared to those obtained using previous models. These results verify the effectiveness and practicability of the proposed fractional-order quadratic time-varying parameter discrete grey model.**

***Index Terms*—Fractional-order quadratic time-varying parameters, discrete grey prediction model, least-square estimation, optimization model**

I. Introduction

THE model GM(1, 1) is the basis of grey system prediction theory, which is widely used in data analysis and prediction in industry, agriculture, social economy, and other fields. The GM (1, 1) model has been universally used in scientific research because it does not need samples to show regular distribution and is easy to calculate and test [1],[2]. With its continuous application in different research areas, the model shows many defects, such as inaccurate background value reconstruction, inaccurate model description, and even great differences between the analysis results and the actual data. Therefore, scholars have modified the GM (1, 1) to improve the prediction accuracy [3], [4], [5], [6], [7], [8], [9]. To verify the feasibility of China's total poverty alleviation in 2020, Liu et al. [10] analyzed the exogenous influence and improved the GM (1,1) model. Based on the GM (1,1, t^{α}) model and the fractional accumulated generating operation, Wu et al. [11] proposed a fractional grey model FAGM (1,1, t^{α}). A reciprocal accumulating and optimized background value were used in [12] to establish a grey new information GRM (1, 1) to improve the fitting accuracy of monotonic decreasing data. Moreover, the non-equidistant cumulative grey theory model NNFGM (1, 1) was established to develop a GM (1, 1) suitable for non-uniform (or nearly non-uniform) systems with non-equidistant sequences [13]. The developed model relied on the non-liquidity cumulative generation operator and the inverse cumulative generation operator.

Parameters with integer-order characteristics have been used in several studies to establish grey theoretical models to analyze systems with fractional-order. Indeed, choosing parameters with fractional-order properties to describe the object could better reveal its essential properties and behavior. The previous studies ignored the actual order (fractional order) of the system, mainly due to the complexity of fractional grey theory modeling and the lack of the corresponding theoretical basis. With the continuous development of grey theory, this "bottleneck" has gradually been overcome, and related achievements have been constantly emerged [14],[15],[16][17]. In the theoretical research and practical application of grey theory, the model parameters describing the object were extended from integral calculus to fractional calculus, substantially improving the accuracy of the model describing the research objects [18],[19],[20],[21]. These models extended the existing integer order grey models and played a role in promoting the development and application of grey theory.

Because the parameters of some research objects in engineering practice are time-varying, describing them accurately using the previous grey theory model is challenging. Therefore, scholars developed fractional-order multivariable grey models [22],[23],[24],[25]. These analysis models have greatly improved the analysis accuracy, but the results still cannot meet the engineering requirements. In this study, a fractional-order quadratic time-varying parameter

Manuscript received June 30, 2020; revised September 26, 2021.This work is supported by the fund items as follows: the science and technology plan project of Changde science and Technology Bureau (CKH (2020) 75|No.2020C084 and No.2020S109), the Dr. Scientific research start project (NO. 403|E0 7016046), the scientific Research Foundation of Hunan Provincial Education Department (17A147), the project of Hunan Provincial Department of Education (19c1271), the provincial specialty disciplines of higher education institutions in Hunan Province (XJT [2018]469), the general project of ChangDe science and Technology Bureau (2018J051), the provincial specialty disciplines of higher education institutions in Hunan Province (XJT [2019]379), the Hunan provincial and Municipal Joint Fund Project (2020JJ6006), the key projects of Hunan Education Department (19A341).

Xiaogao Yang is an associate professor of the College of Mechanical Engineering, Hunan University of Arts and Science, ChangDe, 415000, China (e-mail: yxg_568@126.com).

Deqiong Ding* is an associate professor of the college of Mathematics and Statistics, Hunan University of Finance and Economics, ChangSha, 410205, China (e-mail: ddqshe2010@163.com).

Youxin Luo is a Professor of College of Mechanical Engineering, Hunan University of Arts and Science, ChangDe, 415000, China (e-mail: llyx123@126.com)

discrete grey model, FQDGM (1, 1), is proposed. The minimum relative error and the minimum mean square error are set as optimization objectives in developing FQDGM-I and FQDGM-II. In these models, the fractional-order and initial value correction are used as design variables, and the least-squares method is employed to estimate the model parameters. The model is verified through examples and compared with previous studies. The results show that the background value reconstruction and data-fitting accuracy of the model have been significantly improved, and the model has high theoretical and application value.

II. FRACTIONAL-ORDER QUADRATIC TIME-VARYING PARAMETERS DISCRETE GREY PREDICTION MODEL

A. Fractional-order Generation Sequence

If there is an original sequence as shown below

$$X^{(0)}=[x^{(0)}(1)\quad x^{(0)}(2)\cdots x^{(0)}(n)], \tag{1}$$

the cumulative generation sequence can be expressed as

$$X^{(r)}=[x^{(r)}(1)\quad x^{(r)}(2)\cdots x^{(r)}(n)]. \tag{2}$$

In (2), The $x^{(r)}(n)$ can be obtained by the following expression

$$x^{(r)}(k)=\sum_{i=1}^{k}x^{(r-1)}(i)=x^{(r)}(k-1)+x^{(r-1)}(k)\cdot \tag{3}$$

The $X^{(r)}$ can be obtained according to the principle of matrix operation as follows:

$$X^{(r)}=A_1X^{(r-1)}=A_1A_1X^{(r-2)}=A_1^rX^{(0)}, \tag{4}$$

where A^r represents the cumulative generation matrices $A_1^r=(a_{ij}^r)_{n\times n}$

$$(a_{ij}^r)_{n\times n}=\begin{cases}\dfrac{(i-j+r-1)!}{(i-j)!(r-1)!} & i\geq j\\ 0 & i<j\end{cases}. \tag{5}$$

For r in (4), the combination number is generalized from integer to fraction. Then, the fractional-order cumulative generating matrix is derived. The coefficient of $X^{(r)}(k)$ in the expansion can be obtained using the following method.

Suppose a_k can be expressed as

$$a_k=\frac{(k-i+1)(k-i+2)(k-i+r-1)}{(r-1)!}(k=1,2\cdots n), \tag{6}$$

the a_k can be evolved as

$$a_k=\frac{(r+k-i+1)!}{(k-i)!(r-1)!}=\frac{\Gamma(r+k-1)}{\Gamma(k-i+1)\Gamma(r)}\cdot \tag{7}$$

in which, Γ is a Gamma function. So, $X^{(r)}(k)$ can be expressed with a series as follow,

$$x^{(r)}(k)=\sum_{i=1}^{k}\frac{\Gamma(r+k-1)}{\Gamma(k-i+1)\Gamma(r)}x^{(0)}(i)\cdot \tag{8}$$

Note that when r is an integer, (3) is established. Equation (8) is the general condition in which r expands the fractions with integers. Thus, (6) is used to calculate the fractional-order $X^{(r)}(k)$.

B. Quadratic Time-varying Parameters Discrete Grey Model

If $X^{(1)}(k)$ is the one-time cumulative generation sequence of the non-negative original sequence, the quadratic time-varying parameters of the discrete grey model can be established as follows:

$$x^{(1)}(k+1)=(\beta_1+\beta_2k+\beta_3k^2)x^{(1)}(k)+\beta_4k^2+\beta_5k+\beta_6 \tag{9}$$

in which, k=1, 2, 3… n-1.

The introduction of a quadratic time-varying term is paramount because it causes the parameters to have time-varying characteristics. Using quadratic time-varying parameters in grey theory makes the model consistent with the white exponential law, linear law, quadratic coincidence, and stretching transformation.

The quadratic time-varying parameters extend the application of grey theory to non-isometric sequences. When $\beta_3=\beta_4=0$, the QDGM (1, 1) model is transformed into a linear time varying parameter discrete grey model TDGM (1, 1). When $\beta_2=\beta_3=\beta_4=0$, the QDGM (1, 1) model is transformed into the original discrete grey model DGM (1, 1). The QDGM (1, 1) model is a general form of discrete grey model DGM (1, 1) and linear time-varying parameter discrete grey model TDGM (1, 1). If the initial condition is given as follows:

$$\hat{X}^{(0)}(1)=x^{(0)}(1)+C_1, \tag{10}$$

the optimization model is established with C_1 as the initial value and the minimum average relative error as objective.

It is not difficult to determine that the discrete gray system model QDGM (1, 1) with quadratic time-varying parameters has the first-order cumulative characteristic. The change of initial conditions does not affect the simulation value of the QDGM (1, 1) model. When the actual sequence growth is non-negative, the simulation value is increasingly convex or decreasingly concave. These limitations prevent the QDGM (1, 1) model from achieving high prediction accuracy. Therefore, the QDGM (1, 1) is extended to the FQDGM (1, 1), which more effectively reveals the essential characteristics and behavior of the object.

C. Fractional-order Quadratic Time-varying Parameters Discrete Grey Model

If $X^{(r)}(k)$ is a one-time cumulative generation sequence of the non-negative original sequence, the quadratic time-varying parameter sequence of the discrete grey model can be expressed as follows:

$$x^{(r)}(k+1)=(\beta_1+\beta_2k+\beta_3k^2)x^{(r)}(k)+\beta_4k^2+\beta_5k+\beta_6 \tag{11}$$

where k=1, 2, 3… n-1. In the FQDGM (1,1), $X^{(r)}(k)$ is the r order cumulative generation sequence of the non-negative original sequence.

It is easy to determine that the fractional-order quadratic time-varying parameters discrete grey model FQDGM (1,1) is a general form. The model will correspond to different models for different values of parameters r and β. If the fractional number r=1, the fractional-order quadratic time-varying parameters discrete grey model FQDGM (1, 1) is transformed into the quadratic time-varying parameters discrete grey model QDGM (1, 1). When $\beta_3=\beta_4=0$, the fractional-order quadratic time-varying parameters discrete grey model FQDGM (1, 1) is transformed into a fractional linear time-varying parameter discrete grey model FTDGM (1, 1).

Furthermore, the fractional-order quadratic time-varying parameters discrete grey model FQDGM (1, 1) is an extension of the QDGM (1, 1), FTDGM (1, 1), TDGM (1, 1), FDGM (1, 1), and DGM (1, 1) models. Consequently, the values of r and β play an important role in the proposed model. Specifically, the model has different accuracy for the fractional-order number r. The parameter β can be obtained

$$B=\left|\begin{matrix} X^{(r)}(1) & X^{(r)}(1) & X^{(r)}(1) & 1 & 1 & 1 \\ X^{(r)}(2) & 2X^{(r)}(1) & 4X^{(r)}(1) & 4 & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X^{(r)}(n-1) & (n-1)X^{(r)}(n-1) & (n-1)^2X^{(r)}(n-1) & (n-1)^2 & (n-1) & 1 \end{matrix}\right| \quad (13)$$

with the following equation:

$$P=\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_6 \end{bmatrix}=(B^TB)^TB^TY \quad (12)$$

where the matrix B can be expressed as (13).
The variable Y can be obtained as follows:

$$Y=\left|\begin{matrix} X^{(r)}(2) \\ X^{(r)}(3) \\ \vdots \\ X^{(r)}(n) \end{matrix}\right| \quad (14)$$

The order parameter r of the FQDGM (1,1) can be obtained using the following steps:

①Given the initial conditions $X^{(r)}(1)=X^{(0)}(1)+C_1$ (C_1 is a correction), the estimation can be expressed as:

$$\hat{x}^{(r)}(k+1)=(\hat{\beta}_1+\hat{\beta}_2k+\hat{\beta}_3k^2)\hat{x}^{(r)}(k)+\hat{\beta}_4k^2+\hat{\beta}_5k+\hat{\beta}_6 \quad (15)$$

in which, k=1, 2, 3… n-1.

②For A^r to satisfy $\begin{cases}(A_1^r)^{-1}=A_1^{-r} \\ A_1^rA_1^{-r}=I \\ X^{(0)}A_1^{-r}A_1^rX^{(0)}=A_1^{-r}X^{(r)}\end{cases}$, defining

A_1^{-r} as the subtraction generation matrix, $\hat{x}^{(r)}$ can be restored to the original sequence of $\hat{X}^{(0)}$ as follows:

$$\hat{x}^{(0)}(k)=\sum_{i=0}^{k-1}(-1)^i\frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)}\hat{x}^{(r)}(k-i) \quad (16)$$

III. OPTIMIZATION MODEL

The fractional-order quadratic time-varying parameter discrete grey model FQDGM (1, 1) is established by taking the fractional-order r as design variables and the minimum average relative error as the objective function (note the optimization model of FQDGM-I).

$$\min f(r,C_1)=\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x^{(0)}(k)-\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right| \quad (17)$$

Moreover, if the fractional-order r is taken as a design variable and the minimum mean square error as the objective function, a new fractional-order quadratic time-varying parameter discrete grey model FQDGM (1, 1) is established (namely, FQDGM-II).

$$\min f(r,C_1)=\frac{1}{n}\sum_{k=1}^{n}(x^{(0)}(k)-\hat{x}^{(0)}(k))^2 \quad (18)$$

IV. VERIFICATION AND APPLICATION

The per capita disposable income of urban households and urban built-up areas are important objects of national economic regulation, which are directly related to the well-being of urban residents. It is an important content of economic analysis and regulation to analyze the data of per capita disposable income of urban households and urban built-up area every year and make corresponding decisions. However, because the background value fitting in previous studies was not accurate, the analysis results often do not meet the requirements. Therefore, statistical data analysis and experimental research have become the chosen research methods, but these approaches often lead to growth in the analysis cycle and increased research costs. In this section, we present some FQDGM (1, 1) applications using the proposed approach.

A. Application1

In this section, the per capita disposable income data for urban households in China from 2000 to 2009 were selected for analysis [26]. The statistical data is shown in Table I.

TABLE I
PER CAPITA DISPOSABLE INCOME OF URBAN HOUSEHOLDS IN CHINA FROM 2000 TO 2009

Ordinal	Year	Per capita disposable income (RMB ¥)
1	2000	6280.0
2	2001	6859.6
3	2002	7702.8
4	2003	8472.2
5	2004	9421.6
6	2005	10493.0
7	2006	11759.5
8	2007	13785.8
9	2008	15780.8
10	2009	17174.7

The grey theory analysis models were used to calculate the per capita disposable income of urban households. The comparative analysis of the discrete gray model QDGM in [26], the DGM in [27], and the proposed fractional-order quadratic time-varying parameter discrete gray optimization model is performed. The calculation results have been obtained, as shown in Table II, where $x^{(0)}$(k) is the per capita disposable income.

TABLE II
COMPARISON OF FITTING VALUE ABOUT FOUR MODELS

k	$x^{(0)}$(k)	**Fitting value**			
		DGM	QDGM	FQDGM-I	FQDGM-II
1	6280.0	6425.9	6339.2	6295.9	6295.9
2	6859.6	7122.5	6927.9	6818.0	6817.9
3	7702.8	7894.5	7619.8	7752.7	7752.7
4	8472.2	8750.1	8431.3	8470.6	8470.6
5	9421.6	9698.6	9385.9	9432.1	9432.1
6	10493.0	10749.8	10515.6	10485.0	10485.1
7	11759.5	11914.9	11862.4	11760.5	11760.5
8	13785.8	13206.4	13482.7	13686.6	13686.6
9	15780.8	14637.9	15452.0	15783.3	15783.2
10	17174.7	16224.5	17873.6	17309.5	17309.6

Fig. 1 shows the curves that fit the per capita disposable income data according to three models. The data of the second-order FQDGM (1, 1) recommended in this paper approaches the statistical value (solid black line).

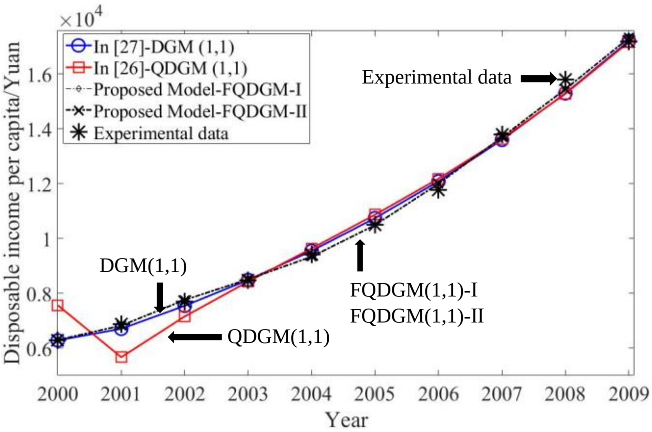

Fig.1 Comparative analysis of data fitting with three mathematical models

The relative error ranges of fitting data of the different gray theory analysis models are listed in Table III.

TABLE III
COMPARISON OF RELATIVE ERROR ABOUT FOUR MODELS

k	$x^{(0)}(k)$	Relative Error%			
		DGM	QDGM	FQDGM-I	FQDGM-II
1	6280.0	2.32	0.94	0.25	0.25
2	6859.6	3.83	1.00	-0.61	-0.61
3	7702.8	2.49	-1.08	0.65	0.65
4	8472.2	3.28	-0.48	0.02	0.02
5	9421.6	2.94	-0.38	0.11	0.11
6	10493.0	2.45	0.22	-0.08	-0.08
7	11759.5	1.32	0.88	-0.01	-0.01
8	13785.8	-4.2	-2.20	-0.72	-0.72
9	15780.8	-7.24	-2.08	0.02	0.02
10	17174.7	-5.53	4.07	0.78	0.78

The results in Table III show that the data obtained using FQDGM-I and FQDGM-II are almost equal.

The error values produced by fitting analysis with the FQDGM-I and FQDGM-II models are smaller than those produced by using the DGM and QDGM models (as shown in Fig. 2).

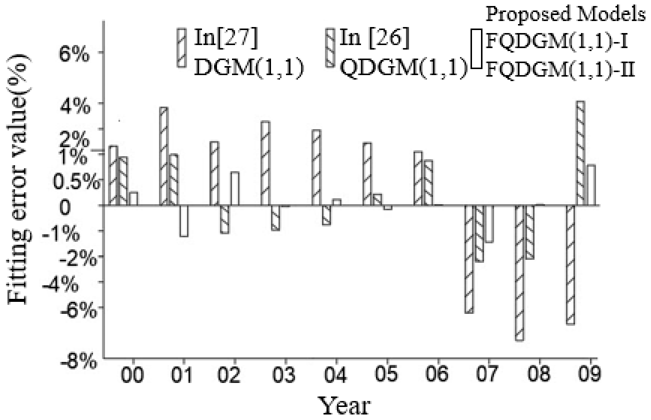

Fig2 Histogram of fitting relative error with three models on per capita disposable income of urban households in China

From the comparative analysis of the mathematical models, the DGM (1, 1) and QDGM (1, 1) fitting errors are higher than the fitting errors for the proposed FQDGM (1, 1), which indicates that the proposed model has good accuracy.

B. *Application 2*

The analysis data of urban building areas are a nonlinear series having the property of fractional order. Due to the inaccuracy of the parameters used in the previous analysis model [26],[27], the fitting accuracy of the background value was low, and there was a large gap between the analysis results and the actual statistical data. Statistical analysis and experimental research should be avoided in this case because of the long analysis cycle and high research costs. Therefore, the proposed FQDGM (1, 1) is used to analyze the urban building area. The variable x(0)(k) is the statistical data of the built-up area in Xinxiang central city. Moreover, the comparative analysis of the discrete gray model QDGM in [26], the DGM in [27], and the proposed fractional-order quadratic time-varying parameter discrete gray optimization model is performed. The built-up areas of downtown Xinxiang during 2001–2008 (data from China Statistical Yearbook 2009) are shown in Table IV.

TABLE IV
STATISTICAL DATA OF BUILT-UP AREA OF DOWNTOWN XINXIANG FROM2001 TO 2008

Ordinal	Year	Built-up area of downtown(km^2)
1	2001	60.0
2	2002	62.0
3	2003	68.0
4	2004	73.5
5	2005	76.8
6	2006	89.5
7	2007	90.9
8	2008	94.6

The fitting data of four grey theory analysis models is shown in Table V, where x(0)(k) corresponds to the statistical data of the built-up area of downtown Xinxiang from 2001 to 2008.

TABLE V
COMPARISON OF FITTING VALUE ABOUT FOUR MODELS

k	$x^{(0)}(k)$	Fitting value			
		DGM	QDGM	FQDGM-I	FQDGM -II
1	60.0	60.00	60.00	60.00	60.00
2	62.0	63.48	61.87	62.00	62.00
3	68.0	68.16	69.27	68.00	68.00
4	73.5	73.17	68.05	73.50	73.501
5	76.8	78.56	89.09	76.78	76.78
6	89.5	84.34	75.89	89.48	89.48
7	90.9	90.55	95.24	90.90	90.90
8	94.6	97.22	94.73	94.59	94.59

Through the fitting analysis of the four models on the building area, it is found that the accuracy of the four models is quite different, as shown in Table VI.

TABLE VI
COMPARISON OF RELATIVE ERROR ABOUT FOUR MODELS

k	$x^{(0)}(k)$	Fitting error			
		DGM	QDGM	FQDGM-I	FQDGM -II
1	60.0	0	0	0	0
2	62.0	-2.3914	0.2049	0.0005	0.004
3	68.0	-0.2289	-1.8691	0.0013	0.0008
4	73.5	0.4455	7.4173	0.0011	0.0011
5	76.8	-2.3167	-16.0387	0.0010	0.0009
6	89.5	5.7426	15.1908	0.0007	0.0009
7	90.9	0.3852	-4.7732	0.0011	0.0007
8	94.6	-2.7754	-0.1464	0.0006	0.0010

The fitting curves of the four grey theory models applied to the Xinxiang urban built-up area are shown in Fig. 3.

As shown in the figure, the DGM and QDGM fitting data of the discrete gray optimization model deviated from the statistical data more than the values derived using the proposed optimization model.

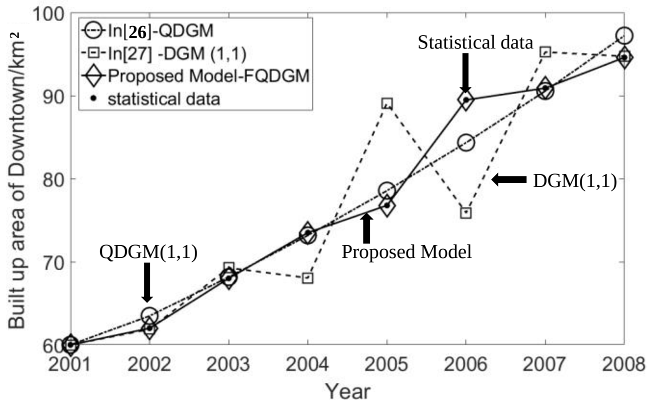

Fig.3 Comparison of simulated values of four models of built-up area in Xinxiang central city

This example shows that the background reconstruction ability and data fitting accuracy of the proposed model is better than the reference models, especially when a small amount of information is available and data analysis is poor. The proposed method is adaptive and scientific.

V. CONCLUSIONS

In scientific research and engineering applications, several research objects present fractional-order time-varying characteristics. Moreover, establishing an accurate analysis model with the previous grey theory is challenging due to the fractional time-varying property of the parameters. This attribute brings inconvenience and even interference to data analysis and decision-making. For example, inaccurate parameter descriptions and difficulty in determining modeling related factors often occur in the modeling-process; In the process of analysis, the analysis results are inconsistent with the actual statistical data. This study proposed two optimization models, namely FQDGM-I and FQDGM-II, to address the limitations mentioned above. In the process, the minimum relative error and minimum mean square error were employed as optimization objectives, and the least square method was applied to estimate the model parameters. In addition, we considered the fractional-order and initial value correction as design variables to assess the fractional-order time-varying characteristics of the parameters. The example shows that the background value reconstruction accuracy of the analysis model recommended in this paper is high, and the fitting accuracy was improved compared with previous approaches. These results verify the effectiveness and practicability of the proposed FQDGM (1, 1).

The FQDGM (1, 1) is an extension of the QDGM (1, 1), FTDGM (1, 1), TDGM (1, 1), FDGM (1, 1), and DGM (1, 1) models. The proposed model addresses the problems of the low background value reconstruction, and the data fitting accuracy of the previous model cannot meet the actual engineering requirements. The model established in this paper opens up a new idea for studying objects with fractional time-varying characteristics in business, industry and science.

REFERENCES

[1] Subing Liu, Chunwu Yin, Dazhi Cao, "Weapon equipment management cost prediction based on forgetting factor recursive GM (1,1) model," Grey Systems: vol.10, no. 1, pp38-45, 2020

[2] Youxin Luo, "Non-equidistant new information GM (1, 1) model and its application". Journal of Shen-yang University of Technology, vol. 32, no. 5, pp550-554, 2010

[3] XiaoGao Yang, Deqion Ding, Youxin Luo, "Non-equidistance Fractional-order Accumulation Grey Power Model NFGPM (1, 1) and Its Application," Engineering Letters, vol. 28, no. 3, pp827-832, 2020

[4] Zhen Liu, Yumei Xie, Yaoguo Dang, "A progress analysis of Chinese poverty alleviation based on improved GM (1,1)," Systems Engineering- Theory & Practice, vol. 39, no. 10, pp2476-2486, 2019

[5] Yuxiao Kang, Xinping Xiao, "Morbidity analysis of the grey GM (1,1) derived models," Systems Engineering- Theory & Practice, vol. 39, no. 10, pp2610-2618, 2019

[6] Zhengxin Wang, Yufeng Zhao, "GM (1,1) model with seasonal dummy variables and its application," Systems Engineering- Theory & Practice, vol. 40, no.11, pp2981-2990, 2020

[7] Dang Luo, Xiaolei Wang, "Grey GM (1,1, T) model coupled with trigonometric function and its application," Systems Engineering- Theory & Practice, vol. 40, no. 7, pp1906-1917, 2020

[8] Wenzhan Dai, Junfeng Li, "Modeling Research on Non-equidistance GM (1,1) Model," Systems Engineering - Theory & Practice, vol. 25, no. 9, pp 89-93, 2005

[9] Lei Huang, Shubi Zhang, "The application of accumulating method PGM (1, 1) model for deformation prediction," Urban Geotechnical Investigation & Surveying, no. 4, pp152-155, 2012

[10] Zhen Liu, Yumei Xie, Yaoguo Dang, "A progress analysis of Chinese poverty alleviation based on improved GM (1,1)," Systems Engineering - Theory & Practice, vol. 39, no. 10, pp2476-2486, 2019

[11] Wenging Wu, Xin Ma, Yuanyuan Zhang, Yong Wang, Xinxing Wu, "Analysis of novel FAGM (1,1, t^{α}) model to forecast health expenditure of China," Grey Systems: Theory and Application, vol. 9, no. 2, pp 232-250, 2019

[12] Yuehua CAI, Weiyue XIAO, Bin ZENG, "Grey New Information GRM (1,1) and Its Application based on Reciprocal Accumulated Generating and Background Value Optimization," Applied Mechanics and Materials, vol.364, pp188-191, 2013

[13] Qiyuan Liu, Dejie Yu, "Non-equidistance and nonhomogeneous grey model NNFGM (1, 1) with the fractional-order accumulation and its application," Journal of Interdisciplinary Mathematics, vol. 20, no. 6-7, pp1423-1426, 2017

[14] Lifeng Wu, Sifeng Liu, Ligen Yao, Shuli Yan, Dinglin Liu, "Grey system model with the fractional-order accumulation," Communications in Nonlinear Science and Numerical Simulation, vol. 18, no. 7, pp1775-1785, 2013

[15] Wenqing Wu, Xin Ma, Yong Wang, Yuanyuan Zhang, Bo Zeng, "Research on a novel fractional GM (α, n) model and its applications," Grey Systems: Theory and Application, vol. 9, no.3, pp356-373, 2019

[16] Lifeng Wu, Sifeng Liu, Zhigeng Fang, Haiyan Xu, Properties of the GM (1,1) with fractional-order accumulation. Applied Mathematics and Computation, vol. 252, pp287-293, 2015

[17] Lifeng Wu, Sifeng Liu, Ligeng Yao, "Discrete grey model based on fractional-order accumulates," Systems Engineering - Theory & Practice, vol. 34, no. 7, pp 34(7):1822-1827, 2014

[18] Xin Ma, Wenqing Wu, Bo Zeng, Yong Wang, Xinxing Wu, "The conformable fractional grey system model," ISA Transactions, vol. 96, pp255-271, 2020

[19] Shuhua Mao, Min Zhu, Xinping Yan, Mingyun Gao, Xinping Xiao, "Modeling mechanism of a novel fractional grey model based on matrix analysis,". Journal of Systems Engineering and Electronics, vol.27, no.5, pp 1040-1053, 2016

[20] Baohua Yang, Jinshuai Zhao, "Fractional-order discrete grey GM (1, 1) power model and its application," Control and Decision, vol.30, no.7, pp1264-1268, 2015

[21] Youxin Luo, "Multivariable grey model FMGM (1, n) with fractional-order accumulation and its application," Journal of Central South University (Science and Technology), vol. 48, no.10, pp 2686-2290, 2017

[22] Song Ding, Roujin Li, Zui Tao, "A novel adaptive discrete grey model with time-varying parameters for long-term photovoltaic power generation forecasting," Energy Conversion and Management, vol. 227, pp113644-113659, 2021

[23] Liang Zeng, Yanan Li, "Opposite-Direction Accumulative Discrete

Grey Prediction Model with Linear Time-Varying Parameters," Journal of Chongqing University of Technology (Natural Science), vol.34, no.7, pp238-244, 2020
[24] Yingjun Tian, Lizhong Song, Miaofeng Ruan, "Design of online fault prediction system based on linear time-varying parameters discrete grey model," Journal of naval university of engineering, vol.26, no.1, pp17-22, 2014
[25] Shiquan Jiang, Sifeng Liu, Zhongxia Liu, Zhigeng Fang, "Cubic time-varying parameters discrete grey forecasting model and its properties," Control and Decision, vol.31, no.2, 2016
[26] Liyun Wu, Zhengpeng Wu, Mei Li, "Quadratic time-varying parameters discrete grey model," Systems Engineering- Theory & Practice, vol. 33, no. 11, pp2887-2893, 2013
[27] LanXiang LUO, "Progressive Optimization GM (1,1) Model based on Direct Modeling for Demand Forecast of Urban Construction Land and Its Empirical Research," Electronic Journal of Geotechnical Engineering, vol.20, no.12, pp5023-5028, 2015

XiaoGao Yang is working in College of Mechanical Engineering, Hunan University of Arts and Science. He is the Hunan Provincial General University Subject Leader. The main research areas include mechanical design and manufacturing, system research, uncertainty system theory in mechanical engineering, modern intelligent optimization method, computational mechanism and mechanical transmission. Email: yxg_568@126.com

DeQiong Ding is working in Hunan University of Finance and Economics. The main research areas include research work of mathematics statistics and modeling, data mining and financial and economic analysis. Email: ddqshe2010@163.com

YouXin Luo is working in College of Mechanical Engineering, Hunan University of Arts and Science. He is the Hunan Provincial General University Subject Leader, and the National Natural Science Foundation Letter Review Expert, in china. The main research areas include mechanical design and manufacturing, system research, uncertainty system theory in mechanical engineering, modern intelligent optimization method, computational mechanism. Email: llyx123@126.com