Duality for Multiobjective Optimization with Generalized Invex Functions

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Abstract—**Based on the definition of d-ρ-(η,θ)-invex, we presents several generalized invex concepts, which include d-ρ-(η,θ)-quadiinvex, d-ρ-(η,θ)-psedoinvex and strictly d-ρ-(η,θ)-psedoinvex. Meanwhile, we discuss and derive some Mond-Weir and Wolf type duality theorems.**

Index Terms—**Generalized invex; Multiobjective optimization; Mond-Weir duality; Wolf type duality**

I. Introduction

In optimization theory, the study of different convexity is a very important branch. During the past years, many authors have presented various generalized convex functions. For example, Hanson presented the definition of η-invexity, and derived some optimality result under η-invexity conditions [1]. Subsequently, many research results on η-invex have been proposed, including properties, extensions and applications [2-6]. Later, Zalma presented ρ-(η,θ)-invexity, and discussed some optimality conditions [7]. In 2009, d-ρ-(η,θ)-invex was introduced by Nahak and Mohapatra [8]. In this paper, they obtained the related properties of two types of duality: Mond-Weir and Wolf type duality. For an optimization problem in Banach space, Behera et al. derived the Karush-Kuhn-Tucker (KKT) conditions based on ρ-(η,θ)-invex function in 2016 [9]. In 2017, Nahak et al. proposed ρ-(η,θ)-B-invex function firstly, and then they generalized ρ-(η,θ)-B-invex function in Banach space [10]. Furthermore, for single objective optimization problem, some optimality results were obtained, and furthermore some duality results were established in Banach space. In the same year, in the quotient space of fuzzy numbers, the concepts of convexity, quasiconvexity and pseudoconvexity were proposed firstly, and then he optimality results of convexity and generalized convexity on fuzzy mappings were derived [11]. Motivated by the work in [8] and the importance of convexity and generalized convexity, this paper presents the concepts of d-ρ-(η,θ)-quasiinvex, d-ρ-(η,θ)-psedoinvex and strictly d-ρ-(η,θ)-invex. Furthermore, duality results about Mond-Weir under d-ρ-(η,θ)-invex are discussed, which are different from [6]. At the same time, duality results for Mond-Weir with d-ρ-(η,θ)-psedoinvex assumption are establishd.

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II. Preliminaries

Generally speaking, the definition of multiobjective programming problem is given as follows:

$$(VP)\begin{cases}\min & f(x)=(f_1(x),\cdots,f_k(x))\\ \text{s.t.} & g(x)=(g_1(x),\cdots,g_m(x))\leqq 0,\\ & x\in X,\end{cases}$$

where $X\subseteq R^n$ is not empty. Suppose η and θ are two functions from $X\times X\to R^n$. $f^{'}(u,\eta(x,u))$ is defined as follows:

$$f^{'}(u,\eta(x,u))=\lim_{\lambda\to 0^+}\frac{f(u+\lambda\eta(x,u))-f(u)}{\lambda},$$

which is called the directional derivative.

Considering problem (VP), for convenience in expression, the feasible region $\{x \mid g(x)\leqq 0, x\in X\}$ is represented by D. Next, some related concepts are briefly introduced.

Definition 1[6]. For $\bar{x}\in D$, if it satisfies $f(x)\not< f(\bar{x})(x\in D)$, then we call $\bar{x}$ a weak Pareto solution for (VP).

Definition 2[6]. Suppose $X\subseteq R^n$ is not empty, $u\in X$ and f is directionally differentiable at u from X to R. For $x\in X$, if $\exists\eta,\ \theta$ and ρ meet

$$f(x)-f(u)\geqq f^{'}(u,\eta(x,u))+\rho\parallel\theta(x,u)\parallel^2,$$

we call $f(x)$ d-ρ-(η,θ)-invex.

Definition 3. Suppose $X\subseteq R^n$ is not empty, $u\in X$ and f is directionally differentiable at u from X to R. For $\forall x\in X$, $\exists\eta,\ \theta$ and ρ,

(1) when $f(x)\leqq f(u)$, we have $f^{'}(u,\eta(x,u))+\rho\parallel\theta(x,u)\parallel^2\leqq 0$, then we call $f(x)$ d-ρ-(η,θ)-quasiinvex.

(2) when $f(x)<f(u)$, we have $f^{'}(u,\eta(x,u))+\rho\parallel\theta(x,u)\parallel^2<0$, then we call $f(x)$ d-ρ-(η,θ)-psedoinvex.

(3) when $f(x)\leqq f(u)$, we have $f^{'}(u,\eta(x,u))+\rho\parallel\theta(x,u)\parallel^2<0$, then we call $f(x)$ strictly d-ρ-(η,θ)-psedoinvex.

III. Mond-Weir duality

Consider *(VP)* and $x\in D$, this section presents the Mond-weir dual problem below:

$$(VD)\begin{cases}\max & f(y)=(f_1(y),\cdots,f_k(y))\\ \text{s.t.} & \mu^Tg(y)\geqq 0,\\ & (\lambda^Tf^{'}+\mu^Tg^{'})(y,\eta(x,y))\geqq 0,\\ & \lambda^Te=1,\ \lambda\in R_k^+,\ \mu\in R_m^+.\end{cases}$$

Let W be the feasible region of (VD), and Pr_XW be the projection from W to X.

Lemma. For $i = 1, \cdots, k$, and $\lambda \in R_k^+$, if f_i is d-ρ_i-(η,θ)-invex from X to R, then

$$\sum_{i=1}^{k} \lambda_i f_i \text{ is } d - \sum_{i=1}^{k} \lambda_i \rho_i - (\eta, \theta) - \text{invex}.$$

Proof. Since f_i is a d-ρ_i-(η,θ)-invex function, by the definition, we have

$$f_i^{'}(u, \eta(x,u)) + \rho_i \parallel \theta(x,u) \parallel^2 \leqq f_i(x) - f_i(u).$$

Since $\lambda_i \geqq 0$, we can derive that

$$\lambda_i f_i^{'}(u, \eta(x,u)) + \lambda_i \rho_i \parallel \theta(x,u) \parallel^2 \leqq \lambda_i f_i(x) - \lambda_i f_i(u).$$

Furthermore, we can obtain that

$$\begin{aligned} &\sum_{i=1}^{k} \lambda_i f_i^{'}(u, \eta(x,u)) + \sum_{i=1}^{k} \lambda_i \rho_i \parallel \theta(x,u) \parallel^2 \\ &\leqq \sum_{i=1}^{k} \lambda_i f_i(x) - \sum_{i=1}^{k} \lambda_i f_i(u). \end{aligned}$$

So

$$\sum_{i=1}^{k} \lambda_i f_i \text{ is } d - \sum_{i=1}^{k} \lambda_i \rho_i - (\eta, \theta) - \text{invex},$$

and the proof is completed.

Theorem 1. For $x \in D$, $(y, \lambda, \mu) \in W$, and $\forall y \in D \bigcup Pr_X W$, if

(1) $f_i (i = 1, \cdots, k)$ meets d-ρ_i-(η,θ)-invex,

(2) $g_j (j = 1, \cdots, m)$ meets d-$\bar{\rho}_j$-(η,θ)-invex,

(3) $\sum\limits_{i=1}^{k} \lambda_i \rho_i + \sum\limits_{j=1}^{m} \mu_j \bar{\rho}_j \geqq 0$,

we have

$$\lambda^T f(x) \geqq \lambda^T f(y).$$

Proof. By Lemma, we know that

$$\sum_{i=1}^{k} \lambda_i f_i(x) \text{ is } d - \sum_{i=1}^{k} \lambda_i \rho_i - (\eta, \theta) - \text{invex}.$$

Similarly, we can derive that $\mu^T g$ is

$$d - \sum_{j=1}^{m} \mu_j \bar{\rho}_j - (\eta, \theta) - \text{invex}.$$

To be contrary, suppose that the conclusion is incorrect, i.e.

$$\sum_{i=1}^{k} \lambda_i f_i(y) > \sum_{i=1}^{k} \lambda_i f_i(x).$$

Because $\sum\limits_{i=1}^{k} \lambda_i f_i$ is

$$d - \sum_{i=1}^{k} \lambda_i \rho_i - (\eta, \theta) - \text{invex},$$

the following inequality holds:

$$\begin{aligned} &\sum_{i=1}^{k} \lambda_i f_i^{'}(y, \eta(x,y)) + \sum_{i=1}^{k} \lambda_i \rho_i \parallel \theta(x,y) \parallel^2 \\ &\leqq \sum_{i=1}^{k} \lambda_i f_i(x) - \sum_{i=1}^{k} \lambda_i f_i(y) < 0. \end{aligned} \tag{1}$$

Considering $x \in D$, $(y, \lambda, \mu) \in W$ and $\mu \in R_m^+$, it is easy to derive the following relationship:

$$\mu^T g(x) \leqq 0 \leqq \mu^T g(y).$$

Combining that $\sum\limits_{j=1}^{m} \mu_j g_j(x)$ is

$$d - \sum_{j=1}^{m} \mu_j \bar{\rho}_j - (\eta, \theta) - \text{invex},$$

we can obtain that

$$\begin{aligned} 0 &\geqq \mu^T g(x) - \mu^T g(y) \\ &\geqq \sum_{j=1}^{m} \mu_j g_j^{'}(y, \eta(x,y)) + \sum_{j=1}^{m} \mu_j \bar{\rho}_j \parallel \theta(x,y) \parallel^2 . \end{aligned} \tag{2}$$

From (1) and (2), we have the relationship below:

$$\begin{aligned} &\sum_{i=1}^{k} \lambda_i f_i^{'}(y, \eta(x,y)) + \sum_{j=1}^{m} \mu_j g_j^{'}(y, \eta(x,y)) \\ &+ (\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j=1}^{m} \mu_i \bar{\rho}_j) \parallel \theta(x,u) \parallel^2 < 0. \end{aligned}$$

Since

$$\lambda^T \rho + \mu^T \bar{\rho} \geqq 0,$$

it is easy to derive the following inequality:

$$\sum_{i=1}^{k} \lambda_i f_i^{'}(y, \eta(x,y)) + \sum_{j=1}^{m} \mu_j g_j^{'}(y, \eta(x,y)) < 0.$$

Obviously, it is incompatible with the following dual constraint:

$$0 \leqq \lambda^T f^{'}(y, \eta(x,y)) + \mu^T g^{'}(y, \eta(x,y)).$$

It implies that the conclusion of this theorem is correct.

Corollary For $(y, \lambda, \mu) \in W$ and $\bar{x} \in D$, if the conditions (1)-(3) of Theorem 1 hold, and $\lambda^T f(\bar{x}) = \lambda^T f(y)$, then $\bar{x}$ will be a weak Pareto solution for (VP).

Proof. To be contrary, assume that the conclusion is incorrect. According to the definition of weak Pareto solution, $\exists \hat{x} \in D$, which satisfies

$$f(\bar{x}) > f(\hat{x}).$$

According to

$$\lambda \in R_k^+$$

and

$$\lambda^T e = 1,$$

it is easy to derive the following inequality relationship:

$$\sum_{i=1}^{k} \lambda_i f_i(\hat{x}) < \sum_{i=1}^{k} \lambda_i f_i(\bar{x}). \tag{3}$$

However, by Theorem 1, the following inequality should hold:

$$\sum_{i=1}^{k} \lambda_i f(\hat{x}) \geqq \sum_{i=1}^{k} \lambda_i f(y) = \sum_{i=1}^{k} \lambda_i f(\bar{x}).$$

It contracts the formula (3), and this implies that the proof is completed.

Theorem 2. Assume $\bar{x}$ is a weak Pareto solution of (VP). For all $x \in D$, if $\exists \bar{\lambda} \in R_k^+$, and $\bar{\mu} \in R_m^+$ satisfy (i)-(iii) below:

(i) $\bar{\mu}^T g(\bar{x}) = 0$,

(ii) $\bar{\lambda}^T e = 1$, and $g(\bar{x}) \leqq 0$,

(iii) $(\bar{\lambda}^T f^{'} + \bar{\mu}^T g^{'})(\bar{x}, \eta(x, \bar{x})) \geqq 0$.

then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ will be feasible for (VD). Furthermore, if the conditions of Theorem 1 are met, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ will also be a weak Pareto solution for (VD).

Proof. According to the structure of (VD), for $\bar{x}$, since it satisfies conditions (i)-(iii), it is obviously $(\bar{x}, \bar{\lambda}, \bar{\mu})$ meets all the requests of (VD).

From Theorem 1, for $\forall (y, \lambda, \mu) \in W$, it implies

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(y) \leqq \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}). \qquad (4)$$

Now, suppose the conclusion is not correct, that is $\exists (\hat{y}, \hat{\lambda}, \hat{\mu}) \in W$ meets

$$f(\hat{y}) > f(\bar{x}).$$

Furthermore, by

$$\bar{\lambda}^T e = 1, \ \bar{\lambda} \in R_k^+,$$

the following inequality holds:

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\hat{y}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}),$$

which is incompatible with (4), and the proof is completed.

Theorem 3. Assume that $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is a weak Pareto solution of (VD), and (1)-(3) of Theorem 1 hold, then $\bar{y}$ will be a weak Pareto solution of (VP).

Proof. To be contrary, assume the conclusion is not correct. This means that $\exists \bar{x} \in D$, which makes the following relationship hold:

$$f(\bar{y}) > f(\hat{x}).$$

According to

$$\bar{\lambda}^T e = 1,$$

and

$$\bar{\lambda} \in R_k^+,$$

we can get

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\hat{x}).$$

By Theorem 1, since

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(x)$$

is

$$d - \sum_{i=1}^{k} \bar{\lambda}_i \rho_i - (\eta, \theta) - \text{invex at } \bar{y},$$

we have

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + \sum_{i=1}^{k} \bar{\lambda}_i \rho_i \parallel \theta(\hat{x}, \bar{y}) \parallel^2 \leqq \sum_{i=1}^{k} \bar{\lambda}_i f_i(\hat{x} - \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}).$$

Since

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\hat{x}),$$

it follows that

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + \sum_{i=1}^{k} \bar{\lambda}_i \rho_i \parallel \theta(\hat{x}, \bar{y}) \parallel^2 < 0. \qquad (5)$$

In addition, because $\sum_{j=1}^{m} \bar{\mu}_i g_i(x)$ is

$$d - \sum_{j=1}^{m} \bar{\mu}_j \bar{\rho}_j - (\eta, \theta) - \text{invex},$$

we can derive that

$$\sum_{j=1}^{m} \bar{\mu}_j g_j^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + \sum_{j=1}^{m} \bar{\mu}_j \bar{\rho}_j \parallel \theta(\hat{x}, \bar{y}) \parallel^2 \leqq \sum_{j=1}^{m} \bar{\mu}_j g_j(\hat{x}) - \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}).$$

From

$$\hat{x} \in D$$

and

$$(\bar{y}, \bar{\lambda}, \bar{\mu}) \in W,$$

we can obtain that

$$\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) \geqq 0 \geqq \sum_{j=1}^{m} \bar{\mu}_j g_j(\hat{x}).$$

Therefore,

$$\sum_{j=1}^{m} \bar{\mu}_j g_j^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + \sum_{j=1}^{m} \bar{\mu}_j \bar{\rho}_j \parallel \theta(\hat{x}, \bar{y}) \parallel^2 \leqq 0. \qquad (6)$$

From (5) and (6), we can get

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + \sum_{j=1}^{m} \bar{\mu}_j g_j^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + (\sum_{i=1}^{k} \bar{\lambda}_i \rho_i + \sum_{j=1}^{m} \bar{\mu}_j \bar{\rho}_j) \parallel \theta(\hat{x}, \bar{y}) \parallel^2 < 0.$$

Since $\sum_{i=1}^{k} \bar{\lambda}_i \rho_i + \sum_{j=1}^{m} \bar{\mu}_j \bar{\rho}_j \geqq 0$, we have the following inequality:

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + \sum_{j=1}^{m} \bar{\mu}_j g_j^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) < 0,$$

which is incompatible with

$$0 \leqq \sum_{i=1}^{k} \bar{\lambda}_i f_i^{'}(\bar{y}, \eta(\hat{x}, \bar{y})) + \sum_{j=1}^{m} \bar{\mu}_j g_j^{'}(\bar{y}, \eta(\hat{x}, \bar{y})).$$

The proof is completed.

IV. Wolf type duality

In this section, by using d-ρ-(η, θ)-psedoinvex, the Wolf type duality of (VP) will be discussed. For $\forall x \in D$, the Wolf type dual form of problem (VP) is as follows:

$$(VD) \begin{cases} \max & \phi(y, \lambda, \mu) = f(y) + \mu^T g(y) e \\ \text{s.t.} & \mu^T g(y) \geqq 0, \\ & (\lambda^T f^{'} + \mu^T g^{'})(y, \eta(x, y)) \geqq 0, \\ & \lambda^T e = 1, \ \mu \in R_m^+ \ \lambda \in R_k^+. \end{cases}$$

Let W be the feasible set of (VD) and $Pr_X W$ be a projection from W to X.

Theorem 4. Assume that x and (y, λ, μ) are feasible points of (VP) and (VD), respectively. Since $\lambda^T f + \mu^T g$ is

$$d - \rho - (\eta, \theta) - \text{psedoinvex}$$

and $\rho \geqq 0$, then we have

$$\phi(y,\lambda,\mu) \not> f(x).$$

Proof. To be contrary, assume that the conclusion is incorrect, that is

$$\phi(y,\lambda,\mu) > f(x).$$

Since

$$\lambda \in R_k^+,\ \lambda^T e = 1,\ \mu \in R_m^+,$$

and x is feasible for rpoblem (VP), we can obtain

$$\sum_{i=1}^{k}\lambda_i f_i(y)+\sum_{j=1}^{m}\mu_j g_j(y) > \sum_{i=1}^{k}\lambda_i f_i(x)+\sum_{j=1}^{m}\mu_j g_j(x). \quad (7)$$

For $\forall y \in D\bigcup Pr_X W$, by (7), since

$$\lambda^T f + \mu^T g$$

is

$$d-\rho-(\eta,\theta)-\text{psedoinvex},$$

we can derive that

$$(\lambda^T f^{'} + \mu^T g^{'})(y,\eta(x,y)) < -\rho \parallel \theta(x,y) \parallel^2 .$$

Furthermore, from $\rho \geqq 0$, it is not difficult to derive

$$(\lambda^T f^{'} + \mu^T g^{'})(y,\eta(x,y)) < 0.$$

Obviously, it does not meet the following inequality:

$$0 \leqq (\lambda^T f^{'} + \mu^T g^{'})(y,\eta(x,y)),$$

which implies

$$\phi(y,\lambda,\mu) \not> f(x).$$

Theorem 5. Assume $\bar{x}$ is a weak Pareto point for (VP), and the conditions of Theorem 2 can be satisfied, then $(\bar{x},\bar{\lambda},\bar{\mu})$ will be feasible (VD). Furthermore, for $y \in D\bigcup Pr_X W$ and $\forall(y,\lambda,\mu) \in W$, if

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i + \sum_{j=1}^{m}\bar{\mu}_j g_j$$

is

$$d-\rho-(\eta,\theta)-\text{psedoinvex},$$

then $(\bar{x},\bar{\lambda},\bar{\mu})$ will be a weak Pareto point for (VD).

Proof. Because $\bar{x}$ meets the requests of Theorem 2, $(\bar{x},\bar{\lambda},\bar{\mu})$ meets the requests of (VD), and is feasible for (VD). According to Theorem 4, for $\forall(y,\lambda,\mu) \in W$, we have

$$f(y) + \sum_{j=1}^{m}\bar{\mu}_j g_j(y)e \leqq f(\bar{x}).$$

Since

$$\sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x}) = 0,$$

we have

$$f(y) + \sum_{j=1}^{m}\bar{\mu}_j g_j(y)e \leqq f(\bar{x}) + \sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x}),$$

and this implies the conclusion of this theorem is correct.

Theorem 6. Suppose $(\bar{y},\bar{\lambda},\bar{\mu})$ satisfies all the conditions of problem (VD), and $\bar{\lambda}^T f + \bar{\mu}^T g$ is

$$d-\rho-(\eta,\theta)-\text{psedoinvex}$$

and $\rho \geq 0$, then $\bar{y}$ is a weak Pareto solution for (VP).

Proof. To be contrary, assume that $\exists \bar{x} \in D$, which satisfies the following relationship:

$$f(\bar{y}) > f(\bar{x}).$$

From

$$\bar{\lambda}^T e = 1,$$

and

$$\bar{\lambda} \geqq 0,$$

we have

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y}) > \sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x}). \quad (8)$$

Furthermore, since

$$\bar{x} \in D, \bar{\mu} \geqq 0,\ (\bar{y},\bar{\lambda},\bar{\mu}) \in W,$$

we can derive the inequality as follows:

$$\sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{y}) \geqq 0 \geqq \sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x}). \quad (9)$$

By adding (8) and (9), we have

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{y}) > \sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x}).$$

Since $\sum_{i=1}^{k}\bar{\lambda}_i f_i + \sum_{j=1}^{m}\bar{\mu}_j g_j$ is

$$d-\rho-(\eta,\theta)-\text{psedoinvex},$$

we can get that

$$(\sum_{i=1}^{k}\bar{\lambda}_i f_i^{'} + \sum_{j=1}^{m}\bar{\mu}_j g_j^{'})(\bar{y},\eta(\bar{x},\bar{y})) < -\rho \parallel \theta(\bar{x},\bar{y}) \parallel^2 .$$

Furthermore, from $\rho \geqq 0$, we can obtain that

$$(\sum_{i=1}^{k}\bar{\lambda}_i f_i^{'} + \sum_{j=1}^{m}\bar{\mu}_j g_j^{'})(\bar{y},\eta(\bar{x},\bar{y})) < 0.$$

Obviously, it is incompatible with the inequality below:

$$0 \leqq (\sum_{i=1}^{k}\bar{\lambda}_i f_i^{'} + \sum_{j=1}^{m}\bar{\mu}_j g_j^{'})(\bar{y},\eta(\bar{x},\bar{y})).$$

The proof is completed.

Theorem 7. Suppose $(\bar{y},\bar{\lambda},\bar{\mu})$ is a weak Pareto point of (VD), and for $\bar{y} \in D\bigcup Pr_X W$, $\bar{\lambda}^T f$ and $\bar{\mu}^T g$ are d-ρ-(η,θ)-psedoinvex and d-$\bar{\rho}$-(η,θ)-quasiinvex at $\bar{y}$, respectively, and $\rho + \bar{\rho} \geq 0$, then $\bar{y}$ will be a weak Pareto solution for (VP).

Proof. To be contrary, assume that the conclusion is incorrect, which means that $\exists \bar{x} \in D$ meets

$$f(\bar{x}) < f(\bar{y}).$$

From

$$\bar{\lambda}^T e = 1$$

and
$$\bar{\lambda} \in R_k^+,$$
we can get
$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) < \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}).$$
Considering $\sum_{i=1}^{k} \bar{\lambda}_i f_i$ is
$$d - \rho - (\eta, \theta) - \text{psedoinvex},$$
we have
$$\sum_{i=1}^{k} \bar{\lambda}_i f_i^{'}(\bar{y}, \eta(\bar{x}, \bar{y})) < -\rho \parallel \theta(\bar{x}, \bar{y}) \parallel^2 . \qquad (10)$$
By
$$\bar{\mu} \in R_m^+, \bar{x} \in D, (\bar{y}, \bar{\lambda}, \bar{\mu}) \in W,$$
we can obtain that
$$\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}) \geqq 0 \geqq \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x})$$
Since $\sum_{j=1}^{m} \bar{\mu}_j g_j$ is
$$d - \bar{\rho} - (\eta, \theta) - \text{quasiinvex},$$
we have
$$\sum_{j=1}^{m} \bar{\mu}_j g_j^{'}(\bar{y}, \eta(\bar{x}, \bar{y})) < -\bar{\rho} \parallel \theta(\bar{x}, \bar{y}) \parallel^2 . \qquad (11)$$
Furthermore, by (10)-(11), it is not difficult to drive
$$(\bar{\lambda}^T f^{'} + \bar{\mu}^T g^{'})(\bar{y}, \eta(\bar{x}, \bar{y})) < -(\rho + \bar{\rho}) \parallel \theta(\bar{x}, \bar{y}) \parallel^2 .$$
By combining
$$\rho + \bar{\rho} \geqq 0,$$
we can get that
$$(\bar{\lambda}^T f^{'} + \bar{\mu}^T g^{'})(\bar{y}, \eta(\bar{x}, \bar{y})) < 0.$$
Obviously, it is incompatible with the following inequality:
$$(\bar{\lambda}^T f^{'} + \bar{\mu}^T g^{'})(\bar{y}, \eta(\bar{x}, \bar{y})) \geqq 0.$$
This implies that the conclusion of this theorem is correct.

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