Recursive Linear Bounds for the Vertex Chromatic Number of the Pancake Graph

Aldrich Ellis C. Asuncion, Renzo Roel P. Tan*, Christian Paul O. Chan Shio, and Kazushi Ikeda

Abstract—**The pancake graph has been the subject of research. While studies on the various aspects of the graph are abundant, results on the chromatic properties may be further enhanced. Revolving around such context, the paper advances an alternative method to produce novel linear bounds for the vertex chromatic number of the pancake graph. The accompanying demonstration takes advantage of symmetries inherent to the graph, capturing the prefix reversal of subsequences through a homomorphism. Contained within the argument is the incorporation of known vertex chromatic numbers for certain orders of pancake graphs, rendering tighter bounds possible upon the release of new findings. In closing, a comparison with existing bounds is done to establish the relative advantage of the proposed technique.**

Index Terms—**chromatic number, linear bound, pancake graph, vertex coloring.**

I. Introduction

The pancake graph of order n is defined as the Cayley graph on the symmetric group of order n generated by prefix reversals. Initially proposed as an interconnection network architecture [1], several graph-theoretic properties and combinatorial problems involving the pancake graph have been studied. These include the embedding [2], cycle structure [3], planarity [4], and diameter [5]. Further, the pancake graph has been a subject of graph-theoretical and group-theoretical interest for its connectivity properties and automorphism groups [6]; notwithstanding, progress on chromatic properties is limited.

Concisely, the paper proposes a new approach to generate bounds for the vertex chromatic number of the pancake graph based on graph homomorphisms. By the procedure, a tight linear bound that proffers a general improvement over known bounds is acquired.

The subsequent sections are organized as follows. The second and third sections inspect preliminary concepts on vertex coloring and the pancake graph. The fourth examines existing bounds for the chromatic number of the pancake graph based on its maximum degree and triangle-free property. In the fifth chapter, one discusses the proposed recursive bound and its advantages. To close, a recapitulation and some recommendations are contained in the sixth chapter.

*Author to whom correspondence should be addressed.

Manuscript received July 24, 2021; revised November 12, 2021. The Japan Society for the Promotion of Science supports the study through the Grants-in-Aid for Scientific Research Program (KAKENHI 18K19821).

A. E. C. Asuncion is lecturer at the Department of Mathematics, School of Science and Engineering, Ateneo de Manila University, National Capital Region, Philippines e-mail: aeasuncion@ateneo.edu.

R. R. P. Tan is assistant professor at the Mathematical Informatics Laboratory, Division of Information Science, Nara Institute of Science and Technology, Nara Prefecture, Japan and the Department of Mathematics, School of Science and Engineering, Ateneo de Manila University, National Capital Region, Philippines e-mail: rr.tan@is.naist.jp, rrtan@ateneo.edu.

C. P. O. Chan Shio is assistant professor at the Department of Mathematics, School of Science and Engineering, Ateneo de Manila University, National Capital Region, Philippines e-mail: cchanshio@ateneo.edu.

K. Ikeda is professor at the Mathematical Informatics Laboratory, Division of Information Science, Nara Institute of Science and Technology, Nara Prefecture, Japan e-mail: kazushi@is.naist.jp.

Basic notation used throughout the paper is below.

$V(G)$	The vertex set of graph G.
$E(G)$	The edge set of graph G.
$\Delta(G)$	The maximum degree of a vertex in graph G.
$\delta(G)$	The minimum degree of a vertex in graph G.
$\omega(G)$	The clique number of graph G.
$\alpha(G)$	The independence number of graph G.
$\chi(G)$	The vertex chromatic number of graph G.
$u \sim v$	Vertex u is adjacent to vertex v.
K_n	The complete graph of n vertices.
C_n	The cycle graph of n vertices.
S_n	The symmetric group of order n.
$Cay(\Gamma, S)$	The Cayley graph generated by the group Γ and generating set S.
P_n	The pancake graph of order n.

II. Vertex Coloring

The study puts focus on vertex coloring, a domain under graph coloring. Although various types of coloring such as edge coloring, total coloring, and unlabeled coloring exist, one refers to vertex coloring when either coloring or graph coloring is mentioned [7]. A proper vertex k-coloring is briefly defined [8].

Definition 1 (Proper Vertex k-Coloring)**.** A proper vertex k-coloring of a graph G is a function $c : V(G) \to \{1, 2, \cdots, k\}$ such that $c(u) \neq c(v)$ if $u \sim v$ in G.

A graph G is k-colorable if there exists a proper vertex k-coloring of G. The minimum positive integer k for which G is k-colorable is called the vertex chromatic number of G and is denoted by $\chi(G)$. In the research, one refers to $\chi(G)$ as simply the chromatic number of G.

An alternative definition of the chromatic number involving graph homomorphisms is also used in succeeding sections; thus, the definition for a graph homomorphism and a related lemma are provided [9].

Definition 2 (Graph Homomorphism)**.** Let G and H be two graphs. A graph homomorphism from G to H is a function ϕ from $V(G)$ to $V(H)$ such that if $u \sim v$ in G then $\phi(u) \sim \phi(v)$ in H.

It is said that G is homomorphic to H if there exists a graph homomorphism from G to H. A graph homomorphism ϕ from a graph G to a graph H is often denoted by $\phi : G \to H$. That the composition of two graph homomorphisms is also a graph homomorphism is then formally stated. As convention, $g \circ f$ is understood to mean the composition of two homomorphisms f and g [9].

Lemma 1. Let G, H, and K be graphs. If $f : G \to H$ and $g : H \to K$ are graph homomorphisms then the composition $g \circ f : G \to K$ is also a graph homomorphism.

Graph homomorphisms may be viewed as a generalization of the concept of vertex coloring. The problem of finding a proper k-coloring of a graph G is equivalent to the problem of finding a graph homomorphism from G to a complete graph [9].

Lemma 2. The chromatic number of a graph G is the least integer r such that there exists a homomorphism from G to K_r, the complete graph on r vertices.

By relating graph homomorphisms to vertex coloring, it is reasonable to expect that bounds on the chromatic number of a graph may be obtained using graph homomorphisms. The following theorem shows the said fact.

Theorem 1. Let G and H be two graphs. If G is homomorphic to H then $\chi(G) \leq \chi(H)$.

Theorem 1 provides a method to bound the chromatic number of a graph. Given a graph G, if a graph H whose chromatic number is known for which there exists a graph homomorphism $\phi : G \to H$ is found then the chromatic number of G may be bounded. Homomorphisms involving the pancake graphs are constructed in such a manner to produce bounds for the chromatic number.

III. Pancake Graph

The pancake graph is a type of undirected Cayley graph, which is a graph constructed from a group and a subset of its elements closed under taking inverses. Requisite to the definition of the graph, therefore, are the definitions of a symmetric subset and a Cayley graph, with the former serving as foundation for the construction of the latter [10].

Definition 3 (Symmetric Subset)**.** Let Γ be a group. A subset $S \subseteq \Gamma$ is said to be symmetric if for every $x \in S$, $x^{-1} \in S$.

Definition 4 (Cayley Graph)**.** Let Γ be a group and S be a symmetric subset of Γ. The Cayley graph of Γ on S, denoted by $Cay(\Gamma, S)$, is the simple undirected graph such that:

1) The vertices of $Cay(\Gamma, S)$ are the elements of Γ; and
2) For any $a, b \in \Gamma$, a and b are adjacent in $Cay(\Gamma, S)$ if $xa = b$ for some $x \in S$.

The set S is called the generating set for $Cay(\Gamma, S)$ and its elements are called the generators of $Cay(\Gamma, S)$. Another necessary concept for the definition of the pancake graph is the prefix reversal, defined below [11].

Definition 5 (Prefix Reversal)**.** The prefix reversal of order k in S_n, with $2 \leq k \leq n$, is a permutation r_i which inverts the order of the first i elements of a permutation when multiplied on the left. That is, if $\pi = [\pi_1\pi_2 \cdots \pi_n]$ then $r_i[\pi_1\pi_2 \cdots \pi_n] = [\pi_i\pi_{i-1} \cdots \pi_1\pi_{i+1} \cdots \pi_n]$.

An explicit formula for r_i may be computed. It follows that the prefix reversal of order i in the symmetric group S_n, with $2 \leq i \leq n$, is the permutation obtained by reversing the first i elements of the identity permutation

$$r_i = [i, i-1, \cdots, 2, 1, i+1, i+2, \cdots, n-1, n].$$

TABLE I: Prefix reversals in S_n for $1 \leq n \leq 5$.

n	Prefix Reversals
1	None
2	[21]
3	[213] , [321]
4	[2134] , [3214] , [4321]
5	[21345] , [32145] , [43215] , [54321]

Table I contains the prefix reversals in S_n for small n.

A Cayley graph of a symmetric group on a set of prefix reversals is then known to be a pancake graph [11]. A formal definition is infra.

Definition 6 (Pancake Graph)**.** The pancake graph P_n of order n for $n \geq 1$ is the Cayley graph $Cay(S_n, PR_n)$, where PR_n is the set of all prefix reversals in S_n.

The pancake graph is typically defined for order 2 or higher [5], [11]. In the paper, nonetheless, the definition is extended to include the pancake graph of order 1 for proof simplification. As illustration, refer to Figures 1 to 4 for pancake graphs P_1, P_2, P_3, and P_4. In the figures, the vertices are permutations written in one-line notation. It is seen that P_1 is the edgeless graph with one vertex, P_2 is the graph K_2, and P_3 is the graph C_6.

Basic properties pertaining to the degree and cycle structure of the pancake graph follows. The degree of the pancake graph, foremost, may be easily inferred as it is a Cayley graph – a graph whose edges are solely determined by a generating set. Additionally, the number of vertices and edges in a pancake graph of given order is established through the handshaking lemma.

Proposition 1. For $n \geq 1$, the pancake graph P_n is a simple $(n-1)$-regular graph.

Corollary 1. The pancake graph P_n has number of vertices $|V(P_n)| = n!$ and edges $|E(P_n)| = \frac{(n-1)\cdot n!}{2}$ for $n \geq 1$.

Results on the cycle structure of the pancake graph is then laid out towards presenting the triangle-free property of the graph [3].

Theorem 2. For $n \geq 3$, there exists a cycle of length k in P_n for any k, where $6 \leq k \leq n!$. There are no cycles of other lengths in P_n.

The proof of Theorem 2 involves showing that the pancake graph exhibits a recursive structure. Any vertex of P_n is a permutation on n symbols $1, 2, ..., n$ which can be formed by taking a permutation of $n-1$ of these symbols and adding the remaining symbol at the end. The vertices in P_n may then be grouped based on the last element in the permutation. Figures 5 and 6 show sample arrangements.

Each of the subgraphs induced by the groups of vertices have vertices determined by permutations on $n-1$ symbols and edges determined by the prefix reversals r_k for $2 \leq k \leq n-1$. Every subgraph is hence a pancake graph of order $n-1$; for $n \geq 2$, P_n contains n copies of P_{n-1}. The pancake graph may be recursively constructed in such a manner.

Since P_3 is the cycle graph C_6, all cycles in P_n are at least of length 6. Through using induction and taking advantage of the recursive structure of the pancake graph, cycles of length k may even be constructed for $6 \leq k \leq n!$ [3].

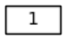

Fig. 1: $P_1 = Cay\left(S_1, \varnothing\right)$.

Fig. 2: $P_2 = Cay\left(S_2, \{[21]\}\right)$.

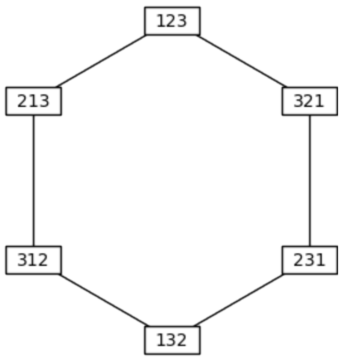

Fig. 3: $P_3 = Cay\left(S_3, \{[213], [321]\}\right)$.

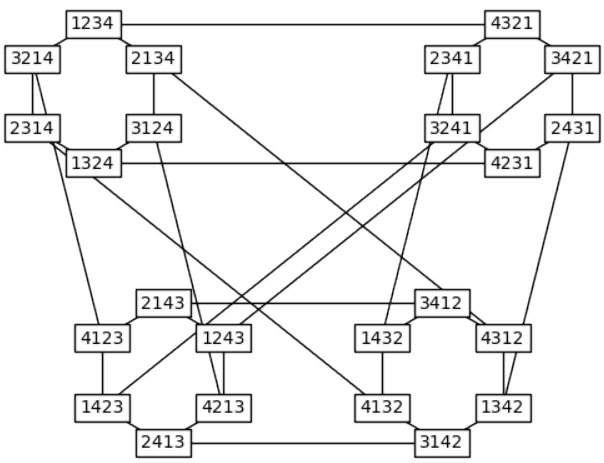

Fig. 4: $P_4 = Cay\left(S_4, \{[2134], [3214], [4321]\}\right)$.

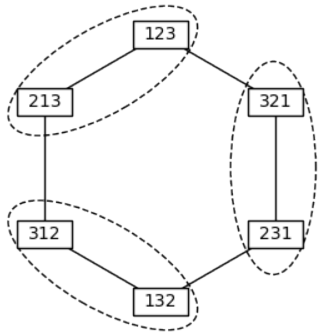

Fig. 5: Vertices of P_3 grouped according to the last element in the permutation associated with each vertex.

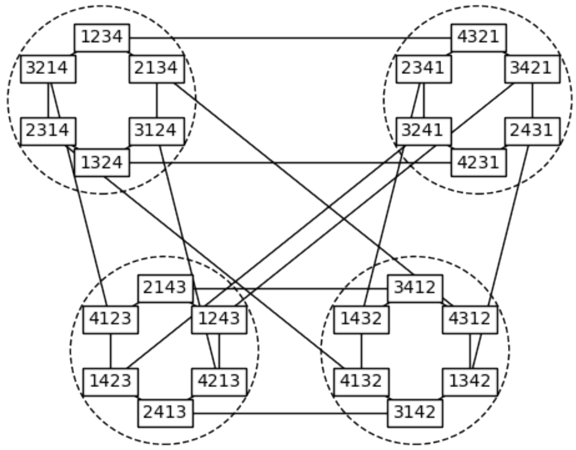

Fig. 6: Vertices of P_4 grouped according to the last element in the permutation associated with each vertex.

That a pancake graph is triangle free is an important consequence of Theorem 2. As the name suggests, a graph is said to be triangle-free if it does not contain any triangles, which are defined as subgraphs that are isomorphic to C_3. Note that the clique number of P_n, the size of the largest complete subgraph in the graph, is easily determined by Theorem 2 as well.

Corollary 2. For $n \geq 1$, P_n is triangle-free.

Corollary 3. The clique number of P_n is

$$\omega\left(P_n\right) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n > 1 \end{cases}.$$

Regarding the exact chromatic number of the pancake graph, a basic result is that $\chi(P_n)$ is nondecreasing with respect to n. This is due to the pancake graph of order $n-1$ being contained in a pancake graph of order n for all $n \geq 2$. With the observation in mind, Table II compiles the following results on $\chi(P_n)$.

- Since $P_1 = K_1$, $P_2 = K_2$, and $P_3 = C_6$, the first three chromatic numbers are trivial – $\chi(P_1) = 1$, $\chi(P_2) = 2$, and $\chi(P_3) = 2$.
- Since P_4 contains an odd cycle of length 7, Theorem 2 states that it is not 2-colorable. A 3-coloring of P_4 is presented in Figure 7.

TABLE II: Known values of $\chi(P_n)$.

n	$\chi(P_n)$
1	1
2	2
3	2
4	3
5	3
6	4
7	4
8	4
9	4

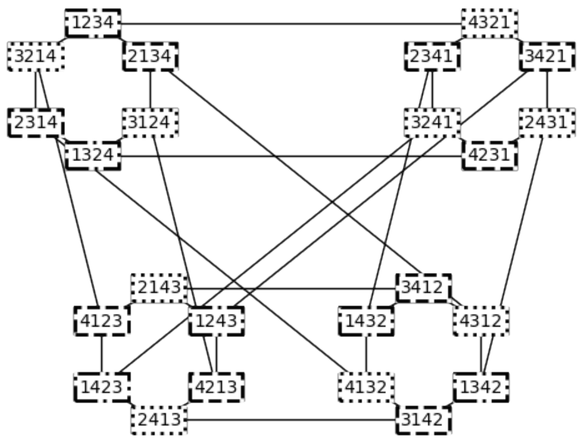

Fig. 7: A 3-coloring of P_4.

- Presented in [11] is a proper 3-coloring of P_5. This proves that $\chi(P_5) = 3$.
- A proper 4-coloring of P_6 is generated [11]. The coloring is computationally determined to be optimal, proving that $\chi(P_6) = 4$.
- A theorem in [11] implies that $\chi(P_7) = 4$.
- With a computational approach, it is found that $\chi(P_8) = 4$ and $\chi(P_9) = 4$ [12].

IV. Existing Bounds

In the section, existing bounds on the chromatic number of the pancake graph are presented.

A. Bounds by Maximum Degree

One begins with a famous result known as Brooks' theorem [13], [14]. The theorem bounds the chromatic number of a graph based on the maximum degree of its vertices.

Theorem 3. For every graph G, $\chi(G) \leq \Delta(G) + 1$. Moreover, $\chi(G) = \Delta(G) + 1$ if and only if $\Delta(G) \neq 2$ and G has the complete graph $K_{\Delta(G)+1}$ as a connected component or $\Delta(G) = 2$ and G has an odd cycle as a connected component.

Since the pancake graph P_n is $(n-1)$-regular, $\chi(P_n) \leq n - 1$ for all $n \geq 3$ by Brooks' theorem. Extensions to Brooks' theorem that give bounds based on the maximum degree and clique number of a graph are published both in [15] and [16]. The results below are achieved independently by the two.

Theorem 4. Suppose G is a connected graph with maximum degree $\Delta(G)$ and clique number $\omega(G)$. The following hold.

1) If G is not an odd cycle then
$$\chi(G) \leq \Delta(G) - \left\lfloor \frac{\Delta(G) - \max(3, \omega(G))}{1 + \max(3, \omega(G))} \right\rfloor.$$
2) If $\Delta(G) \geq 7$ and $\omega(G) \leq \left\lfloor \frac{\Delta(G)-1}{2} \right\rfloor$ then
$$\chi(G) \leq \Delta(G) - 1.$$
3) If $\omega(G) \leq 3$, then
$$\chi(G) \leq \left\lceil \frac{3}{4}(\Delta(G) + 1) \right\rceil.$$

In the case of the pancake graphs, $\Delta(P_n) = n - 1$ and $\omega(P_n) = 2$ for $n \geq 2$. The tightest general bound that can be obtained directly from Theorem 4 is $\chi(P_n) \leq \frac{3n}{4}$, obtained from the third item.

B. Bounds for Triangle-Free Graphs

The bound shown in the third item of Theorem 4 holds for graphs G with $\omega(G) \leq 3$. A related condition is when $\omega(G) \leq 2$, which is equivalent to G being triangle-free. A stronger linear bound on $\chi(G)$ for triangle-free graphs G, called Kostochka's bound, is provided [14]. As a remark, this is the tightest among all known closed-form linear bounds for triangle-free graphs in general [14].

Theorem 5. For every triangle-free graph G,
$$\chi(G) \leq \frac{2}{3}\Delta(G) + 2.$$

There is evidence that this bound may be further improved. A probabilistic approach shows an asymptotic bound for the chromatic number of triangle-free graphs [17].

Theorem 6. For every triangle-free graph G,
$$\chi(G) \leq O\left(\frac{\Delta(G)}{\ln(\Delta(G))}\right).$$

As $\Delta(G)$ increases, however, the chromatic number of triangle-free graphs is bounded by a sublinear expression. This means that no bound on $\chi(P_n)$ that is linear in terms of $\Delta(G)$ is optimal. Furthermore, since $P_n = n - 1$ for all $n \geq 1$, it follows that any bound on $\chi(P_n)$ which is linear in terms of n is not tight.

C. Bounds by Graph Order

Most basic bounds that may be imposed on $\chi(P_n)$ are linear in terms of the pancake graph order n. This is not to be confused with the conventional definition of order as the number of vertices. A recent development is below, referred to as Konstantinova's bound [11].

Theorem 7. The following are true.

- If $5 \leq n \leq 8$ then
$$\chi(P_n) \leq \begin{cases} n-2 & \text{if } n \equiv 0 \pmod 4 \\ n-1 & \text{if } n \equiv 1 \pmod 4 \\ n-2 & \text{if } n \equiv 2 \pmod 4 \\ n-3 & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

- If $9 \le n \le 16$ then
$$\chi(P_n) \le \begin{cases} n-4 & \text{if } n \equiv 0 \pmod 4 \\ n-3 & \text{if } n \equiv 1 \pmod 4 \\ n-4 & \text{if } n \equiv 2 \pmod 4 \\ n-5 & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$
- If $n \ge 17$ then
$$\chi(P_n) \le \begin{cases} n-8 & \text{if } n \equiv 0 \pmod 4 \\ n-5 & \text{if } n \equiv 1 \pmod 4 \\ n-6 & \text{if } n \equiv 2 \pmod 4 \\ n-7 & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

Taking into account that chromatic numbers must be integers, it may be shown that Konstantinova's bound is tighter than Kostochka's bound for $n \le 28$. Another with the same author is alluded to as Droogendijk's bound [12]. The result is stated as follows.

Theorem 8. For $n \ge 9$,
$$\chi(P_n) \le 4\left\lfloor \frac{n}{9} \right\rfloor + \chi\left(P_{n \bmod 9}\right).$$

A pertinent open problem which would be part of the ensuing discussion is then briefly reviewed. This is known as Reed's conjecture [18].

Conjecture 1. For all graphs G,
$$\chi(G) \le \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil.$$

Reed's conjecture has been proven for various classes of graphs but not for triangle-free graphs. In the case of the pancake graph, Reed's conjecture would imply
$$\chi(P_n) \le \left\lceil \frac{n}{2} + 1 \right\rceil$$
since $\Delta(P_n) = n - 1$ and $\omega(P_n) \le 2$ for $n \ge 1$.

V. Discussion

A new procedure to generate upper bounds for $\chi(P_n)$ which are linear in terms of n is presented. The method is then used to prove the main contribution of the paper, that $\chi(P_n) \le \frac{4n}{9} + \frac{4}{3}$ for all $n \ge 1$.

A. Partitioning the Graph

First, a generalized version of an unpublished insight is stated and proven [19]. The result bounds $\chi(P_n)$ by first partitioning the vertices of P_n into subgraphs which are homomorphic to smaller pancake graphs. The homomorphisms are then applied to generate inequalities involving the chromatic number of multiple pancake graphs.

Lemma 3. Let S be a nonempty subset of $\{1, 2, ..., n\}$. Let P_n^S be the subgraph of P_n induced by the vertices representing permutations σ such that $\sigma(1) \in S$. Then P_n^S is homomorphic to $P_{|S|}$, the pancake graph of order $|S|$.

Proof: Let $S = \{s_1, s_2, ..., s_m\}$ be a nonempty subset of $\{1, 2, ..., n\}$ and let $m = |S|$. Since the vertices of P_m are permutations of the elements $\{1, 2, ..., m\}$, the vertices of P_m may be relabeled by replacing the element i with s_i for all i, $1 \le i \le m$. Let P'_m denote the relabeled graph. It is enough to show that P_n^S is graph homomorphic to P'_m.

Let $\phi_S : V(P_n^S) \to V(P'_m)$ be a function which takes a permutation in $V(P_n^S)$ in one-line notation and outputs a new permutation in one-line notation by removing every symbol not in S from the input string while preserving the order of the remaining elements. Here are some examples.

- Taking $S = \{1, 2, 5\}$, $\phi_S(15324) = 152$.
- Taking $S = \{2, 4\}$, $\phi_S(41532) = 42$.

One shows that ϕ_S is a graph homomorphism from P_n^S onto P'_m. By the definition of ϕ_S, the output of ϕ_S expresses a permutation of the elements of S in one-line notation, hence $\phi_S(v) \in V(P'_m)$ for all $v \in V(P_n^S)$. To show that ϕ_S is onto, let $v \in V(P'_{|S|})$. This means that v is a permutation of the elements of S. Without loss of generality, suppose $v = [s_1 s_2 \cdots s_m]$. Let $t_1, t_2, ..., t_{n-m}$ be the integers from 1 to n which are not in S, in any order. The permutation $u = [s_1 s_2 \cdots s_m t_1 t_2 \cdots t_{n-m}]$ satisfies $\phi_S(u) = v$ by definition of ϕ_S. Accordingly, ϕ_S is onto.

Justifying that ϕ_S is a graph homomorphism, suppose u and v are adjacent vertices in P_n^S. By the definition of P_n^S, it follows that $u(1)$ and $v(1)$ are both in S. Since u and v are adjacent in P_n^S, a subgraph of P_n, there exists a prefix reversal $r_i \in PR_n$ such that $r_i u = v$. The first and last permutation elements of the reversed prefix must be elements of S as $u(i) = (r_i u)(1) = v(1) \in S$. By considering only the elements of u and v which are in S, it follows that $\phi_S(v)$ may be obtained from $\phi_S(u)$ by applying a prefix reversal in the generating set of P'_m. This is illustrated with examples.

- Taking $S = \{1, 2, 5\}$, the edge $15324 \to 23514$ corresponds to the edge $152 \to 251$ in P'_m, which is also generated by a prefix reversal.
- Taking $S = \{1, 2, 5\}$, the edge $15324 \to 51324$ corresponds to the edge $152 \to 512$ in P'_m, which is also generated by a prefix reversal.

On the above account, $\phi_S(u)$ is adjacent to $\phi_S(v)$, thereby making ϕ_S a graph homomorphism. The subgraph P_n^S is then homomorphic to P'_m, which in turn is isomorphic to P_m, the pancake graph of order $m = |S|$.

■

The lemma allows for a coloring of the subgraphs of a pancake graph based on a homomorphism to a smaller pancake graph. As illustrated in Theorem 1, given two graphs G and H and a graph homomorphism ϕ from G to H, a proper k-coloring of G may be generated from a proper k-coloring of H.

Example 1. Take P_4 and consider the set $S = \{1, 3, 4\}$. The homomorphism from P_4^S to a relabeling of P_3 constructed by Lemma 3 is shown in Figures 8 and 9. Vertices in P_4 are mapped to the vertex of the same color in P_3. Vertices of P_4 not in the domain of the homomorphism are colored white.

By partitioning the vertex set of P_n into multiple subgraphs then applying Lemma 3, the following general bound on the chromatic number is obtained.

Theorem 9. Let $(r_1, r_2, \cdots, r_m)$ be a partition of n. That is, suppose $r_1, r_2, \cdots, r_m$ are positive integers such that $r_1 + r_2 + \cdots + r_m = n$. As a consequence, $\chi(P_n) \le \chi(P_{r_1}) + \chi(P_{r_2}) + \cdots + \chi(P_{r_m})$.

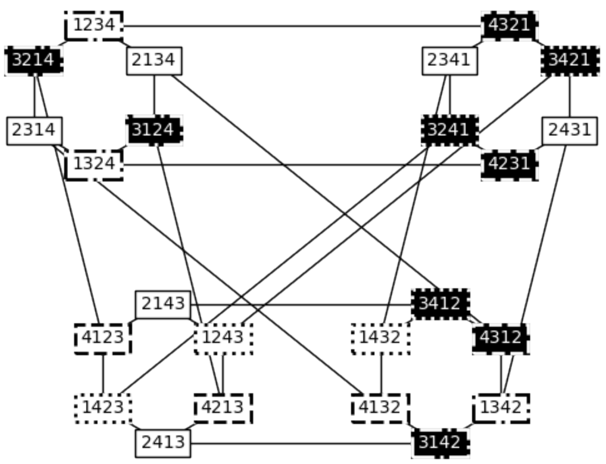

Fig. 8: The subgraph P_4^S of P_4 colored.

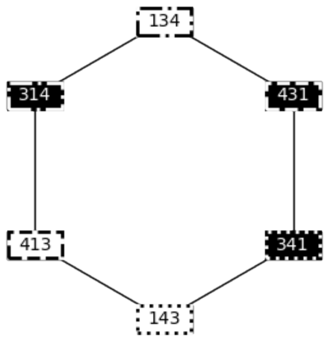

Fig. 9: The relabeled P_3 with corresponding color-coding.

Proof: One denotes by $[i, j]$ the set of integers $\{x \in \mathbb{Z} \mid i \leq x \leq j\}$. Following the convention in the proof of Lemma 3, the subgraph of P_n induced by the vertices representing the permutations σ where $i \leq \sigma(1) \leq j$ may thus be denoted by $P_n^{[i,j]}$. That being so, $P_n^{[i,j]}$ is the subgraph of P_n induced by the vertices corresponding to permutations whose first element when expressed in one-line notation is an integer from i to j.

Let $(r_1, r_2, \cdots, r_m)$ be a partition of n. Consider the vertex-disjoint subgraphs $G_1, G_2, \ldots, G_m$ of P_n given by $G_1 = P_n^{[1,r_1]}$ and $G_k = P_n^{[1+\sum_{i=1}^{k-1} r_i, \sum_{i=1}^{k} r_i]}$, with $2 \leq k \leq m$. As an example, the following 3 subgraphs of P_{12} may be identified given the partition $(3, 4, 5)$ of 12.

- The subgraph induced by vertices whose permutations have 1, 2, or 3 in the first position is $G_1 = P_{12}^{[1,3]}$.
- The subgraph induced by vertices whose permutations have 4, 5, 6, or 7 in the first position is $G_2 = P_{12}^{[4,7]}$.
- The subgraph induced by vertices whose permutations have 8, 9, 10, 11, or 12 in the first position is $G_3 = P_{12}^{[8,12]}$.

It is clear that $G_1, G_2, \ldots, G_m$ are vertex-disjoint subgraphs of P_n which span the entire vertex set of P_n. One may form a proper coloring of P_n using $\chi(G_1)+\chi(G_2)+\cdots+\chi(G_m)$ colors by coloring each graph G_k with a proper $\chi(G_k)$-coloring using a different set of colors for each subgraph G_k, hence

$$\chi(P_n) \leq \chi(G_1) + \chi(G_2) + \cdots + \chi(G_m).$$

By Lemma 3, there is a graph homomorphism from G_k to P_{r_k} for all k; therefore, $\chi(G_i) \leq \chi(P_{r_k})$ for all k. Finally,

$$\chi(P_n) \leq \chi(P_{r_1}) + \chi(P_{r_2}) + \cdots + \chi(P_{r_m}).$$

■

Theorem 9 gives a general method to bound the chromatic number of a given pancake graph using the chromatic numbers of pancake graphs of smaller order. Before proceeding, several corollaries which capture useful properties of $\chi(P_n)$ are presented.

Corollary 4. For any integer $n \geq 2$, $\chi(P_{n-1}) \leq \chi(P_n) \leq \chi(P_{n-1}) + 1$.

Proof: Suppose $n \geq 2$. Since P_n contains P_{n-1} as a subgraph, $\chi(P_{n-1}) \leq \chi(P_n)$. By Theorem 9, $\chi(P_n) \leq \chi(P_{n-1}) + \chi(P_1)$. As P_1 has no edges, $\chi(P_1) = 1$ and the result follows. ■

It is also convenient to state the following special case of Theorem 9, obtained by considering a partition (r_1, r_2) of n.

Corollary 5. For any positive integers n and m, $\chi(P_{n+m}) \leq \chi(P_n) + \chi(P_m)$.

Corollary 5 equivalently states that $\chi(P_n)$ is a subadditive function of n.

One seeks to obtain the tightest bound attainable by the graph homomorphism approach illustrated in Theorem 9. The following theorem shows that in order to minimize the resulting on $\chi(P_n)$, it suffices to consider partitions where the vertices are partitioned into two sets.

Theorem 10. Let $\mathcal{P}$ be the set of all partitions $P = (r_1, r_2, \cdots, r_m)$ of n. Then

$$\min_{P \in \mathcal{P}} \sum_{i=1}^{m} \chi(P_{r_i}) = \min_{1 \leq k \leq n-1} \chi(P_k) + \chi(P_{n-k}).$$

Proof: Since all partitions of the form $(k, n-k)$, where $1 \leq k \leq n$, are contained in $\mathcal{P}$,

$$\min_{P \in \mathcal{P}} \sum_{i=1}^{m} \chi(P_{r_i}) \leq \min_{1 \leq k \leq n-1} \chi(P_k) + \chi(P_{n-k}).$$

To prove the reverse inequality, since

$$\chi(P_{n-r_1}) = \chi(P_{\sum_{k=2}^{m} r_k}) \leq \chi(P_{r_2}) + \cdots + \chi(P_{r_m})$$

for any partition $P = (r_1, r_2, ..., r_m)$,

$$\min_{P \in \mathcal{P}} \sum_{i=1}^{m} \chi(P_{r_i}) \geq \min_{1 \leq r_1 \leq n-1} \chi(P_{r_1}) + \chi(P_{n-r_1}).$$

■

In the original insight, only the case of 2-partitions are considered [19]. Theorem 10 is formally proven, which implies that in order to find the tightest bounds for $\chi(P_n)$ attainable using the graph homomorphism approach in Theorem 9, it suffices to consider two-set partitions of the vertex set. The subadditive property given in Corollary 5 is thus used almost exclusively in the next parts.

B. Generalized Linear Bound

Over the course of the subsection, work that builds on the subadditivity insight is developed towards the principal result. Using the subadditivity of $\chi(P_n)$, the following recursive bound on $\chi(P_n)$ which relies on known exact values may be formulated.

Theorem 11. Let m be a positive integer. The function $f_m(n)$ for positive integers n is defined as follows:

$$f_m(n) = \begin{cases} \chi(P_n) & \text{if } 1 \leq n \leq m \\ f_m(n-m) + \chi(P_m) & \text{if } n > m \end{cases}.$$

Then $\chi(P_n) \leq f_m(n)$.

Proof: Proof by strong induction is used.

Base cases. By the definition of f_m, $\chi(P_n) \leq f_m(n)$ for all n with $1 \leq n \leq m$.

Inductive step. Let $r \geq m$. Suppose $\chi(P_n) \leq f_m(n)$ for all n with $n \leq r$. Since $m < r+1$,

$$f_m(r+1) = f_m(r+1-m) + \chi(P_m)$$

by definition of f_m. As $r+1-m \leq r$,

$$\chi(P_{r+1-m}) \leq f_m(r+1-m)$$

by inductive assumption. Since $r+1-m > 0$, Corollary 5 may be applied to obtain

$$\chi(P_{r+1}) \leq \chi(P_{r+1-m}) + \chi(P_m).$$

Referring to the inductive assumption,

$$\chi(P_{r+1-m}) + \chi(P_m) \leq f_m(r+1-m) + \chi(P_m).$$

With $f_m(r+1-m)+\chi(P_m) = f_m(r+1)$, $\chi(P_{r+1}) \leq f_m(r+1)$ is proven.

As result, $\chi(P_n) \leq f_m(n)$ for all $n \geq 1$. ■

Theorem 11 provides a recursively-defined upper bound on $\chi(P_n)$. One now shows that the obtained upper bound is always linear in terms of n.

Theorem 12. Let m be a positive integer. Then, there exists a nonnegative real constant c such that for every positive integer n,

$$\chi(P_n) \leq \frac{\chi(P_m)}{m} n + c.$$

Proof: Let m be a positive integer. Define the linear nonhomogenous recurrence f_m as

$$f_m(n) = \begin{cases} \chi(P_n) & \text{if } 1 \leq n \leq m \\ f_m(n-m) + \chi(P_m) & \text{if } n > m \end{cases}.$$

From Theorem 11, $\chi(P_n) \leq f_m(n)$ for all positive integers n. To solve the recurrence, it is first converted to an equivalent linear homogenous recurrence. Take the two equations

$$f_m(n) = f_m(n-m) + \chi(P_m)$$

and

$$f_m(n-1) = f_m(n-m-1) + \chi(P_m)$$

for $n > m+1$. Subtracting the latter equation from the former yields

$$f_m(n) - f_m(n-1) = f_m(n-m) - f_m(n-m-1),$$

and so

$$f_m(n) = f_m(n-1) + f_m(n-m) - f_m(n-m-1).$$

Using the equation and the base case $\chi(P_1) = 1$, f_m may be rewritten as the homogeneous linear recurrence shown in Table III.

The characteristic polynomial of the recurrence is $z^{m+1} = z^m + z - 1$. Rearranging and factoring yields $(z-1)(z^m-1) = 0$. This means that the roots of the characteristic polynomial are the mth roots of unity. The root $z = 1$ has multiplicity 2 while all other roots have multiplicity 1. The closed form of the recurrence then has the form

$$\begin{aligned} f_m(n) &= c_m n(1)^n + \sum_{k=0}^{m-1} c_k (e^{2k\pi i/m})^n \\ &= c_m n + \sum_{k=0}^{m-1} c_k e^{2nk\pi i/m} \end{aligned} \tag{1}$$

for some constants $c_0, c_1, ..., c_m \in \mathbb{C}$.

The value of c_m may be explicitly solved for by substituting the base cases $f_m(1) = 1$ and $f_m(m+1) = \chi(P_m) + 1$ into Equation 1. From $f_m(1) = 1$,

$$c_m + \sum_{k=0}^{m-1} c_k e^{2k\pi i/m} = 1. \tag{2}$$

From $f_m(m+1) = \chi(P_m) + 1$, one obtains

$$\begin{aligned} c_m(m+1) + \sum_{k=0}^{m-1} c_k e^{2(m+1)k\pi i/m} &= \chi(P_m) + 1 \\ c_m(m+1) + \sum_{k=0}^{m-1} c_k e^{2k\pi i} e^{2k\pi i/m} &= \chi(P_m) + 1 \\ c_m(m+1) + \sum_{k=0}^{m-1} c_k e^{2k\pi i/m} &= \chi(P_m) + 1. \end{aligned} \tag{3}$$

Subtracting Equation 3 from Equation 2,

$$\begin{aligned} c_m(m+1) - c_m &= \chi(P_m) + 1 - 1 \\ c_m m &= \chi(P_m) \\ c_m &= \frac{\chi(P_m)}{m}. \end{aligned} \tag{4}$$

Substituting Equation 4 into Equation 1, the closed form of $f_m(n)$ may be expressed as

$$f_m(n) = \frac{\chi(P_m)}{m} n + \sum_{k=0}^{m-1} c_k e^{2nk\pi i/m}$$

for some constants $c_0, c_1, ..., c_{m-1} \in \mathbb{C}$.

From the recursive definition of f_m, $f_m(n)$ is a real number if n is a positive integer; hence, if n is a positive integer,

$$f_m(n) = \text{Re}(f_m(n)) \leq |f_m(n)|,$$

One hires the triangle inequality as seen in Table IV. Since $c_0, c_1, ..., c_{m-1}$ are complex constants, by taking $c \in \mathbb{R}$ where $c = \sum_{k=0}^{m-1} |c_k|$,

$$f_m(n) \leq \frac{\chi(P_m)}{m} n + c.$$

There exists, therefore, a constant c such that for every positive integer n,

$$\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c.$$

■

The existence of a nonnegative constant c such that $\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c$ for all integers n is shown. To find an appropriate constant c for a given m, it is enough to find a c such that $\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c$ for all integers n where $1 \leq n \leq m$. This is proven this in the theorem below.

Theorem 13. Let m be a positive integer and c be a nonnegative real number. Suppose $\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c$ for all integers n where $1 \leq n \leq m$. Then the inequality also holds for all integers n where $n > m$.

Proof: That the inequality holds for all $n \geq 1$ is proven by strong induction. Let f_m be the recursive function

$$f_m(n) = \begin{cases} \chi(P_n) & \text{if } 1 \leq n \leq m \\ f_m(n-m) + \chi(P_m) & \text{if } n > m \end{cases}.$$

From Theorem 11, $\chi(P_n) \leq f_m(n)$ for all $n \geq 1$.

Base cases. Since $f_m(n) = \chi(P_n)$ by definition and $\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c$ by assumption, the claim is true for the base cases $1 \leq n \leq m$.

Inductive step. Let $r \geq m$. Suppose $f_m(n) \leq \frac{\chi(P_m)}{m}n + c$ for all $n \leq r$. Since $r+1 > m$, by the definition of f_m,

$$f_m(r+1) = f_m(r+1-m) + \chi(P_m).$$

Since $1 \leq r+1-m \leq r$, one may write as indicated in Table V by inductive assumption; ergo, $f_m(r+1) \leq \frac{\chi(P_m)}{m}(r+1) + c$.

By strong induction, it follows that $\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c$ for all $n \geq 1$. ■

Using Theorems 12 and 13, a general method for generating a linear bound for $\chi(P_n)$ may be established if for some integer m, the exact values of $\chi(P_n)$ for the first m pancake graphs are known. A summary of the new technique follows.

1) Suppose the exact values of the chromatic numbers $\chi(P_1), \chi(P_2), ..., \chi(P_m)$ are known. By Theorem 12, there exists a nonnegative constant c such that for every positive integer n,

$$\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c.$$

2) To find an appropriate value for c, Theorem 13 states it is sufficient to find a c such that the above inequality holds for all n, with $1 \leq n \leq m$.

As organized in Table II, the chromatic number of the pancake graph up to order 9 are found in literature. Since the chromatic numbers of the first 9 pancake graphs are known, one may choose $m = 9$. Such an assignment proves the primary contribution of the paper, which is stated further on.

Theorem 14. For every positive integer n,

$$\chi(P_n) \leq \frac{4n}{9} + \frac{4}{3}.$$

Proof: By Theorem 12, there exists a nonnegative constant c such that for every positive integer n,

$$\chi(P_n) \leq \frac{\chi(P_9)}{9}n + c = \frac{4n}{9} + c.$$

Given the known values of $\chi(P_n)$, it is easily verified that this inequality is satisfied for $1 \leq n \leq 9$ when $c = \frac{4}{3}$. It is then confirmed that by Theorem 13, $\chi(P_n) \leq \frac{4n}{9} + \frac{4}{3}$ for every positive integer n.

■

As mentioned, tighter bounds may be achieved with the discovery of the vertex chromatic numbers of pancake graphs with order greater than 9. An immediate example is the conjecture below that is true if $\chi(P_{10}) = 4$.

Conjecture 2. For every positive integer n,

$$\chi(P_n) \leq \frac{2n}{5} + \frac{8}{5}.$$

Based on the findings in [11] and Corollary 4, one may produce additional linear bounds of the form

$$\chi(P_n) \leq \frac{\chi(P_m)}{m}n + c.$$

To summarize, two key results are utilized:

- That for a pancake graph P_m with $10 \leq m \leq 16$, $4 \leq \chi(P_m) \leq 6$; and
- That the chromatic numbers of a pancake graph P_n and a pancake graph P_{n-1} would differ by at most 1.

Seen in Table VI are the computed values for c given $(m, \chi(P_m))$ pairs with $10 \leq m \leq 16$ and $4 \leq \chi(P_m) \leq 6$. One then proceeds to the best-case and worst-case analyses.

Clearly, the best scenario would be if $\chi(P_m) = 4$ for $10 \leq m \leq 16$. This means that $\chi(P_{16}) = 4$ in this case, yielding the following conjecture.

Conjecture 3. For every positive integer n,

$$\chi(P_n) \leq \frac{n}{4} + \frac{5}{2}.$$

Since the chromatic numbers increase by at most 1, the worst case would be if $\chi(P_{10}) = 5$ and $\chi(P_m) = 6$ for $11 \leq m \leq 16$. In this case, the following conjecture is made given $\chi(P_{16}) = 6$.

Conjecture 4. For every positive integer n,

$$\chi(P_n) \leq \frac{3n}{8} + \frac{7}{4}.$$

C. Comparison and Remarks

Theorem 14 is the tightest linear bound attainable by the proposed graph homomorphism method as there is no $c < 1$ which satisfies the conditions of Theorem 13. Several remarks are now made regarding the results. It is first recognized that this is a relative improvement with respect to those proven so far. Figure 10 presents a comparison.

Remark 1. The bound attained in Theorem 14 is better than both the bound in [14] and the bound in [11]. It proves to be comparable to the bound in [12].

That the bound has potential for refinement is recognized next. An improved bound based on Conjecture 2 is shown in Figure 11.

Remark 2. Should Conjecture 2 hold true, the acquired bound would be better than the bound in [12].

Another remark summarizes Conjectures 3 and 4 from the previous subsection.

TABLE III: Resulting homogenous linear recurrence in the Theorem 12 proof.

$$f_m(n) = \begin{cases} \chi(P_n) & \text{if } 1 \le n \le m \\ \chi(P_m) + 1 & \text{if } n = m+1 \\ f_m(n-1) + f_m(n-m) - f_m(n-m-1) & \text{if } n > m+1 \end{cases}$$

TABLE IV: Applying the triangle inequality in proving Theorem 12.

$$|f_m(n)| = \left| \frac{n}{m}\chi(P_m) + \sum_{k=0}^{m-1} c_k e^{2nk\pi i/m} \right| \le \left| \frac{n}{m}\chi(P_m) \right| + \sum_{k=0}^{m-1} \left| c_k e^{2nk\pi i/m} \right| = \frac{n}{m}\chi(P_m) + \sum_{k=0}^{m-1} |c_k| \left| e^{2nk\pi i/m} \right| = \frac{n}{m}\chi(P_m) + \sum_{k=0}^{m-1} |c_k|$$

TABLE V: Argument in the inductive step for the proof of Theorem 13.

$$f_m(r+1-m) + \chi(P_m) \le \left(\frac{r+1-m}{m}\chi(P_m) + c \right) + \chi(P_m) = \frac{r+1}{m}\chi(P_m) - \chi(P_m) + c + \chi(P_m) = \frac{r+1}{m}\chi(P_m) + c$$

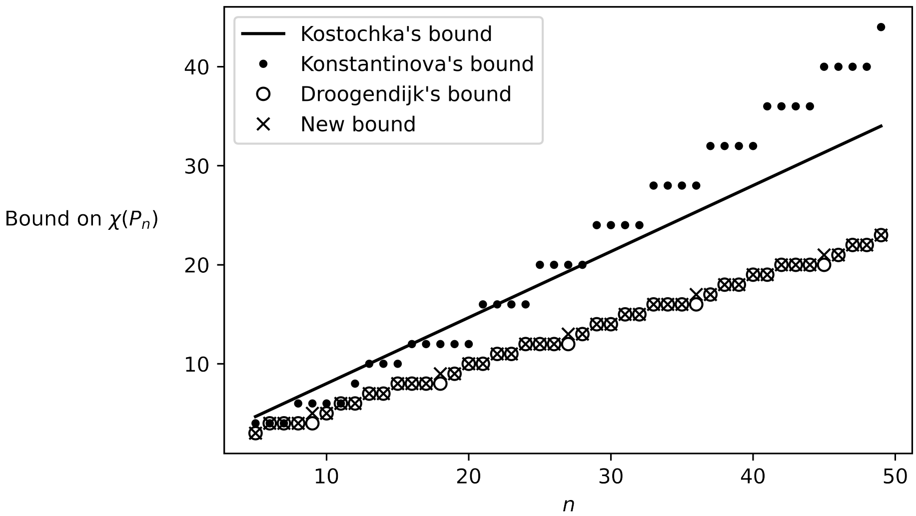

Fig. 10: Comparison of Kostochka's bound, Konstantinova's bound, and Droogendijk's bound to the new bound.

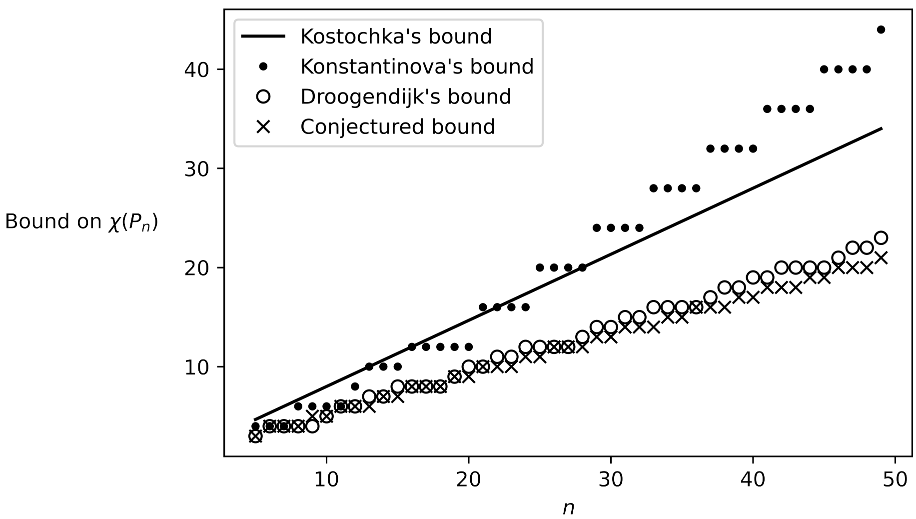

Fig. 11: Comparison of Kostochka's bound, Konstantinova's bound, and Droogendijk's bound to the conjectured bound.

TABLE VI: Calculated values for c given possible $\chi(P_m)$ for $10 \leq m \leq 16$.

$(m, \chi(P_m))$	c
$(10, 4)$	$\frac{8}{5}$
$(10, 5)$	1
$(10, 6)$	$\frac{4}{5}$
$(11, 4)$	$\frac{20}{11}$
$(11, 5)$	$\frac{14}{11}$
$(11, 6)$	$\frac{9}{10}$
$(12, 4)$	2
$(12, 5)$	$\frac{3}{2}$
$(12, 6)$	1
$(13, 4)$	$\frac{28}{13}$
$(13, 5)$	$\frac{22}{13}$
$(13, 6)$	$\frac{16}{13}$
$(14, 4)$	$\frac{16}{7}$
$(14, 5)$	$\frac{13}{7}$
$(14, 6)$	$\frac{10}{7}$
$(15, 4)$	$\frac{12}{5}$
$(15, 5)$	2
$(15, 6)$	$\frac{8}{5}$
$(16, 4)$	$\frac{5}{2}$
$(16, 5)$	$\frac{17}{8}$
$(16, 6)$	$\frac{7}{4}$

Remark 3. The range of linear bounds attainable with the method is the region bounded by $y = \frac{x}{4} + \frac{5}{2}$ and $y = \frac{3x}{8} + \frac{7}{4}$.

The attained bound is also applicable to a known open problem in graph theory. Reed's conjecture claims that for any graph G, the following inequality holds:

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil .$$

As basic properties, $\Delta(P_n) = n - 1$ and $\omega(P_n) = 2$ for any P_n. Substituting these into Reed's conjecture yields

$$\chi(P_n) \leq \left\lceil \frac{n - 1 + 2 + 1}{2} \right\rceil = \left\lceil \frac{n}{2} + 1 \right\rceil ,$$

which is implied by Theorem 14.

Remark 4. The pancake graph satisfies Reed's conjecture.

VI. Conclusion

A general bound for the vertex chromatic number of the pancake graph based on a graph homomorphism approach is advanced. As the bound is linear in terms of the pancake graph order, conciseness is a defining characteristic. In addition, the bound takes known pancake graph chromatic numbers as input, exhibiting a recursive element. Overall, the bound is an ample improvement to bounds in literature. Its relative superiority hinges on the possibility of an update once more vertex chromatic numbers are revealed. As a preview, nevertheless, a conjecture on the revision of the new bound is made in anticipation.

Acknowledgment

Gratitude is expressed to Elena V. Konstantinova for going through the initial draft of the manuscript and more importantly, being a source of inspiration for the research. E. V. Konstantinova is with the Sobolev Institute of Mathematics, Novosibirsk State University, and the Mathematical Center in Akademgorodok, Novosibirsk, Russia.

References

[1] S. Akers and B. Krishnamurthy, "A group-theoretic model for symmetric interconnection networks," *IEEE Transactions on Computers*, vol. 38, no. 4, pp. 555–566, 1989.

[2] L. Morales and I. Sudborough, "Comparing star and pancake networks," in *The Essence of Computation*, G. Goos, J. Hartmanis, J. van Leeuwen, T. Mogensen, D. Schmidt, and I. Sudborough, Eds. Springer Berlin Heidelberg, 2002, vol. 2566.

[3] A. Kanevsky and C. Feng, "On the embedding of cycles in pancake graphs," *Parallel Computing*, vol. 21, no. 6, pp. 923–936, 1995.

[4] Y. Yang, B. Lv, B. Zheng, X. Xu, and K. Zhang, "The crossing number of pancake graph P_4 is six," *arXiv:1211.4642*, 2012. [Online]. Available: http://arxiv.org/abs/1211.4642

[5] E. Konstantinova, "Some problems on cayley graphs," *Linear Algebra and its Applications*, vol. 429, no. 11-12, pp. 2754–2769, 2008.

[6] Y. Deng and X. Zhang, "Automorphism groups of the pancake graphs," *arXiv:1201.0452*, 2012. [Online]. Available: http://arxiv.org/abs/1201.0452

[7] F. Wang and C. Hsu, "Rainbow colorings on pyramid networks," *IAENG International Journal of Computer Science*, vol. 46, no. 3, pp. 439–444, 2019.

[8] G. Chartrand and P. Zhang, *Chromatic graph theory*, ser. Discrete mathematics and its applications. Chapman & Hall/CRC, 2009.

[9] C. Godsil and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. Springer New York, 2001, vol. 207.

[10] J. Bondy and U. Murty, *Graph Theory*, ser. Graduate Texts in Mathematics, S. Axler and K. Ribet, Eds. Springer London, 2008, vol. 244.

[11] E. Konstantinova, "Chromatic properties of the pancake graphs," *Discussiones Mathematicae Graph Theory*, vol. 37, no. 3, pp. 777–787, 2017.

[12] L. Droogendijk and E. Konstantinova, "An improved bound on the chromatic number of the pancake graphs," *arXiv:2103.11092*, 2021. [Online]. Available: http://arxiv.org/abs/2103.11092

[13] R. Brooks, "On colouring the nodes of a network," in *Classic Papers in Combinatorics*, I. Gessel and G. Rota, Eds. Birkhäuser Boston, 2009, pp. 118–121.

[14] T. Jensen and B. Toft, *Graph Coloring Problems*. John Wiley & Sons, Inc., 1994.

[15] P. Catlin, "A bound on the chromatic number of a graph," *Discrete Mathematics*, vol. 22, no. 1, pp. 81–83, 1978.

[16] O. Borodin and A. Kostochka, "On an upper bound of a graph's chromatic number, depending on the graph's degree and density," *Journal of Combinatorial Theory, Series B*, vol. 23, no. 2-3, pp. 247–250, Oct. 1977.

[17] M. Molloy and B. Reed, *Graph colouring and the probabilistic method*, ser. Algorithms and combinatorics. Springer, 2002, no. 23.

[18] B. Reed, "ω, δ, and χ," *Journal of Graph Theory*, vol. 27, no. 4, pp. 177–212, 1998.

[19] L. Droogendijk, "Chromatic number of the pancake graph is subadditive," Mathematics Stack Exchange. [Online]. Available: https://math.stackexchange.com/q/3057032