List 2-Distance Coloring of Planar Graphs with Neither 3-Cycles Nor Intersect 4-Cycles

Wen-juan Zhou and Lei Sun*

***Abstract*—A coloring $\varphi : V(G) \to \{1, 2, ..., k\}$ of G is 2-distance if any two vertices at distance at most two from each other get different colors. If every vertex v of G has its own set $L(v)$ of admissible colors where $|L(v)| \geq k$, then we say that $V(G)$ has a list L of size k. A graph G is said to be list 2-distance k-colorable if any list L of size k allows a 2-distance coloring φ such that $\varphi(v) \in L(v)$ whenever $v \in V(G)$. In this paper, we proved that: Every planar graph with neither 3-cycles nor intersect 4-cycles and $\Delta(G) \geq 18$ is list 2-distance $(\Delta + 8)$-colorable.**

***Index Terms*—list 2-distance k-colorable, cycle, intersect, maximum degree.**

I. Introduction

A Coloring $\varphi : V(G) \to \{1, 2, ..., k\}$ of G is 2-distance if any two vertices at distance at most two from each other get different colors. The minimum number of colors in 2-distance coloring of G is its 2-distance chromatic number, denoted by $\chi_2(G)$. If every vertex v of G has its own set $L(v)$ of admissible colors where $|L(v)| \geq k$, then we say that $V(G)$ has a list L of size k. A graph G is said to be list 2-distance k-colorable if any list L of size k allows a 2-distance coloring φ such that $\varphi(v) \in L(v)$ whenever $v \in V(G)$. The least k for which G is list 2-distance k-colorable is the list 2-distance chromatic number of G, denoted by $\chi_2^l(G)$.

Regarding 2-distance coloring of planar graphs, in 1977, Wegner[8] conjectured that for a graph G: (1)$\chi_2(G) \leq 7$ if $\Delta(G) = 3$. (2)$\chi_2(G) \leq \Delta(G) + 5$ if $4 \leq \Delta(G) \leq 7$. (3)$\chi_2(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor + 1$ if $\Delta(G) \geq 8$. Wegner also claimed that the upper bounds are tight if the conjecture is true. The conjecture remains open till this moment.

For 2-distance coloring problem of planar graphs without 4,5-cycles, Haiyang Zhu et al. [9] proved that: (1)$\chi_2(G) \leq \Delta(G) + 7$ if $\Delta(G) \leq 7$. (2)$\chi_2(G) \leq 14$ if $\Delta(G) = 8$. (3)$\chi_2(G) \leq \Delta(G) + 5$ if $\Delta(G) \geq 9$.

For every planar graph without 4-cycles, Haiyang Zhu et al.[10] proved that: $\chi_2(G) \leq \Delta(G) + 12$. And for every planar graph without 3,4,7-cycles and $\Delta(G) \geq 15$, Bu and Lv[5] proved that: $\chi_2(G) \leq \Delta(G) + 4$.

At the same time, list 2-distance chromatic number of planar graphs have been extensively studied. Borodin and Ivanova[3,4] proved that: If G is a planar graph which satisfies $g(G) \geq 6$ and $\Delta(G) \geq 24$, then $\chi_2^l(G) \leq \Delta(G)+2$. Then Bu and Yan[6,7] proved that: (1)For every planar graph without 3,5-cycles and intersect 4-cycles, $\chi_2^l(G) \leq \Delta(G)+6$. (2)If G is a planar graph which satisfies $g(G) \geq 5$ and $\Delta(G) \leq 5$, then $\chi_2^l(G) \leq 13$.

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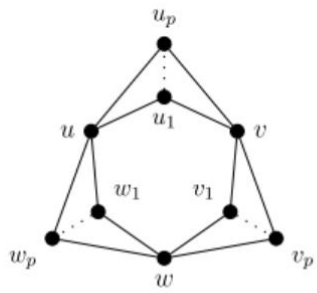

Fig. 1. A Graph G_p with Girth 4, $\Delta(G_p) = 2p$, and $\chi_2^l(G_p) = 3p$.

In this paper, we shall prove the following Theorem 1.

Theorem 1: For every planar graph with neither 3-cycles nor intersect 4-cycles and $\Delta(G) \geq 18$, $\chi_2^l(G) \leq \Delta(G) + 8$.

We note that the condition in Theorem 1 cannot be deleted. In [1], Marthe Bonamy et al. found that there are some planar graphs with intersect 4-cycles satisfying that the difference between $\chi_2^l(G)$ and $\Delta(G)$ can be arbitrarily large (see Fig.1).

To prove this result, we suppose that it is false. In Section 3, we exhibit some properties of a minimal counterexample (regarding the number of vertices and edges) that contradicts Theorem 1. Relying on these properties, we use the Discharging Method in Section 4 to obtain a contradiction.

II. Notations

In this section, we introduce notation used throughout the paper. Given a graph G, we use V, E, F, Δ and δ to denote the vertex set, edge set, face set, maximum degree and minimum degree in G, respectively. For $v \in V(G)$, let $d_G(v)$ denote the degree of v in G, simply $d(v)$. A vertex of degree k (resp. at least k, at most k) will be called k-vertex (resp. k^+-vertex, k^--vertex). For $f \in F(G)$, let $d_G(f)$ denote the degree of f in G, simply $d(f)$. A face of degree k (resp. at least k, at most k) will be called k-face (resp. k^+-face, k^--face).

Let $N_G(v)$ be the set of v's neighbors. Let v's k-neighbor be a vertex adjacent to v with degree k. Let $n_k(v)$ denote the number of v's k-neighbors. For a k-vertex v, we denote its neighbors in ascending order of degree by $v_1, v_2, ..., v_k$, and for a 2-vertex $v_i \in N_G(v)$, if $v_i^{'}$ is a neighbor of v_i except v, then we call it v's weak adjacent vertex. Let $i(j)$-vertex be a vertex with degree i and with j 2-neighbors. A 3-vertex v is a weak 3-vertex if $d(v_1) = 3, d(v_2) + d(v_3) \leq \Delta + 4$. A 3-vertex v is a strong 3-vertex if $d(v_1) = 3, d(v_2) + d(v_3) \geq$

$\Delta+6$. A v is a special 3-vertex if $d(v_1)=4, d(v_2)+d(v_3)\geq \Delta+5$. A 4-vertex v is a weak 4-vertex if it is adjacent to at least one special 3-vertex. We call two 4-cycles intersect when they have at least one vertex in commom.

In the proof of the theorem, we always get a partial L-2-distance coloring φ of a subgraph G' of G. Then we can extend φ to L-2-distance coloring $\varphi^{'}$ of G. Let $F(v)=\{\varphi(u)|u\in N_1(v)\cup N_2(v)\}$ be the set of disable coloring, and $|F(v)|$ be the number of v's disable coloring.

III. Structural Properties

To begin, we prove five structural properties about minimal counterexample. Many of our arguments count the number of colors restricted from use on some uncolored element.

Property 1: $\delta(G)\geq 2$.

The proof is obvious.

Property 2: A 2-vertex is not adjacent to 2-vertices.

Proof: Assume that a 2-vertex u is adjacent to a 2-vertex v. By the minimality of G, $G-uv$ has an $L-2-distance$ coloring $\varphi^{'}$. By recoloring u and v successively, we get $|F(u)|\leq\Delta+1$, $|F(v)|\leq\Delta+2$. Thus $\varphi^{'}$ is a $2-distance$ coloring of G with $(\Delta+8)$ colors, a contradiction. ■

Property 3: There is no k-vertex v in G with one of the following properties:

(1) Let $3\leq k\leq 7$, $d(v_i)=2$ for all $1\leq i\leq k-2$, then $d(v_{k-1})+d(v_k)\leq\Delta-k+9$.
(2) Let $k=3$, $d(v_1)=3$, then $d(v_2)+d(v_3)\leq\Delta+4$ and $d(u)+d(w)\leq\Delta+5$ where u and w are v_1's neighbors except v.
(3) Let $k=4$, $d(v_1)=2$, then $d(v_2)+d(v_3)+d(v_4)\leq\Delta+6$.
(4) Let $k=5$, $d(v_i)=2$ for all $1\leq i\leq 2$, then $d(v_3)+d(v_4)+d(v_5)\leq\Delta+5$.
(5) Let $k=6$, $d(v_i)=2$ for all $1\leq i\leq j$ where $j\in\{2,3\}$, then $\sum_{m=j+1}^{6}d(v_m)\leq\Delta+7-j$.
(6) Let $k=7$, $d(v_i)=2$ for all $1\leq i\leq j$ where $j\in\{1,2,3,4\}$, then $\sum_{m=j+1}^{7}d(v_m)\leq\Delta+7-j$.
(7) Let $k=8$, $d(v_i)=2$ for all $1\leq i\leq 8$, then there is an index $i\in\{1,2,...,8\}$, $d(v_i^{'})\leq\Delta-1$.
(8) Let $k=8$, $d(v_i)=2$ for all $1\leq i\leq 7$, then $d(v_8)\leq\Delta-7$ and there is an index $i\in\{1,2,...,7\}$, $d(v_i^{'})\leq\Delta-1$.
(9) Let $k=8$, $d(v_i)=2$ for all $1\leq i\leq j$ where $j\in\{3,4,5,6\}$, then $\sum_{m=j+1}^{8}d(v_m)\leq\Delta+7-j$ and there is an index $i\in\{1,2,...,j\}$, $d(v_i^{'})\leq\Delta-1$.
(10) Let $k=9$, $d(v_i)=2$ for all $1\leq i\leq 9$, then there is an index $i\in\{1,2,...,9\}$, $d(v_i^{'})\leq\Delta-1$.
(11) Let $k=9$, $d(v_i)=2$ for all $1\leq i\leq 8$, then $d(v_9)\leq\Delta-9$ and there is an index $i\in\{1,2,...,8\}$, $d(v_i^{'})\leq\Delta-1$.
(12) Let $k=9$, $d(v_i)=2$ for all $1\leq i\leq j$ where $j\in\{6,7\}$, then $\sum_{m=j+1}^{9}d(v_m)\leq\Delta-j$ and there is an index $i\in\{1,2,...,j\}$, $d(v_i^{'})\leq\Delta-1$.
(13) Let $k=10$, $d(v_i)=2$ for all $1\leq i\leq 10$, then there are indices $i,j\in\{1,2,...,10\}$, $d(v_i^{'})\leq\Delta-2$, $d(v_j^{'})\leq\Delta-1$, and $i\neq j$.
(14) Let $k=10$, $d(v_i)=2$ for all $1\leq i\leq 9$, then $d(v_{10})\leq\Delta-11$ and there are indices $i,j\in\{1,2,...,9\}$, $d(v_i^{'})\leq\Delta-2$, $d(v_j^{'})\leq\Delta-1$, and $i\neq j$.

Proof:

(1) Assume that $d(v_{k-1})+d(v_k)\leq\Delta-k+9$ for all $3\leq k\leq 7$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v and v_i for all $1\leq i\leq k-2$ where $3\leq k\leq 7$ successively, we get $|F(v)|\leq k-2+(d(v_{k-1})+d(v_k))\leq\Delta+7$, $|F(v_i)|\leq\Delta+i+2$ for all $1\leq i\leq k-2$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.
(2) Assume that $d(v_1)=3$, u and w are its neighbors except v, $d(v_2)+d(v_3)\leq\Delta+4$ and $d(u)+d(w)\leq\Delta+5$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_1 and v successively, we get $|F(v_1)|\leq\Delta+7$, $|F(v)|\leq\Delta+7$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.
(3) The proof is similar to (1).
(4) The proof is similar to (1).
(5) Assume that $d(v_i)=2$ for all $1\leq i\leq j$ where $j\in\{2,3\}$, and $\sum_{m=j+1}^{6}d(v_m)\leq\Delta-j+7$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v and v_i for all $1\leq i\leq j$ where $j\in\{2,3\}$ successively, we get $|F(v)|\leq\Delta+7$, $|F(v_i)|\leq\Delta-j+i+6$ for all $1\leq i\leq j$ where $j\in\{2,3\}$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.
(6) The proof is similar to (5).
(7) Assume that $d(v_1^{'})\leq\Delta-1$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1 and v successively, we get $|F(v_i)|\leq\Delta+i-2$ for all $2\leq i\leq 8$, $|F(v_1)|\leq\Delta+6$, $|F(v)|=2\times 8=16$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.
(8) Assume that $d(v_8)\leq\Delta-7$ and $d(v_1^{'})\leq\Delta-1$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_2, v_3, v_4, v_5, v_6, v_7, v_1 and v successively, we get $|F(v_i)|\leq\Delta+i-1$ for all $2\leq i\leq 7$, $|F(v_1)|\leq\Delta+6$, $|F(v)|\leq\Delta+7$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.
(9) Assume that $\sum_{m=j+1}^{8}d(v_m)\leq\Delta+7-j$ where $j\in\{3,4,5,6\}$ and $d(v_1^{'})\leq\Delta-1$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v, v_2, v_3, ..., v_j and v_1 successively, we get $|F(v)|\leq\Delta+7$, $|F(v_i)|\leq\Delta-j+i+7$ for all $2\leq i\leq j$ where $j\in\{3,4,5,6\}$, $|F(v_1)|\leq\Delta+7$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.
(10) The proof is similar to (7).
(11) The proof is similar to (8).
(12) The proof is similar to (9).
(13) Assume that $d(v_1^{'})\leq\Delta-2$, and $d(v_2^{'})\leq\Delta-1$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_2, v_1 and v successively, we get $|F(v_i)|\leq\Delta+i-3$ for all $3\leq i\leq 10$, $|F(v_2)|\leq\Delta+7$, $|F(v_1)|\leq\Delta+7$, $|F(v)|=2\times 10=20$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.
(14) Assume that $d(v_1^{'})\leq\Delta-2$, $d(v_2^{'})\leq\Delta-1$ and $d(v_{10})\leq\Delta-11$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_2, v_1 and v successively, we get $|F(v_i)|\leq\Delta+i-2$ for all $3\leq i\leq 9$, $|F(v_2)|\leq\Delta+7$, $|F(v_1)|\leq\Delta+7$, $|F(v)|\leq\Delta+7$. Therefore, G is list

$2-distance$ $(\Delta+8)$-colorable, a contradiction. ■

Property 4: There is no 4-vertex v in G with one of the following properties:

(1) Let $d(v_i)=3$ for all $1\le i\le 4$, then $d(v_{i1})+d(v_{i2})\le \Delta+4$ for all $i\in\{1,2,3,4\}$ where v_{i1} and v_{i2} are v_i's neighbors except v.

(2) Let $d(v_i)=3$ for all $1\le i\le 3$, then $d(v_{i1})+d(v_{i2})\le \Delta+4$ for all $i\in\{1,2,3\}$ where v_{i1} and v_{i2} are v_i's neighbors except v and $d(v_4)\le\Delta-2$.

Proof:

(1) Assume that $d(v_{i1})+d(v_{i2})\le\Delta+4$ for all $i\in\{1,2,3,4\}$ where v_{i1} and v_{i2} are v_i's neighbors except v. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_1, v_2, v_3, v_4 and v successively, we get $|F(v_i)|\le\Delta+i+3$ for all $1\le i\le 4$, $|F(v)|=3\times 4=12$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.

(2) Assume that $d(v_{i1})+d(v_{i2})\le\Delta+4$ for all $i\in\{1,2,3\}$ where v_{i1} and v_{i2} are v_i's neighbors except v and $d(v_4)\le\Delta-2$. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_1, v_2, v_3 and v successively, we get $|F(v_i)|\le\Delta+i+2$ for all $1\le i\le 3$, $|F(v)|\le\Delta-2+3\times 3=\Delta+7$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction. ■

Property 5: In G, every $7(0)$-vertex is adjacent to at most six $3(1)$-vertices, and when it is adjacent to six $3(1)$-vertices, $d(v_7)\ge\Delta-10$.

Proof: Assume that a $7(0)$-vertex v is adjacent to seven $3(1)$-vertices, v_1, v_2, v_3, v_4, v_5, v_6, v_7 and m is a 2-vertex which is also a neighbor of v_1. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_1, m and v successively, we get $|F(v_1)|\le\Delta+7$, $|F(m)|\le\Delta+2$, and $|F(v)|=3\times 7=21$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction.

Assume that $7(0)$-vertex v is adjacent to six $3(1)$-vertices, v_1, v_2, v_3, v_4, v_5, v_6, $d(v_7)\le\Delta-11$ and m is a 2-vertex which is also a neighbor of v_1. By the minimality of G, $G-vv_1$ has an $L-2-distance$ coloring. By recoloring v_1, m and v successively, we get $|F(v_1)|\le\Delta+7$, $|F(m)|\le\Delta+2$, and $|F(v)|\le\Delta+7$. Therefore, G is list $2-distance$ $(\Delta+8)$-colorable, a contradiction. ■

IV. Proof of Theorem 1

In this section, we give the proof of our main results by discharging method.

Proof: We prove Theorem 1 by contradiction. Let G be a planar graph with neither 3-cycles nor intersect 4-cycles and with fewest sum of the number of vertices and edges such that $\chi_2^l(G)>\Delta(G)+8$. That is to say, there exists a list L of size $\Delta+8$ of $V(G)$, but G is not L-2-distance colorable. By the minimality of G, G is connected.

According to Euler's formula $|V|+|F|-|E|=2$, and $\sum\limits_{v\in V} d(v)=\sum\limits_{f\in F} d(f)=2|E|$, we get:

$$\sum_{v\in V}\left(\tfrac{3}{2}d(v)-5\right)+\sum_{f\in F}(d(f)-5)=-10.$$

For all $x\in V(G)\cup F(G)$, we define an initial weight function ω: if $v\in V$, let $\omega(v)=\frac{3}{2}d(v)-5$; if $f\in F$, let $\omega(f)=d(f)-5$. Then we get $\sum_{x\in V(G)\cup F(G)} w(x)=-10$. If we obtain a new weight $\omega'(x)\ge 0$ for all $x\in V\cup F$ by transferring weights from one element to another, then we get a contradiction:

$$0\le\sum_{x\in V\cup F}\omega'(x)=\sum_{x\in V\cup F}\omega(x)=-10.$$

This contradiction shows the nonexistence of G. Therefore, Theorem 1 is true. ■

We will redistribute the charge according to the eleven discharging rules below. Note that each of the discharging rules preserves the sum of the charges. Hence, the sum of the charges after discharging is negative. By assuming that each of the five structural properties above holds, we prove that after discharging each element has nonnegative charge, this is an obvious contradiction. We use the following discharging rules:

(R1) When $d(f)=4$

If f is incident with 3-vertices, then each 3-vertex sends $\frac{1}{4}$ to f; if f is incident with 4^+-vertices, then each 4^+-vertex sends $\frac{1}{2}$ to f.

(R2) Every 2-vertex v gets 1 from its adjacent 3^+-vertices.

(R3) For a 3-vertex v, $uv\in E(G)$; if $4\le d(u)\le 5$, then v gets $\frac{1}{4}$ from u; if $d(u)=6$, then v gets $\frac{1}{2}$ from u; if $7\le d(u)\le 10$, then v gets $\frac{3}{4}$ from u; if $11\le d(u)\le\Delta-4$, then v gets $\frac{7}{8}$ from u; if $d(u)\ge\Delta-3$, then v gets 1 from u. In particular, each weak 3-vertex gets $\frac{3}{4}$ from each adjacent strong 3-vertex.

(R4) For a 4-vertex v, $uv\in E(G)$; if $d(u)=5$, then v gets $\frac{1}{4}$ from u; if $6\le d(u)\le 10$, then v gets $\frac{1}{2}$ from u; if $d(u)\ge 11$, then v gets 1 from u. In particular, each weak 4-vertex gets $\frac{1}{2}$ from each adjacent special 3-vertex.

(R5) For a 5-vertex v, $uv\in E(G)$; if $d(u)=6$, then v gets $\frac{1}{4}$ from u; if $7\le d(u)\le\Delta-2$, then v gets $\frac{1}{2}$ from u; if $d(u)=\Delta-1$, then v gets $\frac{3}{4}$ from u; if $d(u)=\Delta$, then v gets 1 from u.

(R6) For a 6-vertex v, $uv\in E(G)$; if $7\le d(u)\le\Delta-2$, then v gets $\frac{1}{2}$ from u; if $d(u)=\Delta-1$, then v gets $\frac{3}{4}$ from u; if $d(u)=\Delta$, then v gets 1 from u.

(R7) For a 7-vertex v, $uv\in E(G)$; if $8\le d(u)\le 9$, then v gets $\frac{1}{4}$ from u; if $10\le d(u)\le\Delta-1$, then v gets $\frac{1}{2}$ from u; if $d(u)=\Delta$, then v gets $\frac{3}{4}$ from u.

(R8) For an 8-vertex v, $uv\in E(G)$; if $9\le d(u)\le\Delta-1$, then v gets $\frac{1}{4}$ from u; if $d(u)=\Delta$, then v gets $\frac{1}{2}$ from u. Moreover, v gets $\frac{3}{16}$ from each weak adjacent Δ-vertex.

(R9) For a 9-vertex v, v gets $\frac{1}{9}$ from each weak adjacent Δ-vertex.

(R10) For a 10-vertex v, v gets $\frac{1}{16}$ from each weak adjacent $(\Delta-1)^+$-vertex.

(R11) After R1-R10, if a 2-vertex v's weight is positive. Then v sends its excess weight to the 2-vertex which has negative weight and is incident with the same 5-face with v.

In the following, we will prove that $\omega'(x)\ge 0$ for all $x\in V(G)\cup F(G)$.

First we check $\omega'(f)\ge 0$, $\forall f\in F(G)$.

Case 1: $d(f)=4$.

When f is incident with 2-vertices, but it is not incident with 3-vertices, from property 2, and by R1, $\omega'(f)\ge(4-5)+\frac{1}{2}\times 2=0$.

When f is incident with both 2-vertices and 3-vertices, from property 2, 3(1) and 3(2), and by R1, $\omega'(f)\ge(4-5)+\frac{1}{4}\times 2+\frac{1}{2}=0$.

When f is not incident with 2-vertices, but it is incident with 3-vertices, from property 3(2), and by R1, $\omega'(f) \geq (4-5)+\frac{1}{4}\times 2+\frac{1}{2}\times 2=\frac{1}{2}>0$.

When f is incident with neither 2-vertices nor 3-vertices, and by R1, $\omega'(f) \geq (4-5)+\frac{1}{2}\times 4=1>0$.

Case 2: $d(f)\geq 5$.

$\omega'(f)=d(f)-5\geq 0$.

Then we check $\omega'(v)\geq 0$, $\forall v\in V(G)$.

Case 1: $d(v)=2$.

From property 2 and by R2, $\omega'(v)\geq(\frac{3}{2}\times 2-5)+1\times 2=0$.

Case 2: $d(v)=3$. From property 3(1), $n_2(v)\leq 1$.

Case 2.1: $n_2(v)=1$.

From property 3(1), $d(v_2)+d(v_3)\geq \Delta+7$, so $(d(v_2), d(v_3))\in\{(7,\Delta),(8,(\Delta-1)^+),(9,(\Delta-2)^+),(10,(\Delta-3)^+),(11^+,11^+)\}$. By R3, v_2 and v_3 send at least $\frac{7}{4}$ to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 3-5)-1+\frac{7}{4}-\frac{1}{4}=0$.

Case 2.2: $n_2(v)=0$.

When v is adjacent to a 3-vertex v_1, if $d(v_2)+d(v_3)\leq \Delta+4$, then v is a weak 3-vertex, and from property 3(2), $d(u)+d(w)\geq\Delta+6$ where u and w are v_1's neighbors except v, so v_1 is a strong 3-vertex. At the same time v_2 and v_3 are 3^+-vertices. Again by property 3(2), v_2 and v_3 are not weak 3-vertices. By R1 and R3, $\omega'(v)=(\frac{3}{2}\times 3-5)+\frac{3}{4}-\frac{1}{4}=0$.

If $d(v_2)+d(v_3)=\Delta+5$, then v is neither a strong 3-vertex nor a weak 3-vertex. And $(d(v_2),d(v_3))\in\{$ $(5,\Delta)$, $(6,(\Delta-1)^+)$, $(7,(\Delta-2)^+)$, $(8,(\Delta-3)^+)$, $(9,(\Delta-4)^+)$, $(10,(\Delta-5)^+)$, $(11^+,11^+)$ $\}$. By R3, v_2 and v_3 send at least $\frac{5}{4}$ to v. By R1 and R4, $\omega'(v)\geq(\frac{3}{2}\times 3-5)+\frac{5}{4}-\frac{1}{4}=\frac{1}{2}>0$.

Suppose $d(v_2)+d(v_3)\geq\Delta+6$, then v is a strong 3-vertex. So $(d(v_2),d(v_3))\in\{(6,\Delta),(7,(\Delta-1)^+),(8,(\Delta-2)^+),(9,(\Delta-3)^+),(10,(\Delta-4)^+),(11^+,11^+)\}$. By R3, v_2 and v_3 send at least $\frac{3}{2}$ to v. By R1, $\omega'(v)\geq(\frac{3}{2}\times 3-5)-\frac{3}{4}+\frac{3}{2}-\frac{1}{4}=0$.

When v is not adjacent to a 3-vertex, and by R1 and R3, $\omega'(v)\geq(\frac{3}{2}\times 3-5)+\frac{1}{4}\times 3-\frac{1}{4}=0$. In particular, if v is a special 3-vertex, then by R1, R3 and R4, $\omega'(v)\geq(\frac{3}{2}\times 3-5)+\frac{1}{4}+\frac{1}{4}+1-\frac{1}{2}-\frac{1}{4}=\frac{1}{4}>0$.

Case 3: $d(v)=4$. From property 3(1), $n_2(v)\leq 2$.

Case 3.1: $n_2(v)=2$.

From property 3(1), $d(v_3)+d(v_4)\geq\Delta+6$, so $(d(v_3), d(v_4))\in\{(6,\Delta),(7,(\Delta-1)^+),(8,(\Delta-2)^+),(9,(\Delta-3)^+),(10,(\Delta-4)^+),(11^+,11^+)\}$. By R4, v_3 and v_4 send at least $\frac{3}{2}$ to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 4-5)-1\times 2+\frac{3}{2}-\frac{1}{2}=0$.

Case 3.2: $n_2(v)=1$.

From property 3(3), $d(v_2)+d(v_3)+d(v_4)\geq\Delta+7$. So $(d(v_2),d(v_3),d(v_4))\in\{(3,4,\Delta),(3,5,(\Delta-1)^+),(3,6,(\Delta-2)^+)$, $(3,7,(\Delta-3)^+)$, $(3,8,(\Delta-4)^+)$, $(3,9,(\Delta-5)^+)$, $(3,10,(\Delta-6)^+)$, $(3,11^+,11^+),(4^+,4^+,10^+)\}$. By R3 and R4, v_2, v_3 and v_4 send at least $\frac{3}{4}$ to v. By R1-R3, $\omega'(v)\geq(\frac{3}{2}\times 4-5)-1+\frac{3}{4}-\frac{1}{2}=\frac{1}{4}>0$.

Case 3.3: $n_2(v)=0$.

Case 3.3.1: $n_3(v)=4$.

From property 4(1), there exists a vertex v_i, $d(v_{i1})+d(v_{i2})\geq\Delta+5$ for some $i\in\{1,2,3,4\}$ where v_{i1} and v_{i2} are v_i's neighbors except v. So v_i is a special 3-vertex, by R1, R3 and R4, $\omega'(v)=(\frac{3}{2}\times 4-5)-\frac{1}{4}\times 4+\frac{1}{2}-\frac{1}{2}=0$.

Case 3.3.2: $n_3(v)=3$.

If $d(v_4)\leq\Delta-2$, then from property 4(2), there exists a vertex v_i such that $d(v_{i1})+d(v_{i2})\geq\Delta+5$ for some $i\in\{1,2,3\}$ where v_{i1} and v_{i2} are v_i's neighbors except v. So v_i is a special 3-vertex, by R1, R3 and R4, $\omega'(v)=(\frac{3}{2}\times 4-5)-\frac{1}{4}\times 3+\frac{1}{2}-\frac{1}{2}=\frac{1}{4}>0$. Suppose $d(v_4)\geq\Delta-1$, then by R1, R3 and R4, $\omega'(v)=(\frac{3}{2}\times 4-5)-\frac{1}{4}\times 3+1-\frac{1}{2}=\frac{3}{4}>0$.

Case 3.3.3: $n_3(v)\leq 2$.

By R1 and R3, $\omega'(v)\geq(\frac{3}{2}\times 4-5)-\frac{1}{4}\times 2-\frac{1}{2}=0$.

Case 4: $d(v)=5$. From property 3(1), $n_2(v)\leq 3$.

Case 4.1: $n_2(v)=3$.

From property 3(1), $d(v_4)+d(v_5)\geq\Delta+5$, so $(d(v_4),d(v_5))\in\{(5,\Delta),(6,(\Delta-1)^+),(7^+,7^+)\}$. By R5, v_4 and v_5 send at least 1 to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 5-5)-1\times 3+1-\frac{1}{2}=0$.

Case 4.2: $n_2(v)=2$.

From property 3(4), $d(v_3)+d(v_4)+d(v_5)\geq\Delta+6$. So $(d(v_3), d(v_4), d(v_5))\in\{(3,3,\Delta)$, $(3,4,(\Delta-1)^+)$, $(3,5^+,7^+)$, $(4,4,(\Delta-2)^+),(4,5^+,7^+),(5^+,5^+,7^+)\}$. By R4 and R5, v_3, v_4 and v_5 send at least $\frac{1}{4}$ to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 5-5)-1\times 2+\frac{1}{4}-\frac{1}{2}=\frac{1}{4}>0$.

Case 4.3: $n_2(v)\leq 1$.

Then by R1, R2 and R3, $\omega'(v)\geq(\frac{3}{2}\times 5-5)-1-\frac{1}{4}\times 4-\frac{1}{2}=0$.

Case 5: $d(v)=6$. From property 3(1), $n_2(v)\leq 4$.

Case 5.1: $n_2(v)=4$.

From property 3(1), $d(v_5)+d(v_6)\geq\Delta+4$, so $(d(v_5), d(v_6))\in\{(4,\Delta),(5,(\Delta-1)^+),(6^+,7^+)\}$, by R4, R5 and R6, v_5 and v_6 send at least $\frac{1}{2}$ to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 6-5)-1\times 4+\frac{1}{2}-\frac{1}{2}=0$.

Case 5.2: $n_2(v)=3$.

From property 3(5), $d(v_4)+d(v_5)+d(v_6)\geq\Delta+5$. So $(d(v_4),d(v_5),d(v_6))\in\{(3,3,(\Delta-1)^+)$, $(3,4,(\Delta-2)^+)$, $(3,5,(\Delta-3)^+)$, $(3,6^+,7^+)$, $(4,4,(\Delta-3)^+)$, $(4,5^+,7^+)$, $(5^+,5^+,7^+)$ $\}$. By R3-R6, v_4, v_5 and v_6 send at least $-\frac{1}{2}$ to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 6-5)-1\times 3-\frac{1}{2}-\frac{1}{2}=0$.

Case 5.3: $n_2(v)=2$.

From property 3(5), $d(v_3)+d(v_4)+d(v_5)+d(v_6)\geq\Delta+6$. So ($d(v_3)$, $d(v_4)$, $d(v_5)$, $d(v_6)$) $\in\{$ $(3,3,3,(\Delta-3)^+)$, $(3,3,4,(\Delta-4)^+)$, $(3,3,5,(\Delta-5)^+)$, $(3,3,6^+,7^+)$, $(3,4^+,4^+,7^+)$, $(4^+,4^+,4^+,7^+)$ $\}$. By R3-R6, v_3, v_4, v_5 and v_6 send at least -1 to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 6-5)-1\times 2-1-\frac{1}{2}=\frac{1}{2}>0$.

Case 5.4: $n_2(v)\leq 1$.

Then by R1, R2, R3, R4 and R5, $\omega'(v)\geq(\frac{3}{2}\times 6-5)-1-\frac{1}{2}\times 5-\frac{1}{2}=0$.

Case 6: $d(v)=7$. From property 3(1), $n_2(v)\leq 5$.

Case 6.1: $n_2(v)=5$.

From property 3(1), $d(v_6)+d(v_7)\geq\Delta+3$, so $(d(v_6), d(v_7))\in\{(3,\Delta),(4,(\Delta-1)^+),(5,(\Delta-2)^+),(6,(\Delta-3)^+),(7^+,7^+)\}$, by R3-R7, v_6 and v_7 send at least 0 to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 7-5)-1\times 5-\frac{1}{2}=0$.

Case 6.2: $n_2(v)=4$.

From property 3(6), $d(v_5)+d(v_6)+d(v_7)\geq\Delta+4$. So $(d(v_5),d(v_6),d(v_7))\in\{$ $(3,3,(\Delta-2)^+)$, $(3,4,(\Delta-3)^+)$, $(3,5,(\Delta-4)^+)$, $(3,6,(\Delta-5)^+)$, $(3,7^+,7^+)$, $(4^+,4^+,7^+)$ $\}$, by R3-R7, v_5, v_6 and v_7 send at least -1 to v. By R1 and R2, $\omega'(v)\geq(\frac{3}{2}\times 7-5)-1\times 4-1-\frac{1}{2}=0$.

Case 6.3: $n_2(v)=3$.

From property 3(6), $d(v_4)+d(v_5)+d(v_6)+d(v_7)\geq\Delta+5$. So ($d(v_4)$, $d(v_5)$, $d(v_6)$, $d(v_7)$) $\in\{$ $(3,3,3,(\Delta-4)^+)$, $(3,3,4^+,(\Delta-5)^+)$, $(3,3,5^+,(\Delta-6)^+)$, $(3,3,6^+,(\Delta-7)^+)$, $(3,3,7^+,7^+)$, $(3,4^+,4^+,7^+)$, $(4^+,4^+,4^+,7^+)$ $\}$, by

R3-R7, v_4, v_5, v_6 and v_7 send at least $-\frac{7}{4}$ to v. By R1 and R2, $\omega^{'}(v) \geq (\frac{3}{2} \times 7 - 5) - 1 \times 3 - \frac{7}{4} - \frac{1}{2} = \frac{1}{4} > 0$.

Case 6.4: $n_2(v) = 2$.

From property 3(6), $d(v_3)+d(v_4)+d(v_5)+d(v_6)+d(v_7) \geq \Delta + 6$. So ($d(v_3)$, $d(v_4)$, $d(v_5)$, $d(v_6)$, $d(v_7)$)$\in${(3, 3, 3, 3, $(\Delta-6)^+$), (3, 3, 3, 4^+, $(\Delta-7)^+$), (3, 3, 3, 5^+, 10^+), (3, 3, 3, 6^+, 9^+), (3, 3, 3, 7^+, 7^+), (3, 3, 4^+, 4^+, 7^+), (3, 4^+, 4^+, 4^+, 7^+), (4^+, 4^+, 4^+, 7^+)}, by R3-R7, v_3, v_4, v_5, v_6 and v_7 send at least $-\frac{5}{2}$ to v. By R1 and R2, $\omega^{'}(v) \geq (\frac{3}{2} \times 7 - 5) - 1 \times 2 - \frac{5}{2} - \frac{1}{2} = \frac{1}{2} > 0$.

Case 6.5: $n_2(v) = 1$. From property 3(6), $d(v_2)+d(v_3)+d(v_4)+d(v_5)+d(v_6)+d(v_7) \geq \Delta + 7$.

Case 6.5.1: $n_3(v) = 5$.

So $d(v_7) \geq \Delta - 8$. By R1-R3, $\omega^{'}(v) = (\frac{3}{2} \times 7 - 5) - 1 - \frac{3}{4} \times 5 - \frac{1}{2} = \frac{5}{4} > 0$.

Case 6.5.2: $n_3(v) \leq 4$.

Then by R1-R7, $\omega^{'}(v) \geq (\frac{3}{2}\times 7-5)-1-\frac{3}{4}\times 4-\frac{1}{2}\times 2-\frac{1}{2} = 0$.

Case 6.6: $n_2(v) = 0$.

Since v is adjacent to six 3(1)-vertices, v transfer most weight. From property 5, $d(v_7) \geq \Delta - 10$. By R1 and R4-R7, $\omega^{'}(v) \geq (\frac{3}{2} \times 7 - 5) - \frac{3}{4} \times 6 - \frac{1}{2} = \frac{1}{2} > 0$.

Case 7: $d(v) = 8$. From property 3(7), $n_2(v) \leq 8$.

Case 7.1: $n_2(v) = 8$.

From property 3(7), $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 8$. By R1, R2 and R8, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 8 + \frac{3}{16} \times 8 - \frac{1}{2} = 0$.

Case 7.2: $n_2(v) = 7$.

If $d(v_8) \leq \Delta - 7$. From property 3(8), $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 7$. By R1-R8, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 7 - \frac{3}{4} + \frac{3}{16} \times 7 - \frac{1}{2} = \frac{1}{16} > 0$.

Suppose $d(v_8) \geq \Delta - 6$; by R1, R2 and R8, $\omega^{'}(v) = (\frac{3}{2} \times 8 - 5) - 1 \times 7 + \frac{1}{2} - \frac{1}{2} = 0$.

Case 7.3: $n_2(v) = 6$.

If $d(v_7) + d(v_8) \leq \Delta + 1$, from property 3(9) $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 6$. By R1-R8, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 6 - \frac{3}{4} \times 2 + \frac{3}{16} \times 6 - \frac{1}{2} = \frac{1}{8} > 0$.

Suppose $d(v_7) + d(v_8) \geq \Delta + 2$, then So $(d(v_7), d(v_8)) \in \{(3, (\Delta-1)^+), (4, (\Delta-2)^+), (5, (\Delta-3)^+), (6, (\Delta-4)^+), (7, (\Delta-5)^+), (8^+, 8^+)\}$, by R3-R8, v_7 and v_8 send at least $-\frac{1}{2}$ to v. By R1 and R2, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 6 - \frac{1}{2} - \frac{1}{2} = 0$.

Case 7.4: $n_2(v) = 5$.

If $d(v_6) + d(v_7) + d(v_8) \leq \Delta + 2$, from property 3(9), $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 5$. By R1-R8, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 5 - \frac{3}{4} \times 3 + \frac{3}{16} \times 5 - \frac{1}{2} = \frac{3}{16} > 0$.

Suppose $d(v_6) + d(v_7) + d(v_8) \geq \Delta + 3$, then ($d(v_6)$, $d(v_7), d(v_8)) \in \{(3, 3, (\Delta-3)^+), (3, 4^+, 8^+), (4^+, 4^+, 8^+)\}$, by R3-R8, v_6, v_7 and v_8 send at least $-\frac{5}{4}$ to v. By R1 and R2, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 5 - \frac{5}{4} - \frac{1}{2} = \frac{1}{4} > 0$.

Case 7.5: $n_2(v) = 4$.

If $d(v_5) + d(v_6) + d(v_7) + d(v_8) \leq \Delta + 3$, from property 3(9), $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 4$. By R1-R8, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 4 - \frac{3}{4} \times 4 + \frac{3}{16} \times 4 - \frac{1}{2} = \frac{1}{4} > 0$.

Suppose $d(v_5) + d(v_6) + d(v_7) + d(v_8) \geq \Delta + 4$, then v transfer most weight when $n_3(v) = 3$. By R1, R2 and R3, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 4 - \frac{3}{4} \times 3 - \frac{1}{2} = \frac{1}{4} > 0$.

Case 7.6: $n_2(v) = 3$.

If $d(v_4) + d(v_5) + d(v_6) + d(v_7) + d(v_8) \leq \Delta + 4$, from property 3(9), $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 3$. By R1-R8, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 3 - \frac{3}{4} \times 5 + \frac{3}{16} \times 3 - \frac{1}{2} = \frac{5}{16} > 0$.

Suppose $d(v_4) + d(v_5) + d(v_6) + d(v_7) + d(v_8) \geq \Delta + 5$, then v transfer most weight when $n_3(v) = 4$; by R1-R3, $\omega^{'}(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 3 - \frac{3}{4} \times 4 - \frac{1}{2} = \frac{1}{2} > 0$.

Case 7.7: $n_2(v) \leq 2$.

Then by R1-R8, $\omega^{'}(v) \geq (\frac{3}{2}\times 8-5)-1\times 2-\frac{3}{4}\times 6-\frac{1}{2} = 0$.

Case 8: $d(v) = 9$. From property 3(10), $n_2(v) \leq 9$.

Case 8.1: $n_2(v) = 9$.

From property 3(10), v is weak adjacent to nine Δ-vertices. By R1, R2 and R9, $\omega^{'}(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 9 + \frac{1}{9} \times 9 - \frac{1}{2} = 0$.

Case 8.2: $n_2(v) = 8$.

If $d(v_9) \leq \Delta - 9$, from property 3(11) v is weak adjacent to eight Δ-vertices. By R1-R9, $\omega^{'}(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 8 - \frac{3}{4} + \frac{1}{9} \times 8 - \frac{1}{2} = \frac{5}{36} > 0$.

Suppose $d(v_9) \geq \Delta - 8$, by R1 and R2, $\omega^{'}(v) = (\frac{3}{2} \times 9 - 5) - 1 \times 8 - \frac{1}{2} = 0$.

Case 8.3: $n_2(v) = 7$.

If $d(v_8) + d(v_9) \leq \Delta - 7$, from property 3(12), $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 7$. By R1-R9, $\omega^{'}(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 7 - \frac{3}{4} \times 2 + \frac{1}{9} \times 7 - \frac{1}{2} = \frac{17}{36} > 0$.

Suppose $d(v_8) + d(v_9) \geq \Delta - 6$, then $(d(v_8), d(v_9)) \in \{(3, (\Delta-9)^+), (4^+, 9^+)\}$, by R3-R8, v_8 and v_9 send at least $-\frac{3}{4}$ to v. By R1 and R2, $\omega^{'}(v) \geq (\frac{3}{2}\times 9-5)-1\times 7-\frac{3}{4}-\frac{1}{2} = \frac{1}{4} > 0$.

Case 8.4: $n_2(v) = 6$.

If $d(v_7) + d(v_8) + d(v_9) \leq \Delta - 6$, from property 3(12), $d(v_i^{'}) = \Delta$ for all $1 \leq i \leq 6$. By R1-R9, $\omega^{'}(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 6 - \frac{3}{4} \times 3 + \frac{1}{9} \times 6 - \frac{1}{2} = \frac{5}{12} > 0$.

Suppose $d(v_7)+d(v_8)+d(v_9) \geq \Delta - 5$, then ($d(v_7)$, $d(v_8)$, $d(v_9)$)$\in${(3, 3, $(\Delta-11)^+$), $(3, 4^+, 9^+), (4^+, 4^+, 9^+)$}, by R3-R8, v_7, v_8 and v_9 send at least $-\frac{3}{2}$ to v. By R1 and R2, $\omega^{'}(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 6 - \frac{3}{2} - \frac{1}{2} = \frac{1}{2} > 0$.

Case 8.5: $n_2(v) \leq 5$.

Then by R1-R8, $\omega^{'}(v) \geq (\frac{3}{2}\times 9-5)-1\times 5-\frac{3}{4}\times 4-\frac{1}{2} = 0$.

Case 9: $d(v) = 10$. From property 3(13), $n_2(v) \leq 10$.

Case 9.1: $n_2(v) = 10$.

From property 3(13), v is weak adjacent to ten $(\Delta-1)^+$-vertices or nine Δ-vertices. By R1, R2 and R10, $\omega^{'}(v) \geq (\frac{3}{2} \times 10 - 5) - 1 \times 10 + \frac{1}{16} \times 9 - \frac{1}{2} = \frac{1}{16} > 0$.

Case 9.2: $n_2(v) = 9$.

If $d(v_{10}) \leq \Delta - 11$, from property 3(14), v is weak adjacent to nine $(\Delta-1)^+$-vertices or eight Δ-vertices. By R1, R2, R3 and R10, $\omega^{'}(v) \geq (\frac{3}{2}\times 10-5)-1\times 9-\frac{3}{4}+\frac{1}{16}\times 8-\frac{1}{2} = \frac{1}{4} > 0$.

Suppose $d(v_{10}) \geq \Delta - 10$, by R3, R4, R8 and R9, $\omega^{'}(v) = (\frac{3}{2} \times 10 - 5) - 1 \times 9 - \frac{1}{3} - \frac{1}{2} = \frac{1}{4} > 0$.

Case 9.3: $n_2(v) \leq 8$.

Then by R1-R8, $\omega^{'}(v) \geq (\frac{3}{2}\times 10-5)-1\times 8-\frac{3}{4}\times 2-\frac{1}{2} = 0$.

Case 10: $11 \leq d(v) \leq (\Delta - 2)$.

By R2-R10, v send at most 1 to adjacent 8^--vertices. Then by R1, $\omega^{'}(v) = (\frac{3}{2}d(v) - 5) - d(v) - \frac{1}{2} = \frac{1}{2}d(v) - \frac{11}{2} \geq 0$.

Case 11: $d(v) = (\Delta - 1) \geq 17$.

By R2-R10, v send at most 1 to adjacent 10^--vertices and $\frac{1}{16}$ to each weak 10-vertex. Then by R1, $\omega^{'}(v) = (\frac{3}{2}d(v) - 5) - d(v) - \frac{1}{16}d(v) - \frac{1}{2} = \frac{7}{16}d(v) - \frac{11}{2} \geq \frac{119}{16} - \frac{11}{2} = \frac{31}{16} > 0$.

Case 12: $d(v) = \Delta \geq 18$.

By R2-R10, v send at most 1 to adjacent 10^--vertices and $\frac{3}{16}$ to each weak 8^+-vertex. Then by R1, $\omega^{'}(v) = (\frac{3}{2}d(v) - 5) - d(v) - \frac{3}{16}d(v) - \frac{1}{2} = \frac{5}{16}d(v) - \frac{11}{2} \geq \frac{45}{8} - \frac{11}{2} = \frac{1}{8} > 0$.

This completes the proof of Theorem 1.

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