Dynamics of a Stochastic Consumer-Resource Model with Time-Dependent Delays and Harvesting Terms

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Abstract—**In this paper, we consider a stochastic consumer-resource model with time-dependent delays and harvesting terms. We first establish some sufficient conditions for the existence of the global positive solutions, and then prove stochastic ultimate boundedness and asymptotic path estimation via Lyapunov functions. Also, sufficient criteria for the extinction of the stochastic consumer-resource model are obtained. Finally, an example is given to illustrate the feasibility of our results.**

Index Terms—**Consumer-resource model, Stochastically ultimate boundedness, Asymptotic pathwise estimation, Extinction, Time delays, Harvesting terms.**

I. Introduction

In order to better study the relationships between populations, Holland and DeAngelis predicted the dynamic transitions between the results of interspecific interactions using the consumer-resource (C-R) theory. As the parameter values of the consumer-resource interactions vary, all possible outcomes of species coexistence emerged, such as predation, parasitism, competition, mutualism, commensalism. Therefore, consumer-resource interaction attracted much attention, see [1]–[7] and the reference therein. In [2], the authors introduced the following population dynamics of bi-directional C-R interactions:

$$\begin{cases} z_1'(t) = z_1(t)[r_1 - a_1 z_1(t) + b_1 f_1(z_1, z_2) - c_1 g_1(z_1, z_2)], \\ z_2'(t) = z_2(t)[r_2 - a_2 z_2(t) + b_2 f_2(z_1, z_2) - c_2 g_2(z_1, z_2)], \end{cases}$$

where u_i and r_i represent the number density and the population growth rate of species i, respectively; $b_i f_i(u_i, u_j)$ and $c_i g_i(u_i, u_j)$ represent the bi-directional C-R interaction, $i, j = 1, 2$.

Time delay is involved in more reasonable and meaningful models of population interactions, which is the source of instability in dynamic systems. It may cause population fluctuations, see [7]–[12] and the reference therein. In [7], the authors derived the following almost periodic C-R model with time delays:

$$\begin{cases} z_1'(t) = z_1(t)[r_1(t) - a_1(t)z_1(t) - b_1(t)z_2(t) \\ \qquad\qquad + c_1(t)z_2(t - \vartheta_1(t))], \\ z_2'(t) = z_2(t)[r_2(t) - a_2(t)z_2(t) - b_2(t)z_1(t) \\ \qquad\qquad + c_2(t)z_1(t - \vartheta_2(t))], \end{cases}$$

where $c_i(t)z_j(t - \vartheta_i(t))$ represents a positive feedback term, which is due to gestation; the delay $\vartheta_i(t)$ represents a gestation or maturation period, $i \neq j$, $i, j = 1, 2$.

Also, most population systems are inevitably influenced by environmental noise. In [13], the authors have revealed that all factors relating to the system, including the birth rate, carrying capacity, competition coefficients and other parameters, exhibit random fluctuation more or less. Therefore, deterministic systems with random factors have been widely studied with many achievements made, see [14]–[22] and the reference therein.

Motivated by the above, under the famous catch-per-unit-effort harvesting hypothesis [23]–[25], we are concerned with the following stochastic C-R system with time-dependent delays and harvesting terms:

$$\begin{cases} dz_1(t) = z_1(t)\left[\dfrac{r_1(t)}{z_1(t-\tau_1(t)) + m_1(t)} - a_1(t)z_1(t) \right. \\ \qquad \left. - b_1(t)z_2(t) + c_1(t)z_2(t - \vartheta_1(t)) - h_1(t)\right]dt \\ \qquad + \delta_1(t)z_1(t)dB_1(t), \\ dz_2(t) = z_2(t)\left[\dfrac{r_2(t)}{z_2(t-\tau_2(t)) + m_2(t)} - a_2(t)z_2(t) \right. \\ \qquad \left. - b_2(t)z_1(t) + c_2(t)z_1(t - \vartheta_2(t)) - h_2(t)\right]dt \\ \qquad + \delta_2(t)z_2(t)dB_2(t), \end{cases} \tag{1}$$

where $r_i(t)$, $a_i(t)$, $b_i(t)$, $\tau_i(t)$, $m_i(t)$, $c_i(t)$, $h_i(t)$, $\vartheta_i(t)$, $\delta_i(t)$ are all positive, bounded and continuous functions on $[0, +\infty)$ and $m_i(t) \geq 1$; $B_i(t)$ represents independent Brownian motion; $\sigma_i(t)$ denotes the intensities of the white noises, $i = 1, 2$. Although there exist many paper to investigate the stochastic population model, very little work has been done on the stochastic system with time-dependent delays and harvesting terms. Therefore, it is very meaningful to study the stochastic C-R model (1).

For convenience, we denote

$$g^U = \sup_{t \in [0, +\infty)} g(t), \quad g^L = \inf_{t \in [0, +\infty)} g(t).$$

II. Global positive solutions

To study the dynamical behavior of the stochastic C-R system (1), we will give some conditions under which system (1) has a unique global positive solution.

Denote $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space, and $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration that satisfies the usual conditions. Let

$$z(t) = (z_1(t), z_2(t)) \in R_+^2 = \{z \in R^2 | z_i > 0, \ i = 1, 2\},$$

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and $C([-\hat{\vartheta},0];R_+^2)$ is the family of continuous functions $\phi:[-\hat{\vartheta},0]\to R_+^2$ with the norm $\|\phi\|_1=\max_{1\le i\le 2}\|\phi_i\|$, $\|\phi_i\|=\sup_{-\hat{\vartheta}\le\theta\le 0}|\phi_i(\theta)|$, where

$$\hat{\vartheta}=\max\{\vartheta,\tau\},\ \vartheta=\lim_{1\le i\le 2}\big\{\sup_{t\in R}\vartheta_i(t)\big\},$$
$$\tau=\lim_{1\le i\le 2}\big\{\sup_{t\in R}\tau_i(t)\big\},\ \vartheta'=\max_{1\le i\le 2}\big\{\sup_{t\in R}\vartheta_i'(t)\big\}<1.$$

The initial data for system (1) is

$$\phi(\theta)=(\phi_1(\theta),\phi_2(\theta)),\ -\hat{\vartheta}\le\theta\le 0. \tag{2}$$

Theorem 2.1: System (1) has a unique positive global solution $z(t)=(z_1(t),z_2(t))$ for $t\ge-\hat{\vartheta}$ with any given initial value (2), and the solution will remain in R_+^2 a.s. (almost surely).

Proof: According to the assume, we know that the coefficients of the stochastic system (1) are locally continuous. For any initial value $\phi(t)=(\phi_1(t),\phi_2(t))\in C([-\hat{\vartheta},0];R_+^2)$, there exists a unique local solution $z(t)$ for $t\in[-\hat{\vartheta},\pi_e)$, where π_e is the explosion time. It is easy to see that the solution is global equivalent to $\pi_e=+\infty$ a.s. Let $\phi(t)\in C([-\hat{\vartheta},0];R_+^2)$ all lie within the interval $[\frac{1}{k_0},k_0]$ for $k_0>0$. Define

$$\pi_k=\inf\left\{t\in[0,\pi_e):z_1(t)\text{ or }z_2(t)\notin\left(\frac{1}{k},k\right)\right\},\ k\ge k_0,$$

where π_k is the stopping time.

Set $\inf\emptyset=\infty$, whence π_k is increasing as $k\to\infty$. Denote $\pi_\infty=\lim_{k\to\infty}\pi_k$, then $\pi_\infty\le\pi_e$ a.s. In order to prove $\pi_e=\infty$ a.s., it is only necessary to prove $\pi_\infty=\infty$ a.s. If $\pi_\infty\ne\infty$ a.s., there exists $T>0$ and $\epsilon\in(0,1)$ such that $P\{\pi_\infty\le T\}>\epsilon$. Therefore, there exists an integer $k_1\ge k_0$ such that $P\{\pi_k\le T\}\ge\epsilon$, where $k\ge k_1$.

Consider a function $V:R_+^2\to R_+$ by $V(z)=V_1(z)+V_2(z)$, where

$$V_1(z)=(z_1-1-\ln z_1)+(z_2-1-\ln z_2),$$
$$V_2(z)=0.5c_1^U\int_{t-\vartheta_1(t)}^t\frac{z_2^2(s)ds}{1-v'}+0.5c_2^U\int_{t-\vartheta_2(t)}^t\frac{z_1^2(s)ds}{1-v'}.$$

Obviously, $V(z)$ is a nonnegative function. By using the Itô's formula to $V(z)$, we have

$$\begin{aligned}
&dV_1(z)\\
=\ &\left(1-\frac{1}{z_1}\right)dz_1+\frac{1}{2z_1^2}(dz_1)^2\\
&+\left(1-\frac{1}{z_2}\right)dz_1+\frac{1}{2z_2^2}(dz_2)^2\\
=\ &(z_1(t)-1)\Bigg[\frac{r_1(t)}{z_1(t-\tau_1(t))+m_1(t)}-a_1(t)z_1(t)\\
&-b_1(t)z_2(t)+c_1(t)z_2(t-\vartheta_1(t))-h_1(t)\Bigg]dt\\
&+(z_2(t)-1)\Bigg[\frac{r_2(t)}{z_2(t-\tau_2(t))+m_2(t)}-a_2(t)z_2(t)\\
&-b_2(t)z_1(t)+c_2(t)z_1(t-\vartheta_2(t))-h_2(t)\Bigg]dt\\
&+0.5(\delta_1^2(t)+\delta_2^2(t))dt+(z_1(t)-1)\delta_1(t)dB_1(t)\\
&+(z_2(t)-1)\delta_2(t)dB_2(t)\\
=\ &LV(z_1,z_2)dt+(z_1(t)-1)\delta_1(t)dB_1(t)\\
&+(z_2(t)-1)\delta_2(t)dB_2(t),
\end{aligned}$$

where

$$\begin{aligned}
&LV_1(z)\\
=\ &(z_1(t)-1)\Bigg[\frac{r_1(t)}{z_1(t-\tau_1(t))+m_1(t)}-a_1(t)z_1(t)\\
&-b_1(t)z_2(t)+c_1(t)z_2(t-\vartheta_1(t))-h_1(t)\Bigg]\\
&+(z_2(t)-1)\Bigg[\frac{r_2(t)}{z_2(t-\tau_2(t))+m_2(t)}-a_2(t)z_2(t)\\
&-b_2(t)z_1(t)+c_2(t)z_1(t-\vartheta_2(t))-h_2(t)\Bigg]\\
&+0.5(\delta_1^2(t)+\delta_2^2(t))\\
\le\ &-a_1^Lz_1^2(t)+[r_1^U+a_1^U+b_2^U]z_1(t)+0.5c_1^Uz_1^2(t)\\
&+0.5c_1^Uz_2^2(t-\vartheta_1(t))-a_2^Lz_2^2(t)+[r_2^U+a_2^U\\
&+b_1^U]z_2(t)+0.5c_2^Uz_2^2(t)+0.5c_2^Uz_1^2(t-\vartheta_2(t))\\
&+0.5(\delta_1^U)^2+0.5(\delta_2^U)^2.
\end{aligned}$$

Compute

$$\begin{aligned}
dV_2(z)\ =\ &\frac{c_1^U}{2(1-\vartheta')}z_2^2(t)-\frac{c_1^U(1-\vartheta_1'(t))}{2(1-\vartheta')}z_2^2(t-\vartheta_1(t))\\
&+\frac{c_2^U}{2(1-\vartheta')}z_1^2(t)-\frac{c_2^U(1-\vartheta_2'(t))}{2(1-\vartheta')}z_1^2(t-\vartheta_2(t))\\
\le\ &\frac{c_1^U}{2(1-\vartheta')}z_2^2(t)-0.5c_1^Uz_2^2(t-\vartheta_1(t))\\
&+\frac{c_2^U}{2(1-\vartheta')}z_1^2(t)-0.5c_2^Uz_1^2(t-\vartheta_2(t)).
\end{aligned}$$

Then

$$\begin{aligned}
LV\ =\ &LV_1+LV_2\\
\le\ &\Big(-a_1^L+0.5c_1^U+\frac{c_2^U}{2(1-\vartheta')}\Big)z_1^2(t)\\
&+[r_1^U+a_1^U+b_2^U]z_1(t)+\Big(-a_2^L+c_2^U\\
&+\frac{c_1^U}{2(1-\vartheta')}\Big)z_2^2(t)+[r_2^U+a_2^U+b_1^U]z_2(t)\\
&+0.5(\delta_1^U)^2+0.5(\delta_2^U)^2\\
\le\ &A,
\end{aligned}$$

where A is the upper bound of LV. Therefore, we have

$$\begin{aligned}
dV(z)\ \le\ &Kdt+(z_1(t)-1)\delta_1(t)dB_1(t)\\
&+(z_2(t)-1)\delta_2(t)dB_2(t).
\end{aligned}$$

A conclusion can be obtained by using the same proof method as [19], so it is omitted. The proof is completed. ■

III. Stochastically ultimate boundedness

Lemma 3.1: Let $\gamma\in(0,1)$, then there exists a constant $N=N(\gamma)>0$, which is independent of $(\phi_1(\theta),\phi_2(\theta))\in C([-\hat{\vartheta},0],R_+^2)$, such that the solution $z=(z_1,z_2)$ of the stochastic model (1) has

$$\limsup_{t\to+\infty}E|z|^\gamma\le N.$$

Proof: To define $V(z) = z_1^\gamma + z_2^\gamma$, applying the Itô formula to the system (1), we have

$$\begin{aligned} dV(z) &= LV(z_1,z_2)dt + \delta_1(t)\gamma z_1^\gamma(t)dB_1(t) \\ &\quad +\delta_2(t)\gamma z_2^\gamma(t)dB_2(t), \end{aligned}$$

where

$$\begin{aligned} & LV(z_1,z_2) \\ = \ & \gamma z_1^\gamma(t)\Big[\frac{r_1(t)}{z_1(t-\tau_1(t))+m_1(t)} - a_1(t)z_1(t) \\ & -b_1(t)z_2(t) + c_1(t)z_2(t-\vartheta_1(t)) - h_1(t)\Big] \\ & +\gamma z_2^\gamma(t)\Big[\frac{r_2(t)}{z_2(t-\tau_2(t))+m_2(t)} - a_2(t)z_2(t) \\ & -b_2(t)z_1(t) + c_2(t)z_1(t-\vartheta_2(t)) - h_2(t)\Big] \\ & +0.5\gamma(\gamma-1)(z_1^\gamma(t)\delta_1^2(t) + z_2^\gamma(t)\delta_2^2(t)) \\ \le \ & \gamma r_1^U z_1^\gamma(t) + \gamma c_1^U z_1^\gamma(t) z_2(t-\vartheta_1(t)) + \gamma r_2^U z_2^\gamma(t) \\ & +\gamma c_2^U z_2^\gamma(t) z_1(t-\vartheta_2(t)) \\ \le \ & \gamma r_1^U z_1^\gamma(t) + 0.25(\gamma c_1^U)^2 z_1^{2\gamma}(t) + z_2^2(t-\vartheta_1(t)) \\ & +\gamma r_2^U z_2^\gamma(t) + 0.25(\gamma c_2^U)^2 z_2^{2\gamma}(t) + z_1^2(t-\vartheta_2(t)) \\ \le \ & G(z) - V(z) + z_2^2(t-\vartheta_1(t)) + z_1^2(t-\vartheta_2(t)) \\ & -\frac{e^\vartheta}{1-\vartheta'}(z_1^2(t)+z_2^2(t)), \end{aligned}$$

$$\begin{aligned} & G(z) \\ = \ & \gamma r_1^U z_1^\gamma(t) + 0.25(\gamma c_1^U)^2 z_1^{2\gamma}(t) + \gamma r_2^U z_2^\gamma(t) \\ & +0.25(\gamma c_2^U)^2 z_2^{2\gamma}(t) + \frac{1-\vartheta'+e^\vartheta}{1-\vartheta'}(z_1^2(t)+z_2^2(t)). \end{aligned}$$

Since $\gamma \in (0,1)$, $G(z)$ is bounded in R_+^2, i.e. $G(z) \le \hat{M}$, where $\hat{M} \ge 0$ for any $z \in R_+^2$.

Hence, we have

$$\begin{aligned} & dV(z) \\ \le \ & \Big[\hat{M} - V(z) + z_2^2(t-\vartheta_1(t)) + z_1^2(t-\vartheta_2(t)) \\ & -\frac{e^v}{1-\vartheta'}(z_1^2(t)+z_2^2(t))\Big]dt \\ & +\delta_1^U\gamma z_1^\gamma(t)dB_1(t) + \delta_2^U\gamma z_2^\gamma(t)dB_2(t). \end{aligned} \tag{3}$$

By virtue of (3), we apply the Itô formula to $e^tV(z)$,

$$\begin{aligned} & d(e^tV(z)) \\ = \ & e^tV(z)dt + e^tdV(z) \\ \le \ & e^t\Big[\hat{M} + z_2^2(t-\vartheta_1(t)) + z_1^2(t-\vartheta_2(t)) \\ & -\frac{e^v}{1-\vartheta'}(z_1^2(t)+z_2^2(t))\Big]dt \\ & +\delta_1^U\gamma e^t z_1^\gamma(t)dB_1(t) + \delta_2^U\gamma e^t z_2^\gamma(t)dB_2(t), \end{aligned}$$

then

$$\begin{aligned} & e^tEV(z) \\ \le \ & V(z_1(0),z_2(0)) + e^t\hat{M} - E\Big[\frac{e^\vartheta}{1-\vartheta'}\int_0^t e^s(z_1^2(s) \\ & +z_2^2(s))ds\Big] + E\int_0^t e^s z_2^2(s-\vartheta_1(s))ds \\ & +E\int_0^t e^s z_1^2(s-\vartheta_2(s))ds\Big]dt \\ \le \ & V(z_1(0),z_2(0)) + e^t\hat{M} + \frac{e^\vartheta}{1-\vartheta'}\int_{-\vartheta_1(t)}^0 e^s z_2^2(s)ds \\ & +\frac{e^\vartheta}{1-\vartheta'}\int_{-\vartheta_2(t)}^0 e^s z_1^2(s)ds. \end{aligned}$$

We easily infer that $\limsup\limits_{t\to+\infty} EV(u) \le \hat{M}$.

On the other hand,

$$|z|^\gamma = (z_1^2+z_2^2)^{\frac{\gamma}{2}} \le 2^{\frac{\gamma}{2}}\max\{z_1^\gamma, z_2^\gamma\} \le 2^{\frac{\gamma}{2}}V(z).$$

Hence,

$$\limsup_{t\to+\infty} E|z|^\gamma \le 2^{\frac{\gamma}{2}}\hat{M} := N.$$

The proof is completed. ■

Theorem 3.1: Let $\gamma \in (0,1)$, then the system (1) is stochastically ultimate boundedness.

Proof: By using of the Lemma 3.1, we obtain $\limsup\limits_{t\to+\infty} E|z|^\gamma \le N$, where $N > 0$ is a constant. For any $\varepsilon > 0$, let $\eta = (N/\varepsilon)^2$. By Chebyshev's inequality, we have

$$P\{|z(t)| > \eta\} \le \sqrt{\frac{|z(t)|}{\eta}},$$

then

$$\limsup_{t\to+\infty} P\{|z(t)| > \eta\} \le \varepsilon.$$

The proof is completed. ■

IV. Asymptotic pathwise estimation

Theorem 4.1: For any $(\phi_1(\theta),\phi_2(\theta)) \in C([-\hat{\vartheta},0],R_+^2)$, the solution $z=(z_1,z_2)$ of system (1) has the properties

$$\limsup_{t\to+\infty}\frac{\ln z_1(t)}{\ln t} \le 1, \ \limsup_{t\to+\infty}\frac{\ln z_2(t)}{\ln t} \le 1.$$

Proof: Define $V(z) = \ln(z_1+z_2)$, by using the Itô formula to system (1), we have

$$\begin{aligned} & d(e^tV(z)) \\ = \ & e^tV(z)dt + e^tdV(z) \\ \le \ & e^t\ln(z_1+z_2)dt + e^td\ln(z_1+z_2) \\ \le \ & e^t\ln(z_1+z_2)dt + \frac{e^tz_1}{z_1+z_2}\Big[\frac{r_1(t)}{z_1(t-\tau_1(t))+m_1(t)} \\ & -a_1(t)z_1(t) - b_1(t)z_2(t) + c_1(t)z_2(t-\vartheta_1(t)) \\ & -h_1(t) - 0.5\frac{e^t\delta_1^2(t)z_1^2}{(z_1+z_2)^2}\Big]dt + \frac{e^t\delta_1(t)z_1}{z_1+z_2}dB_1(t) \\ & +\frac{e^tz_2}{z_1+z_2}\Big[\frac{r_2(t)}{z_2(t-\tau_2(t))+m_2(t)} - a_2(t)z_2(t) \\ & -b_2(t)z_1(t) + c_2(t)z_1(t-\vartheta_2(t)) - h_2(t) \\ & -0.5\frac{e^t\delta_2^2(t)z_2^2}{(z_1+z_2)^2}\Big]dt + \frac{e^t\delta_2(t)z_2}{z_1+z_2}dB_2(t), \end{aligned}$$

thus

$$\begin{aligned} & e^t\ln(z_1+z_2) \\ = \ & \ln(z_1(0)+z_2(0)) + \int_0^t e^s\ln(z_1(s)+z_2(s))ds \\ & +\int_0^t\frac{e^sz_1(s)}{z_1(s)+z_2(s)}\Big[\frac{r_1(s)}{z_1(s-\tau_1(s))+m_1(s)} \end{aligned}$$

$$-a_1(s)z_1(s) - b_1(s)z_2(s) + c_1(s)z_2(s-\vartheta_1(s))$$
$$-h_1(s) - 0.5\frac{z_1^2(s)e^s\delta_1^2(s)}{(z_1(s)+z_2(s))^2}\Big]ds + Q_1(t)$$
$$+\int_0^t \frac{e^s z_2(s)}{z_1(s)+z_2(s)}\Big[\frac{r_2(s)}{z_2(s-\tau_2(s))+m_2(s)}$$
$$-a_2(s)z_2(s) - b_2(s)z_1(s) + c_2(s)z_1(s-\vartheta_2(s))$$
$$-h_2(s) - 0.5\frac{z_2^2(s)e^s\delta_2^2(s)}{(z_1(s)+z_2(s))^2}\Big]ds + Q_2(t), \qquad (4)$$

where $Q_i(t) = \int_0^t \frac{e^s z_i(s)\delta_i(s)}{z_1(s)+z_2(s)}dB_i(t)$, $i = 1, 2$.

Let $Q(t) = Q_1(t)+Q_2(t)$, then $Q(t)$ is a local martingale, and

$$< Q(t), Q(t) > = 0.5\int_0^t \frac{z_1^2(s)e^{2s}\delta_1^2(s)}{(z_1(s)+z_2(s))^2}ds + 0.5\int_0^t \frac{z_2^2(s)e^{2s}\delta_2^2(s)}{(z_1(s)+z_2(s))^2}ds.$$

By employing the exponential martingale inequality and the Borel-Cantelli lemma [3], for any positive constants l, α and $\lambda > 1$, there exists a $l(\omega)$ for almost all $\omega \in \Omega$, such that for all $l \geq l(\omega)$, we have

$$H(t) \leq \frac{e^{-l\alpha}}{2} < H(t), H(t) > +\lambda e^{l\alpha}\ln l,\ 0 \leq t \leq l\alpha. \qquad (5)$$

By means of (5), we have

$$\begin{aligned} & e^t\ln(z_1+z_2) \\ \leq\ & \ln(z_1(0)+z_2(0)) \\ & + \int_0^t e^s\ln(z_1(s)+z_2(s))ds + \lambda e^{l\alpha}\ln l \\ & + \int_0^t \frac{e^s z_1(s)}{z_1(s)+z_2(s)}\Big[r_1^U + c_1^U z_2(s-\vartheta_1(s))\Big]ds \\ & + \int_0^t \frac{e^s z_2(s)}{z_1(s)+z_2(s)}\Big[r_2^U + c_2^U z_1(s-\vartheta_2(s))\Big]ds. \end{aligned} \qquad (6)$$

Since $0 \leq t \leq l\alpha$, $l \geq l(\omega)$, we know $s \leq l\alpha$. Note that

$$\begin{aligned} & \int_0^t e^s z_2(s-\vartheta_1(s))ds \\ \leq\ & \int_0^t \frac{e^{s-\vartheta_1(s)}e^{\vartheta_1(s)}z_2(s-\vartheta_1(s))}{1-\vartheta'}d(s-\vartheta_1(s)) \\ \leq\ & \frac{e^v}{1-\vartheta'}\int_{-\vartheta_1(0)}^{t-\vartheta_1(t)} e^s z_2(s)ds \\ \leq\ & \frac{e^v}{1-\vartheta'}\int_{-\vartheta_1(0)}^{0} e^s z_2(s)ds + \frac{e^v}{1-\vartheta'}\int_0^t e^s z_2(s)ds. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \int_0^t e^s z_1(s-\vartheta_2(s))ds \\ \leq\ & \frac{e^\vartheta}{1-\vartheta'}\int_{-\vartheta_2(0)}^{0} e^s z_1(s)ds + \frac{e^\vartheta}{1-\vartheta'}\int_0^t e^s z_1(s)ds. \end{aligned}$$

Then (6) can be rewritten

$$\begin{aligned} & e^t\ln(z_1+z_2) \\ \leq\ & \ln(z_1(0)+z_2(0)) + \frac{c_1^U e^\vartheta}{1-\vartheta'}\int_{-\vartheta_1(0)}^{0} e^s z_2(s)ds \\ & + \frac{c_2^U e^\vartheta}{1-\vartheta'}\int_{-\vartheta_2(0)}^{0} e^s z_1(s)ds + \int_0^t e^s\Big[\ln(z_1+z_2) \\ & + r_1^U + r_2^U + \frac{\max\{c_1^U, c_2^U\}e^\vartheta}{1-\vartheta'}(z_1+z_2)\Big]ds + \lambda e^{l\alpha}\ln l. \end{aligned}$$

Obviously, there exists a constant $K^* > 0$ such that

$$\ln(z_1+z_2) + r_1^U + r_2^U + \frac{\max\{c_1^U, c_2^U\}e^\vartheta}{1-\vartheta'}(z_1+z_2) \leq K^*.$$

Let

$$\hat{G} = \ln(z_1(0)+z_2(0)) + \frac{c_1^U e^\vartheta}{1-\vartheta'}\int_{-\vartheta_1(0)}^{0} e^s z_2(s)ds + \frac{c_2^U e^\vartheta}{1-\vartheta'}\int_{-\vartheta_2(0)}^{0} e^s z_1(s)ds,$$

then for all $0 \leq t \leq l\alpha$, $l \geq l(\omega)$, we have

$$e^t\ln(z_1+z_2) \leq \hat{G} + \int_0^t e^s K^* ds + \lambda e^{l\alpha}\ln l.$$

If $\alpha(l-1) \leq t \leq l\alpha$ and $l \geq l(\omega)$, then

$$\frac{\ln(z_1+z_2)}{\ln t} \leq \frac{\hat{G}}{e^t\ln t} + \frac{e^t K^* - K^*}{e^t\ln t} + \frac{\lambda e^{l\alpha}\ln l}{e^t\ln t}.$$

That is

$$\limsup_{t\to+\infty}\frac{\ln(z_1+z_2)}{\ln t} \leq \lambda e^\alpha. \qquad (7)$$

Letting $\lambda \to 1$, $\alpha \to 0$, by (7), we have

$$\limsup_{t\to+\infty}\frac{\ln z_1(t)}{\ln t} \leq 1,\ \limsup_{t\to+\infty}\frac{\ln z_2(t)}{\ln t} \leq 1,\ a.s.$$

The proof is completed. ■

V. Extinction

Since natural and anthropogenic factors can cause population declines or extinction, the study of extinction is of great significance for biological populations. In this section, we will mainly investigate the extinction of system (1).

Theorem 5.1: Assume that $r_i^U - h_i^L - 0.5(\delta_i^L)^2 < 0$, and $c_1(t) = 0$ or $c_2(t) = 0$, then for any given initial value satisfying $(\phi_1(\theta), \phi_2(\theta)) \in C([-\hat{\vartheta}, 0], R_+^2)$, the solution $z = (z_1, z_2)$ of system (1) will be extinct with probability one, that is

$$\limsup_{t\to\infty}\frac{\ln z_1(t)}{t} = 0,\ \limsup_{t\to\infty}\frac{\ln z_2(t)}{t} = 0.$$

Proof: If $c_1(t) = 0$, applying the Itô formula to system (1), we have

$$\begin{aligned} d\ln z_1(t) =\ & \Big[\frac{r_1(t)}{z_1(t-\tau_1(t))+m_1(t)} - a_1(t)z_1(t) \\ & -b_1(t)z_2(t) - h_1(t) - 0.5\delta_1^2(t)\Big]dt \\ & +\delta_1(t)dB_1(t), \\ d\ln z_2(t) =\ & \Big[\frac{r_2(t)}{z_2(t-\tau_2(t))+m_2(t)} - a_2(t)z_2(t) \\ & -b_2(t)z_1(t) + c_2(t)z_1(t-\vartheta_2(t)) - h_2(t) \\ & -0.5\delta_2^2(t)\Big]dt + \delta_2(t)dB_2(t), \end{aligned}$$

Integrating both sides from 0 to t, we get

$$\ln z_1(t) \leq \ln z_1(0) + \int_0^t \Big(r_1(s) - h_1(s) - 0.5\delta_1^2(s)\Big)ds$$

$$
\begin{aligned}
& +\int_0^t \delta_1(s)dB_1(s) \\
\leq\ & \ln z_1(0) + (r_1^U - h_1^L - 0.5(\delta_1^L)^2)t \\
& +\int_0^t \delta_1(s)dB_1(s), \\
\ln z_2(t) \leq\ & \ln z_2(0) + \int_0^t \Big(r_2(s) + c_2(s)z_1(s-\vartheta_2(s)) \\
& -h_2(s) - 0.5\delta_2^2(s)\Big] ds + \int_0^t \delta_2(s)dB_2(s) \\
\leq\ & \ln z_2(0) + (r_2^U - h_1^L - 0.5(\delta_2^L)^2)t \\
& +\int_0^t c_2(s)z_1(s-\vartheta_2(s))ds \\
& +\int_0^t \delta_2(s)dB_2(s),
\end{aligned}
$$

then

$$\limsup_{t\to\infty} \frac{\ln z_1(t)}{t} \leq r_1^U - h_1^L - 0.5(\delta_1^L)^2 < 0,$$

that is $\lim_{t\to\infty} z_1(t) = 0$. Therefore,

$$\limsup_{t\to\infty} \frac{\ln z_2(t)}{t} \leq r_2^U - h_1^L - 0.5(\delta_2^L)^2 < 0.$$

In the same way, if $c_2(t) = 0$, we have $\limsup_{t\to\infty} \frac{\ln z_2(t)}{t} < 0$, and obtain $\limsup_{t\to\infty} \frac{\ln z_1(t)}{t} < 0$ a.s. The proof is completed. ■

Remark 5.1: It can be easily seen from Theorem 5.1 that the stochastic perturbation and harvesting terms do affect the extinction of the model (1). Excessive noise and over-harvesting can lead to the extinction of entire species. Therefore, stochastic perturbation and harvesting terms cannot be ignored when considering population dynamics.

VI. An example

Considering the stochastic consumer-resource model (1), we choose the coefficients

$$
\begin{gathered}
r_1(t) = 0.6 - 0.01\sin 2t,\ r_2(t) = 0.7 + 0.02\sin 2t, \\
a_1(t) = 0.04 + 0.01\cos t,\ a_2(t) = 0.05 + 0.02\cos t, \\
b_1(t) = 0.04 + 0.02\cos 2t,\ b_2(t) = 0.06 + 0.01\cos 2t, \\
c_1(t) = 0.03 + 0.02\cos 2t,\ c_2(t) = 0.04 + 0.01\cos 2t. \\
\tau_1(t) = \tau_2(t) = \vartheta_1(t) = \vartheta_2(t) = 0.02(1+\sin t), \\
m_1(t) = m_2(t) = 2
\end{gathered}
$$

One can calculate that

$$
\begin{gathered}
r_1^U = 0.61, \quad r_2^U = 0.72, \quad a_1^U = 0.05, \quad a_2^U = 0.07, \\
c_1^U = 0.05, \quad c_2^L = 0.05, \quad \hat{\vartheta} = 0.04, \quad \vartheta' = 0.02.
\end{gathered}
$$

The initial data $\phi_1(\theta) = 0.5$, $\phi_2(\theta) = 0.5$, where $\theta \in [-0.04, 0]$.

If $\delta_1(t) = 0.56 + 0.02\sin 2t$, $\delta_2(t) = 0.56 + 0.02\cos 2t$, $h_1(t) = 0.55 + 0.01\sin 2t$ and $h_2(t) = 0.61 + 0.01\sin 2t$, then we have

$$
\begin{aligned}
r_1^U - h_1^L - 0.5(\delta_1^L)^2 = -0.08 < 0, \\
r_2^U - h_2^L - 0.5(\delta_2^U)^2 = -0.03 < 0.
\end{aligned}
$$

It is easy to see that the assumptions of Theorem 2.1-Theorem 5.1 hold. Therefore, the solution of system (1) is stochastic boundedness and extinction.

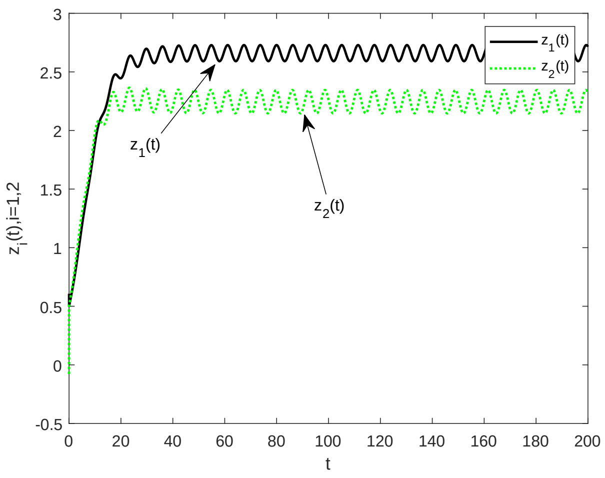

Fig. 1: Dynamic behavior of $z_i(t)$ with $h_i(t) = \delta_i(t) = 0$, and the initial condition $(\phi_1(0), \phi_2(0)) = (0.5, 0, 5)^T$, $i = 1, 2$.

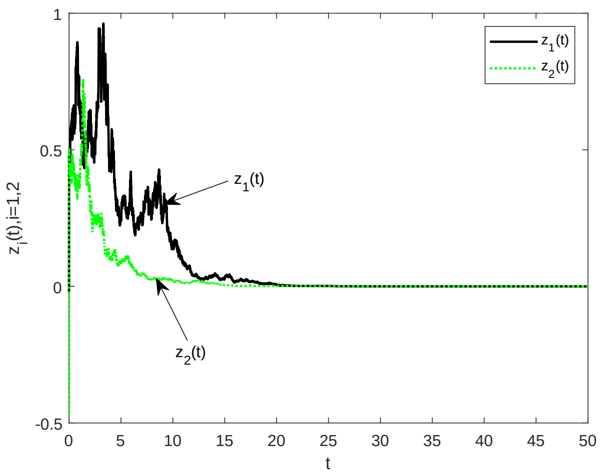

Fig. 2: Dynamic behavior of $z_i(t)$ with $h_1(t) = 0.55 + 0.01\sin 2t$, $h_2(t) = 0.61 + 0.01\sin 2t$, $\delta_1(t) = 0.56 + 0.02\sin 2t$, $\delta_2(t) = 0.56 + 0.02\cos 2t$, $c_1(t) = 0$, where the initial condition $(\phi_1(0), \phi_2(0)) = (0.5, 0, 5)^T$, $i = 1, 2$.

VII. Conclusions

In this paper, we obtain a unique global positive solution to the stochastic C-R model (1), which is stochastically ultimate boundedness. Then, the asymptotic properties of the sample paths and the extinction of the stochastic system are investigated. The results show that the harvesting terms, time delays and stochastic perturbation affect the population model's dynamic behavior, which is vital for population development.

Some valuable questions deserve further study. On the one hand, more realistic but complex models can be proposed. For example, the effect of the impulse factor on the system's dynamic behavior (1) can be taken into consideration, see [26] and [27]. On the other hand, the methods employed in this paper can be used to explore other models, such as Gilpin-Ayala model logistic model, epidemic models, and worm propagation model, see [28]–[31] and the reference therein. We will explore these possibilities in future studies.

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