Existence of Solutions for a System of Coupled Hybrid Fractional Integro-differential Equations

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Abstract—**In this paper, we consider the existence of solutions for a coupled system of hybrid fractional integro-differential equations. As an application of the hybrid fixed point theorem due to Dhage, we obtain the system has a coupled solution.**

Index Terms—**Caputo fractional derivative; coupled system; hybrid fixed point theorem; fractional integro-differential equations.**

I. Introduction

THE fractional differential equations are regarded as an important mathematical branch, because the behavior of many physical systems can be described by applying the fractional order system theory [1-2]. The area of differential equations devoted to quadratic perturbations of nonlinear differential equations (called hybrid differential equations) has attracted much attention to researchers and served as special cases of dynamical systems. Many authors established existence results for hybrid differential equations [3-9].

The authors in [4] investigated the existence and uniqueness of solutions to the first order ordinary hybrid differential equation

$$\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t)),\quad t\in J,$$

$$x(t_0)=x_0\in R.$$

In [10], Zhao et al. considered the existence of solutions to the hybrid differential equations involving Riemann-Liouville fractional differential operators

$$D^q_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t)),\quad t\in[0,T],$$

$$x(0)=0,$$

here $0<q<1$.

In [11], the authors discussed the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential equations

$${}^CD^\alpha\left[\frac{x(t)-\sum\limits_{i=1}^m I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right]=g(t,x(t)),\quad t\in[0,1],$$

$$x(0)=\mu(x),\quad x(1)=A,$$

here ${}^CD^\alpha$ is Caputo fractional derivative, $1<\alpha\le 2$.

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The authors in [12] derived an existence result for the fractional hybrid integro-differential equation

$${}^CD^\omega_0\left[\frac{k(t)-\varphi(t,k(t),I_0^{\gamma_1}k(t),I_0^{\gamma_2}k(t),\cdots,I_0^{\gamma_n}k(t))}{\phi(t,k(t),I_0^{\mu_1}k(t),I_0^{\mu_2}k(t),\cdots,I_0^{\mu_m}k(t))}\right]$$
$$=\gamma(t,k(t)),\quad t\in[0,1].$$

with mixed integral hybrid boundary value conditions.

On the other hand, coupled systems of fractional order differential equations are very meaningful subject of the mathematics and have various applications in disease models, ecological models, nonlocal thermoelasticity and synchronization of chaotic systems [13-16].

Bapurao C. Dhage et al. [17] studied the existence of coupled solutions to the following coupled hybrid integro differential equations of fractional order

$${}^CD^\omega\left(\frac{x(t)-\sum\limits_{i=1}^m I^{\beta_i}h_i(t,x(t),y(t))}{f(t,x(t),y(t)}\right)=\phi(t,x(t),y(t)),\quad t\in J=[0,1],$$

$${}^CD^\delta\left(\frac{y(t)-\sum\limits_{j=1}^n I^{\gamma_j}k_j(t,x(t),y(t))}{g(t,x(t),y(t)}\right)=\psi(t,x(t),y(t)),\quad t\in J=[0,1],$$

$$x(0)=a,\ x(1)=b,\ y(0)=c,\ y(1)=d,$$

here ${}^CD$ is the Caputo fractional derivative.

By mixing idea of the afore works, in this paper, we consider the following system of coupled hybrid fractional integro-differential equations

$${}^CD^\alpha\left(\frac{x(t)-\sum\limits_{i=1}^m I^{\beta_i}h_i(t,x(t),y(t),I^{\theta_1}x(t),I^{\delta_1}y(t))}{f(t,x(t),y(t)}\right)$$
$$=\phi(t,x(t),y(t)),\quad t\in[0,1],\tag{1}$$

$${}^CD^\beta\left(\frac{y(t)-\sum\limits_{j=1}^n I^{\gamma_j}k_j(t,x(t),y(t),I^{\theta_2}x(t),I^{\delta_2}y(t))}{g(t,x(t),y(t)}\right)$$
$$=\psi(t,x(t),y(t)),\quad t\in[0,1],\tag{2}$$

$$x(0)=\mu(x),\quad x(1)=A,\tag{3}$$

$$y(0)=\nu(y),\quad y(1)=B,\tag{4}$$

here ${}^CD$ denotes the Caputo fractional derivative, $\alpha,\beta\in(1,2]$, I^{β_i},I^{γ_j} are the Riemann-Liouville fractional integral of order $\beta_i>0,\gamma_j>0,i=1,2,\cdots,m,\ j=1,2,\cdots,n,\ f,g\in C([0,1]\times R\times R\to R\backslash\{0\}),\phi,\psi\in C([0,1]\times R\times R\to R),h_i,k_j\in C([0,1]\times R\times R\times R\times R\to R)$ for $i=1,2,\cdots,m,\ j=1,2,\cdots,n,\ \theta_1,\theta_2,\delta_1,\delta_2>0,\ \mu,\nu:C([0,1],R)\to R$ and $A,B\in R$.

In this paper, we use hybrid fixed point theorem to establish the sufficient conditions for the existence of solutions to the coupled system (1)-(4). Even though we use the standard tools, its exploration to the problem (1)-(4) is new. Our results have some reference significance to the existing literature on Caputo type nonlocal coupled system of fractional integro-differential equations.

II. The preliminary lemmas

Definition 2.1 [2] For a function $u(t)$, the Caputo derivative of fractional order $\alpha > 0$ is defined as

$$^{C}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds, n = [\alpha]+1,$$

provided that the integral on the right hand side exists.

Definition 2.2 [2] For a function $u(t)$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds, n = [\alpha]+1,$$

provided that the integral on the right hand side exists.

Definition 2.3 [2] For a function $u(t)$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided that the integral on the right hand side exists.

Remark 2.4 If $u(t) \in C^n[0,\infty)$, then

$$\begin{aligned} ^{C}D^{\alpha}u(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds \\ &= I^{n-\alpha}u^{(n)}(t), \quad n = [\alpha]+1. \end{aligned}$$

Lemma 2.5 [2] Let $x \in AC^n[0,1]$ and $\alpha > 0$, then

$$I^{\alpha}\, {^{C}D^{\alpha}}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in R, i = 0, 1, \cdots, n-1, n = [\alpha]+1$.

Let the Banach space $X = C[0,1]$ equipped with standard norm

$$||x|| = \sup_{0\le t\le 1}|x(t)|,$$

which is a Banach algebra with the multiplication "$\cdot$" defined by

$$(x\cdot y)(t) = x(t)\cdot y(t).$$

We consider the product space $E = X \times X$, then the normed linear space $(E, \|(\cdot,\cdot)\|)$ is a Banach space where the norm $\|(\cdot,\cdot)\|$ is given by

$$\|(x,y)\| = \|x\| + \|y\|. \tag{5}$$

Define the multiplication " $\cdot$ " by

$$((x,y)\cdot(u,v))(t) = (x,y)(t)\cdot(u,v)(t) = (x(t)u(t), y(t)v(t)). \tag{6}$$

Lemma 2.6 [17] The Banach space E is a Banach algebra with the norm $\|(\cdot,\cdot)\|$ and the multiplication " $\cdot$ " defined by (5) and (6) respectively.

Definition 2.7 [18] We call that $T: E \to E$ is μ-Lipschitz if there exists a constant μ satisfies

$$\|T(x,y) - T(\bar{x},\bar{y})\| \le \mu(\|x-\bar{x}\| + \|y-\bar{y}\|)$$

for all $(x,y), (\bar{x},\bar{y}) \in E$ with $\mu > 0$. Further if $\mu < 1$, then T is a strict contraction.

Theorem 2.8 [18] Let S be a closed convex bounded nonempty subset of a Banach algebra E, and let $A, C: E \to E$ and $B: S \to E$ be three operators such that
(a) A and C are Lipschitzian with Lipschitz constants δ and ρ;
(b) B is compact and continuous;
(c) $x = AxBy + Cx \quad \Rightarrow x \in S$ for all $y \in S$;
(d) $\delta M + \rho < 1$, where $M = \|B(S)\|$.
Then the operator equation $AxBy + Cx = x$ has a solution in S.

At this point, we give the following hypotheses.
(H_1) The functions $f, g: [0,1]\times R\times R \to R\backslash\{0\}, \phi, \psi: [0,1]\times R\times R \to R, h_i, k_j: [0,1]\times R\times R\times R\times R \to R$ for $i = 1,2,\cdots,m,\ j = 1,2,\cdots,n$ are continuous;
(H_2) There exist positive functions L_f, L_g with bounds $\|L_f\|$ and $\|L_g\|$ respectively such that

$$|f(t,x(t),y(t)) - f(t,\bar{x}(t),\bar{y}(t))| \le L_f(t)(|x-\bar{x}|+|y-\bar{y}|),$$

$$|g(t,x(t),y(t)) - g(t,\bar{x}(t),\bar{y}(t))| \le L_g(t)(|x-\bar{x}|+|y-\bar{y}|),$$

for all $t \in [0,1]$ and $x, \bar{x}, y, \bar{y} \in R$;
(H_3) There exist positive functions L_{h_i}, L_{k_j} with bounds $\|L_{h_i}\|$ and $\|L_{k_j}\|$ respectively such that

$$|h_i(t,k_1,k_2,k_3,k_4) - h_i(t,k_1',k_2',k_3',k_4')| \le L_{h_i}(t)\sum_{s=1}^{4}|k_s - k_s'|,$$

$$|k_j(t,k_1,k_2,k_3,k_4) - k_j(t,k_1',k_2',k_3',k_4')| \le L_{k_j}(t)\sum_{s=1}^{4}|k_s - k_s'|,$$

for all $t \in [0,1]$ and $(k_1,k_2,k_3,k_4), (k_1',k_2',k_3',k_4') \in R^4$;
(H_4) There exist constants $M_0 > 0, M_1 > 0$ such that

$$|h_i(t,x,y,I^{\theta_1}x, I^{\delta_1}y)| \le M_0,$$

$$|k_j(t,x,y,I^{\theta_2}x, I^{\delta_2}y)| \le M_1,$$

for all $(t,x,y,I^{\theta_1}x,I^{\delta_1}y), (t,x,y,I^{\theta_2}x,I^{\delta_2}y) \in [0,1]\times R\times R\times R\times R$ and $i = 1,2,\cdots,m,\ j = 1,2,\cdots,n$;
(H_5) There exist two functions $p_1, p_2 \in L^{\infty}([0,1], R^+)$ and four continuous nondecreasing functions $\psi_1, \psi_2, \xi_1, \xi_2: [0,\infty) \to [0,\infty)$ such that

$$|\phi(t,x,y)| \le p_1(t)\psi_1(|x|)\xi_1(|y|),$$

$$|\psi(t,x,y)| \le p_2(t)\psi_2(|x|)\xi_2(|y|),$$

for all $t \in [0,1], x, y \in R$;
(H_6) There exist constants $M_2, M_3 > 0$ such that

$$\left|\frac{\mu(x)}{f(0,\mu(x),\nu(y))}\right| \le M_2,$$

$$\left|\frac{\nu(y)}{g(0,\mu(x),\nu(y))}\right| \le M_3,$$

for any $x, y \in C([0,1], R)$;
(H_7) There exists $r > 0$ such that

$$\frac{f_0\Lambda + \sum_{i=1}^{m}\frac{H_0}{\Gamma(\beta_i+1)} + g_0\Delta + \sum_{j=1}^{n}\frac{K_0}{\Gamma(\gamma_j+1)}}{\rho} < r,$$

where

$$\rho = 1 - \|L_f\|\Lambda - \|L_g\|\Delta - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}\|L_{h_i}\| \max\{1+\frac{1}{\Gamma(\theta_1+1)}, 1+\frac{1}{\Gamma(\delta_1+1)}\} - \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma_j+1)}\|L_{k_j}\| \max\{1+\frac{1}{\Gamma(\theta_2+1)}, 1+\frac{1}{\Gamma(\delta_2+1)}\},$$

$$\Lambda = \frac{2\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)} + \frac{|A| + M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|} + M_2,$$

$$\Delta = \frac{2\|p_2\|\psi_2(r)\xi_2(r)}{\Gamma(\beta+1)} + \frac{|B| + M_1 \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma_j+1)}}{|g(1,A,B)|} + M_3,$$

$$f_0 = \sup_{t\in[0,1]} f(t,0,0), \quad g_0 = \sup_{t\in[0,1]} g(t,0,0),$$

$$H_0 = \sup_{t\in[0,1]} h_i(t,0,0,0,0), \quad i = 1,2,\cdots,m,$$

$$K_0 = \sup_{t\in[0,1]} K_j(t,0,0,0,0), \quad j = 1,2,\cdots,n.$$

III. Main result

In this section, we present the main existence result for the system of coupled hybrid fractional integro-differential equation (1)-(4).

Lemma 3.1 Let $h \in X$. Then x is a solution of the hybrid fractional integro-differential problem

$$ {}^C D^\alpha \left(\frac{x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t), I^{\theta_1}x(t), I^{\delta_1}y(t))}{f(t,x(t),y(t))} \right) = h(t), \quad t \in [0,1], \tag{7}$$

$$x(0) = \mu(x), \quad x(1) = A, \tag{8}$$

if and only if

$$\begin{aligned} x(t) &= \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t), I^{\theta_1}x(t), I^{\delta_1}y(t)) \\ &+ f(t,x(t),y(t)) \Bigg[\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &+ t \Bigg(\frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))ds}{f(1,A,y(1))} \\ &- \int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \Bigg) + (1-t)\tfrac{\mu(x)}{f(0,\mu(x),y(0))} \Bigg]. \end{aligned} \tag{9}$$

Proof: Applying the Riemann-Liouville fractional integral operator I^α on both sides of the equaiton (7), we obtain

$$\begin{aligned} &\frac{x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t), I^{\theta_1}x(t), I^{\delta_1}y(t))}{f(t,x(t),y(t))} \\ &= \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + c_0 + c_1 t. \end{aligned} \tag{10}$$

Using the boundary conditions (8), we get

$$c_0 = \frac{\mu(x)}{f(0,\mu(x),y(0))}, \tag{11}$$

$$\begin{aligned} &\frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))ds}{f(1,A,y(1))} \\ &= \int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \tfrac{\mu(x)}{f(0,\mu(x),y(0))} + c_1. \end{aligned}$$

So,

$$\begin{aligned} c_1 &= \frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))ds}{f(1,A,y(1))} \\ &- \int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - \tfrac{\mu(x)}{f(0,\mu(x),y(0))}. \end{aligned} \tag{12}$$

Also, by the equations (11) and (12), we find that

$$\begin{aligned} &\frac{x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t), I^{\theta_1}x(t), I^{\delta_1}y(t))}{f(t,x(t),y(t))} \\ &= \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ &+ t \Bigg[\frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))ds}{f(1,A,y(1))} \\ &- \int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - \tfrac{\mu(x)}{f(0,\mu(x),y(0))} \Bigg] + \tfrac{\mu(x)}{f(0,\mu(x),y(0))}. \end{aligned}$$

The proof is completed. ■

By Lemma 3.1, we define an operator $T : E \to E$ associated with the problem (1)-(4) as follows

$$T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t)), \tag{13}$$

where

$$\begin{aligned} T_1(x,y)(t) &= \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t), I^{\theta_1}x(t), I^{\delta_1}y(t)) \\ &+ f(t,x(t),y(t)) \Bigg[\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s,x(s),y(s))ds \\ &+ t \Bigg(\frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))ds}{f(1,A,B)} \\ &- \int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s,x(s),y(s))ds \Bigg) + (1-t)\tfrac{\mu(x)}{f(0,\mu(x),\nu(y))} \Bigg]. \end{aligned} \tag{14}$$

$$\begin{aligned} T_2(x,y)(t) &= \sum_{j=1}^{n} I^{\gamma_j} k_j(t, x(t), y(t), I^{\theta_2}x(t), I^{\delta_2}y(t)) \\ &+ g(t,x(t),y(t)) \Bigg[\tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \psi(s,x(s),y(s))ds \\ &+ t \Bigg(\frac{B - \sum_{j=1}^{n} \int_0^1 \frac{(1-s)^{\gamma_j-1}}{\Gamma(\gamma_j)} k_j(s,x(s),y(s),I^{\theta_2}x(s),I^{\delta_2}y(s))ds}{g(1,A,B)} \\ &- \int_0^1 \tfrac{(1-s)^{\beta-1}}{\Gamma(\beta)} \psi(s,x(s),y(s))ds \Bigg) + (1-t)\tfrac{\nu(y)}{g(0,\mu(x),\nu(y))} \Bigg]. \end{aligned} \tag{15}$$

Observe that coupled system (1)-(4) has a solution if and only if the operator equation $T(x,y) = (x,y)$ has a fixed point.

Theorem 3.2 Assume that the hypotheses (H_1)-(H_7) hold. Furthermore, if

$$\begin{aligned} &(\|L_f\| + \|L_g\|)(\Lambda + \Delta) \\ &+ \sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i+1)} \max\{1+\frac{1}{\Gamma(\theta_1+1)}, 1+\frac{1}{\Gamma(\delta_1+1)}\} \\ &+ \sum_{j=1}^{n} \frac{\|L_{k_j}\|}{\Gamma(\gamma_j+1)} \max\{1+\frac{1}{\Gamma(\theta_2+1)}, 1+\frac{1}{\Gamma(\delta_2+1)}\} \\ &< 1. \end{aligned} \tag{16}$$

Then the coupled system (1)-(4) has a coupled solution.

Proof: Define a subset S of the Banach space E by

$$S = \{(x,y) \in E : \|(x,y)\| \le r\}. \tag{17}$$

It is easy to see that S is a closed, convex and bounded subset of the Banach space E.

Define the operators $A = (A_1, A_2) : E \to E$, $C = (C_1, C_2) : E \to E$ and $B = (B_1, B_2) : S \to E$ by

$$A_1(x,y) = f(t, x(t), y(t)), \quad t \in [0,1],$$

$$A_2(x,y) = g(t, x(t), y(t)), \quad t \in [0,1],$$

$$\begin{aligned} B_1(x,y) &= \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s, x(s), y(s)) ds \\ &+t\Bigg(\frac{A - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s, x(s), y(s), I^{\theta_1} x(s), I^{\delta_1} y(s)) ds}{f(1, A, B)} \\ &- \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s, x(s), y(s)) ds \Bigg) \\ &+(1-t) \frac{\mu(x)}{f(0, \mu(x), \nu(y))}, \qquad t \in [0,1], \end{aligned}$$

$$\begin{aligned} B_2(x,y) &= \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \psi(s, x(s), y(s)) ds \\ &+t\Bigg(\frac{B - \sum_{j=1}^{n} \int_0^1 \frac{(1-s)^{\gamma_j - 1}}{\Gamma(\gamma_j)} k_j(s, x(s), y(s), I^{\theta_2} x(s), I^{\delta_2} y(s)) ds}{g(1, A, B)} \\ &- \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \psi(s, x(s), y(s)) ds \Bigg) \\ &+(1-t) \frac{\nu(y)}{g(0, \mu(x), \nu(y))}, \qquad t \in [0,1], \end{aligned}$$

$$C_1(x,y) = \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t), I^{\theta_1} x(t), I^{\delta_1} y(t)),$$

$$C_2(x,y) = \sum_{j=1}^{n} I^{\gamma_j} k_j(t, x(t), y(t), I^{\theta_2} x(t), I^{\delta_2} y(t)).$$

Then operators T_1, T_2 given by (14),(15) are equivalent to

$$T_1(x(t), y(t)) = A_1(x(t), y(t)) B_1(x(t), y(t)) + C_1(x(t), y(t)),$$

$$T_2(x(t), y(t)) = A_2(x(t), y(t)) B_2(x(t), y(t)) + C_2(x(t), y(t)).$$

Therefore, the operator T given by (13) can be rewritten

$$\begin{aligned} T(x,y)(t) &= (T_1(x,y)(t), T_2(x,y)(t)) \\ &= A(x(t), y(t)) B(x(t), y(t)) + C(x(t), y(t)). \end{aligned}$$

Now we show that the operators A, B and C satisfy all the conditions of Theorem 2.8 in a series of steps.

Step I. We show that A and C are Lipschitzian on E. Let $(x,y), (\bar{x}, \bar{y}) \in E$. Then by $(H_2), (H_3)$, we have

$$\begin{aligned} &|A_1(x,y)(t) - A_1(\bar{x}, \bar{y})(t)| \\ &= |f(t, x(t), y(t)) - f(t, \bar{x}(t), \bar{y}(t))| \\ &\le L_f(t)(|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)|) \\ &\le L_f(t)(\|x - \bar{x}\| + \|y - \bar{y}\|) \end{aligned}$$

for all $t \in [0,1]$. Taking the supremum over t, we obtain

$$\|A_1(x,y) - A_1(\bar{x}, \bar{y})\| \le \|L_f\|(\|x - \bar{x}\| + \|y - \bar{y}\|) \quad (18)$$

for all $(x,y), (\bar{x}, \bar{y}) \in E$. In a similar manner, we can find that

$$\|A_2(x,y) - A_2(\bar{x}, \bar{y})\| \le \|L_g\|(\|x - \bar{x}\| + \|y - \bar{y}\|) \quad (19)$$

for all $(x,y), (\bar{x}, \bar{y}) \in E$. (17) and (18) imply

$$\begin{aligned} &\|A(x,y) - A(\bar{x}, \bar{y})\| \\ &= \|(A_1(x,y), A_2(x,y)) - (A_1(\bar{x}, \bar{y}), A_2(\bar{x}, \bar{y}))\| \\ &= \|(A_1(x,y) - A_1(\bar{x}, \bar{y}), A_2(x,y) - A_2(\bar{x}, \bar{y}))\| \\ &= \|A_1(x,y) - A_1(\bar{x}, \bar{y})\| + \|A_2(x,y) - A_2(\bar{x}, \bar{y})\| \\ &\le \|L_f\|(\|x - \bar{x}\| + \|y - \bar{y}\|) \\ &+\|L_g\|(\|x - \bar{x}\| + \|y - \bar{y}\|) \\ &= (\|L_f\| + \|L_g\|)(\|x - \bar{x}\| + \|y - \bar{y}\|) \end{aligned}$$

for all $(x,y), (\bar{x}, \bar{y}) \in E$, that is, A is a Lipschitzian with Lipschitz constant $L_A = \|L_f\| + \|L_g\|$. Now for all $(x,y), (\bar{x}, \bar{y}) \in E$, by definition of the operaor C, we get

$$\begin{aligned} &|C_1(x,y)(t) - C_1(\bar{x}, \bar{y})(t)| \\ &= |\sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t), I^{\theta_1} x(t), I^{\delta_1} y(t)) \\ &- \sum_{i=1}^{m} I^{\beta_i} h_i(t, \bar{x}(t), \bar{y}(t), I^{\theta_1} \bar{x}(t), I^{\delta_1} \bar{y}(t))| \\ &\le \sum_{i=1}^{m} \int_0^t \Bigg[\frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} L_{h_i}(s) \Bigg((1 + \tfrac{1}{\Gamma(\theta_1 + 1)}) \|x - \bar{x}\| \\ &+(1 + \tfrac{1}{\Gamma(\delta_1 + 1)}) \|y - \bar{y}\| \Bigg) \Bigg] ds \\ &\le \sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)} \max\{1 + \tfrac{1}{\Gamma(\theta_1 + 1)}, 1 + \tfrac{1}{\Gamma(\delta_1 + 1)}\} \\ &(\|x - \bar{x}\| + \|y - \bar{y}\|) \end{aligned}$$

for all $t \in [0,1]$. Taking the supremum over t, we have

$$\begin{aligned} &\|C_1(x,y) - C_1(\bar{x}, \bar{y})\| \\ &\le \sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)} \max\{1 + \tfrac{1}{\Gamma(\theta_1 + 1)}, 1 + \tfrac{1}{\Gamma(\delta_1 + 1)}\} \\ &(\|x - \bar{x}\| + \|y - \bar{y}\|) \end{aligned} \quad (20)$$

for all $(x,y), (\bar{x}, \bar{y}) \in E$. In a similar manner, we obtain

$$\begin{aligned} &\|C_2(x,y) - C_2(\bar{x}, \bar{y})\| \\ &\le \sum_{j=1}^{n} \frac{\|L_{k_j}\|}{\Gamma(\gamma_i + 1)} \max\{1 + \tfrac{1}{\Gamma(\theta_2 + 1)}, 1 + \tfrac{1}{\Gamma(\delta_2 + 1)}\} \\ &(\|x - \bar{x}\| + \|y - \bar{y}\|). \end{aligned} \quad (21)$$

From (20) and (21), we have

$$\begin{aligned} &\|C(x,y) - C(\bar{x}, \bar{y})\| \\ &\le \Bigg[\sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)} \max\{1 + \tfrac{1}{\Gamma(\theta_1 + 1)}, 1 + \tfrac{1}{\Gamma(\delta_1 + 1)}\} \\ &+ \sum_{j=1}^{n} \frac{\|L_{k_j}\|}{\Gamma(\gamma_i + 1)} \max\{1 + \tfrac{1}{\Gamma(\theta_2 + 1)}, 1 + \tfrac{1}{\Gamma(\delta_2 + 1)}\} \Bigg] \\ &(\|x - \bar{x}\| + \|y - \bar{y}\|) \end{aligned}$$

for all $(x,y), (\bar{x}, \bar{y}) \in E$, that is, C is a Lipschitzian with Lipschitz constant

$$\begin{aligned} L_C &= \sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)} \max\{1 + \tfrac{1}{\Gamma(\theta_1 + 1)}, 1 + \tfrac{1}{\Gamma(\delta_1 + 1)}\} \\ &+ \sum_{j=1}^{n} \frac{\|L_{k_j}\|}{\Gamma(\gamma_i + 1)} \max\{1 + \tfrac{1}{\Gamma(\theta_2 + 1)}, 1 + \tfrac{1}{\Gamma(\delta_2 + 1)}\}. \end{aligned}$$

Step II. Now we show that $B = (B_1, B_2)$ is completely continuous from S to E. We first prove that B is continuous. Let (x_n, y_n) be convergent sequences in S such that $x_n \to x, y_n \to y$ as $n \to \infty$. Then by Lebesgue dominated convergence theorem, we have

$$\begin{aligned}&\lim_{n\to\infty} B_1(x_n,y_n)(t)\\&=\lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s,x_n(s),y_n(s))ds\\&+t\lim_{n\to\infty}\Bigg(\frac{A-\sum_{i=1}^m\int_0^1\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x_n(s),y_n(s),I^{\theta_1}x_n(s),I^{\delta_1}y_n(s))ds}{f(1,A,B)}\\&-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s,x_n(s),y_n(s))ds\Bigg)\\&+(1-t)\lim_{n\to\infty}\frac{\mu(x_n)}{f(0,\mu(x_n),\nu(y_n))}\\&=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lim_{n\to\infty}\phi(s,x_n(s),y_n(s))ds\\&+t\Bigg(\frac{A-\sum_{i=1}^m\int_0^1\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}\lim_{n\to\infty}h_i(s,x_n(s),y_n(s),I^{\theta_1}x_n(s),I^{\delta_1}y_n(s))ds}{f(1,A,B)}\\&-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\lim_{n\to\infty}\phi(s,x_n(s),y_n(s))ds\Bigg)\\&+(1-t)\lim_{n\to\infty}\frac{\mu(x_n)}{f(0,\mu(x_n),\nu(y_n))}\\&=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s,x(s),y(s))ds\\&+t\Bigg(\frac{A-\sum_{i=1}^m\int_0^1\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))ds}{f(1,A,B)}\\&-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s,x(s),y(s))ds\Bigg)\\&+(1-t)\frac{\mu(x)}{f(0,\mu(x),\nu(y))}=B_1(x,y)(t).\end{aligned}$$

Analogously, we can prove

$$\lim_{n\to\infty} B_2(x_n,y_n)(t)=B_2(x,y)(t).$$

Hence, $B(x_n,y_n)=(B_1(x_n,y_n),B_2(x_n,y_n))$ converges to $B(x,y)$ pointwise on $[0,1]$.

Next, we will show that $B(S)$ is uniformly bounded in S. For any $(x,y)\in S$, using $(H_4)-(H_6)$, we obtain

$$\begin{aligned}&|B_1(x(t),y(t))|\le\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\phi(s,x(s),y(s))|ds\\&+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|\phi(s,x(s),y(s))|ds\\&+M_2\\&\le\frac{2\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+M_2.\end{aligned}$$

Taking the supremum over t, yields

$$\begin{aligned}\|B_1(x,y)\|&\le\frac{2\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)}\\&+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+M_2\end{aligned}$$

for all $(x,y)\in S$. Hence, B_1 is uniformly bounded on S. In a similar manner, we can obtain B_2 is also uniformly bounded on S. Hence B is uniformly bounded on S.

Now we show that B is equicontinuous. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$, we have

$$\begin{aligned}&|B_1(x,y)(t_2)-B_1(x,y)(t_1)|\\&\le\Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s,x(s),y(s))ds\\&-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s,x(s),y(s))ds\Big|\\&+|t_2-t_1|\Bigg(\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}\\&+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s,x(s),y(s))ds\Bigg)\\&+|t_2-t_1|M_2\\&\le\|p_1\|\psi_1(r)\xi_1(r)\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\&+\|p_1\|\psi_1(r)\xi_1(r)\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\&+|t_2-t_1|\Bigg(\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+\frac{\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)}\Bigg)\\&+|t_2-t_1|M_2.\end{aligned}$$

Obviously, $|B_1(x,y)(t_2)-B_1(x,y)(t_1)|\to0$ as $t_2\to t_1$. Similarly, we can prove $|B_2(x,y)(t_2)-B_2(x,y)(t_1)|\to0$ as $t_2\to t_1$, these imply that B is equicontinuous from the equicontinuity of B_1 and B_2. With the help of Arzela-Ascoli Theorem, B is completely continuous.

Step III. We now prove the condition (c) of Theorem 2.8 is satisfied. Let $x,y\in E$ and $\bar{x},\bar{y}\in E$ such that

$$(x,y)=A(x,y)B(x,y)+C(x,y).$$

Using $(H_2),(H_4),(H_5),(H_6)$, we have

$$\begin{aligned}&|x(t)|=|A_1(x,y)(t)B_1(\bar{x},\bar{y})(t)+C_1(x,y)(t)|\\&\le|A_1(x,y)(t)||B_1(\bar{x},\bar{y})(t)|+|C_1(x,y)(t)|\\&\le|f(t,x(t),y(t))|\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|\phi(s,\bar{x}(s),\bar{y}(s))|ds\\&+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|\phi(s,\bar{x}(s),\bar{y}(s))|ds\\&+M_2\Bigg]\\&+\sum_{i=1}^m\int_0^t\frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}|h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))|ds\\&\le[|f(t,x(t),y(t))-f(t,0,0)|+|f(t,0,0)|]\\&\left(\frac{2\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+M_2\right)\\&+\sum_{i=1}^m\int_0^t\frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\Big(|h_i(s,x(s),y(s),I^{\theta_1}x(s),I^{\delta_1}y(s))\\&-h_i(s,0,0,0,0)|+H_0\Big)ds\\&\le[\|L_f\|(\|x\|+\|y\|)+f_0]\\&\left(\frac{2\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+M_2\right)\\&+\sum_{i=1}^m\int_0^t\frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\Bigg\{\Bigg(\|L_{h_i}\|\\&\left[(1+\frac{1}{\Gamma(\theta_1+1)})\|x\|+(1+\frac{1}{\Gamma(\delta_1+1)})\|y\|\right]\Bigg)+H_0\Bigg\}ds\\&\le[\|L_f\|(\|x\|+\|y\|)+f_0]\\&\left(\frac{2\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+M_2\right)\\&+\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}(\|L_{h_i}\|\max\{1+\frac{1}{\Gamma(\theta_1+1)},1+\frac{1}{\Gamma(\delta_1+1)}\}\\&(\|x\|+\|y\|)+H_0).\end{aligned}$$

Taking the supremum over t, we can get

$$\begin{aligned}&\|x(t)\| \le [\|L_f\|(\|x\|+\|y\|)+f_0]\\&\left(\frac{2\|p_1\|\psi_1(r)\xi_1(r)}{\Gamma(\alpha+1)}+\frac{|A|+M_0\sum\limits_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}}{|f(1,A,B)|}+M_2\right)\\&+\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}(\|L_{h_i}\|\max\{1+\frac{1}{\Gamma(\theta_1+1)},1+\frac{1}{\Gamma(\delta_1+1)}\}\\&(\|x\|+\|y\|)+H_0).\end{aligned} \tag{22}$$

In a similar manner, we can get

$$\begin{aligned}&\|y(t)\| \le [\|L_g\|(\|x\|+\|y\|)+g_0]\\&\left(\frac{2\|p_2\|\psi_2(r)\xi_2(r)}{\Gamma(\beta+1)}+\frac{|B|+M_1\sum\limits_{j=1}^{n}\frac{1}{\Gamma(\gamma_j+1)}}{|g(1,A,B)|}+M_3\right)\\&+\sum_{j=1}^{n}\frac{1}{\Gamma(\gamma_j+1)}(\|L_{k_j}\|\max\{1+\frac{1}{\Gamma(\theta_2+1)},1+\frac{1}{\Gamma(\delta_2+1)}\}\\&(\|x\|+\|y\|)+K_0).\end{aligned} \tag{23}$$

Adding (22) and (23), we have

$$\begin{aligned}&\|x\|+\|y\|\\&\le\left[\|L_f\|\Lambda+\sum_{i=1}^{m}\frac{\|L_{h_i}\|}{\Gamma(\beta_i+1)}\max\{1+\frac{1}{\Gamma(\theta_1+1)},1+\frac{1}{\Gamma(\delta_1+1)}\}\right]\\&(\|x\|+\|y\|)\\&+\left[\|L_g\|\Delta+\sum_{j=1}^{n}\frac{\|L_{k_j}\|}{\Gamma(\gamma_j+1)}\max\{1+\frac{1}{\Gamma(\theta_2+1)},1+\frac{1}{\Gamma(\delta_2+1)}\}\right]\\&(\|x\|+\|y\|)\\&+f_0\Lambda+\sum_{i=1}^{m}\frac{H_0}{\Gamma(\beta_i+1)}+g_0\Delta+\sum_{j=1}^{n}\frac{K_0}{\Gamma(\gamma_j+1)},\end{aligned}$$

therefore, from (H_7), we have

$$\|x\|+\|y\|\le\frac{f_0\Lambda+\sum\limits_{i=1}^{m}\frac{H_0}{\Gamma(\beta_i+1)}+g_0\Delta+\sum\limits_{j=1}^{n}\frac{K_0}{\Gamma(\gamma_j+1)}}{\rho}<r.$$

Step IV. We show that the condition (d) of Theorem 2.8 hold.

$$\begin{aligned}M&=\|B(S)\|=\sup\{\|B(x,y)\|\}\\&=\sup\{\|B_1(x,y)\|+\|B_2(x,y)\|\}\le\Lambda+\Delta.\end{aligned}$$

From the above equations, we get

$$\begin{aligned}&L_AM+L_C\le(\|L_f\|+\|L_g\|)(\Lambda+\Delta)\\&+\sum_{i=1}^{m}\frac{\|L_{h_i}\|}{\Gamma(\beta_i+1)}\max\{1+\frac{1}{\Gamma(\theta_1+1)},1+\frac{1}{\Gamma(\delta_2+1)}\}\\&+\sum_{j=1}^{n}\frac{\|L_{k_j}\|}{\Gamma(\gamma_j+1)}\max\{1+\frac{1}{\Gamma(\theta_2+1)},1+\frac{1}{\Gamma(\delta_2+1)}\}\\&<1.\end{aligned}$$

Thus all the conditions of Theorem 2.8 are satisfied. Therefore, the equation $(x,y)=T(x,y)$ has a solution in S. Consequently, the coupled system of (1)-(4) has a coupled solution, the proof is completely. ■

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