

# Spatiotemporal Dynamics of a Phytoplankton-zooplankton System with Spatial Diffusion and Toxic Effect

Kejun Zhuang, Peixian Cui

**Abstract**—In this paper, a reaction-diffusion phytoplankton-zooplankton model with toxic effect is investigated. Firstly, the basic properties of solutions of the model and the existence of coexistent equilibrium are established. Secondly, the local stability of nonnegative equilibrium solutions in the absence of time delay is analyzed. Thirdly, delay-induced Hopf bifurcation is derived by using the bifurcation theory. Finally, some numerical simulations and conclusions are given.

**Index Terms**—phytoplankton-zooplankton system, reaction diffusion, delay, stability, Hopf bifurcation.

## I. INTRODUCTION

MARINE ecosystem plays a great role in human beings, its service function and ecological value are the important components of the earth life support system and the basic elements of the sustainable development of society and environment.

In marine ecological environment, the autotrophic phytoplankton are the primary producers. The growth of phytoplankton is not only directly controlled by photosynthesis, but also restricted by the concentration of various nutrients. To describe the interactions of nutrient and phytoplankton, some basic mathematical models have been developed in recent decades, see [1], [2], [3], [4], [5], [6]. The dynamic behaviors were investigated, such as local stability and Hopf bifurcation, global stability, travelling waves and so on. This kind of model emphasizes the cycling process between nutrient and autotrophic phytoplankton in low food web, but ignores the effect of herbivorous zooplankton on the ecosystem. In some cases, the predation of phytoplankton by zooplankton is indirectly considered by introducing the loss of autotrophic phytoplankton. Zooplankton do not participate in the nutrient reproduction cycle and the food web cycle is incomplete, so the nutrient-phytoplankton model is oversimple. Moreover, zooplankton not only promote phytoplankton growth by adjusting nutrients, but also limit the growth of zooplankton through the feeding process. Therefore, it is significant to study the phytoplankton-zooplankton model to better understand the role of zooplankton in the food web circulation system.

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From above theoretical basis, the following model was formulated by means of ordinary differential equations in [7]:

$$\begin{cases} \frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) - \alpha PZ, \\ \frac{dZ}{dt} = \beta PZ - \mu Z - \frac{\theta P^2 Z}{\gamma^2 + P^2}, \end{cases} \quad (1)$$

where  $P$  and  $Z$  are the densities of toxin-producing phytoplankton population and zooplankton population, respectively. All the parameters are positive constants,  $r$  is the intrinsic growth rate of phytoplankton,  $K$  is the environmental carrying capacity,  $\alpha$  is the predation rate of zooplankton to phytoplankton,  $\beta$  ( $\beta < \alpha$ ) denotes the ratio of biomass consumed by zooplankton,  $\mu$  is the mortality rate of zooplankton due to natural death as well as due to higher predation,  $\theta$  denotes the rate of toxin liberation by toxic-producing phytoplankton, and  $\gamma$  is the half-saturation constant. It is observed that toxin-producing plankton may terminate the planktonic blooms by decreasing the grazing pressure of zooplankton and thus acts as a biological control. Subsequently, some modified models are proposed in [8], [9], [10], [11], [12] and complex dynamic behaviors are obtained therein.

In the oceans, phytoplankton and zooplankton can move with the tide. So it is more realistic to use reaction-diffusion equations to model the aquatic system [13], [14], [15], [16]. Motivated by the work aforementioned, in the present paper, we first consider the reaction-diffusion phytoplankton-zooplankton model as follows:

$$\begin{cases} \frac{\partial P(x,t)}{\partial t} = d_1 \Delta P(x,t) + rP(x,t) \left(1 - \frac{P(x,t)}{K}\right) - \alpha P(x,t)Z(x,t), & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial Z(x,t)}{\partial t} = d_2 \Delta Z(x,t) + \beta P(x,t)Z(x,t) - \mu Z(x,t) - \frac{\theta P^2(x,t)Z(x,t)}{\gamma^2 + P^2(x,t)}, & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial P(x,t)}{\partial \nu} = \frac{\partial Z(x,t)}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, +\infty), \\ P(x,0) = P_0(x) \geq 0, Z(x,0) = Z_0(x) \geq 0 & x \in \bar{\Omega}. \end{cases} \quad (2)$$

Here  $\Omega \subset \mathbb{R}^n$  ( $n = 1, 2, 3$ ) is a bounded domain with smooth boundary  $\partial\Omega$  and  $\nu$  is the outward unit normal on  $\partial\Omega$ .  $\Delta$  is the usual Laplace operator.  $d_1$  and  $d_2$  are the diffusion coefficients of phytoplankton and zooplankton, respectively. It is assumed that the action of toxin is instantaneous in system (2). However, this is not accurate enough. Time delay can bring rich dynamic behaviors and cannot be ignored [17], [18]. We further consider the following time-delayed

reaction-diffusion model:

$$\begin{cases} \frac{\partial P(t, x)}{\partial t} = d_1 \Delta P(x, t) + rP(x, t) \left(1 - \frac{P(x, t)}{K}\right) - \alpha P(x, t)Z(x, t), & (x, t) \in \Omega \times (0, +\infty), \\ \frac{\partial Z(x, t)}{\partial t} = d_2 \Delta Z(x, t) + \beta P(x, t)Z(x, t) - \mu Z(x, t) - \frac{\theta P^2(x, t - \tau)Z(x, t)}{\gamma^2 + P^2(x, t - \tau)}, & (x, t) \in \Omega \times (0, +\infty), \\ \frac{\partial P(x, t)}{\partial \nu} = \frac{\partial Z(x, t)}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, +\infty), \\ P(x, t) = P_1(x) \geq 0, Z(x, t) = Z_1(x) \geq 0, & (x, t) \in \Omega \times [-\tau, 0], \end{cases} \quad (3)$$

where  $\tau$  is the time needed for zooplankton from ingesting toxic phytoplankton to dying. System (3) is more comprehensive and realistic. It incorporates the special case in [7].

It was pointed out that the population outbreak may happen for the species with periodic fluctuation and it is of great significance to investigate the periodic solutions of biological systems [19]. As we know, Hopf bifurcation is an interesting and significant phenomenon in the time-delayed system, and time-periodic solutions can be generated by time delay [20], [21]. So, the aim of this paper is to investigate the stability of nonnegative solutions and the existence of periodic solutions for phytoplankton-zooplankton systems (2) and (3).

The rest of this paper is organized as follows. In Section 2, we establish the basic properties of solutions. In Section 3, we present the existence of coexistent equilibrium about the system. In Section 4, we analyze the stability of nonnegative equilibria in the absence of time delay. In Section 5, we study the effect of time delay and the existence of Hopf bifurcation. In Section 6, we conduct some numerical simulations to support the theoretical results. In Section 7, we give the conclusions.

## II. FUNDAMENTAL PROPERTIES OF SOLUTIONS

In this section, we establish the well-posedness of solutions for system (2), including the existence, uniqueness, positivity of the solutions, and the nonpersistence of the system.

**Lemma 2.1.** System (2) has the unique and continuous solution  $(P(x, t), Z(x, t))$  for all  $t > 0$  in  $\bar{\Omega}$ , and

$$0 \leq P(x, t) \leq B_1, \quad 0 \leq Z(x, t) \leq B_2,$$

where

$$B_1 = \max\{K, \|P_0\|_\infty\}, \quad B_2 = \max\left\{\frac{\beta\gamma B_1}{\alpha\mu}, \|Z_0\|_\infty\right\}, \\ \|P_0\|_\infty = \sup_{x \in \bar{\Omega}} P_0(x), \quad \|Z_0\|_\infty = \sup_{x \in \bar{\Omega}} Z_0(x).$$

Moreover, the solution is strictly positive when the initial functions are not identical to zero.

The proof process of Lemma 2.1 is similarly to the method in [22], so we omit it.

**Lemma 2.2.** If  $\beta K < \mu + \theta$ , then system (2) is not persistent.

*Proof:* From the nonnegativity of solution and the first equation of (2), we have

$$\frac{\partial P(x, t)}{\partial t} + d_1 \Delta P(x, t) \leq rP \left(1 - \frac{P}{K}\right).$$

Then the simple comparison argument for parabolic problem in [23] shows that  $\limsup_{t \rightarrow +\infty} P(x, t) \leq K$ . Thus there exists  $T_1 \in (0, +\infty)$  such that  $P(x, t) \leq K + \varepsilon$  in  $\Omega \times [T_1, +\infty)$  for an arbitrary constant  $\varepsilon > 0$ . Combining the second equation of (2), we get

$$\frac{\partial Z(x, t)}{\partial t} - d_2 \Delta Z(x, t) \leq (\beta(K + \varepsilon) - \mu - \theta) Z(x, t),$$

and then

$$\limsup_{t \rightarrow +\infty} Z(x, t) \leq 0$$

by the continuity as  $\varepsilon \rightarrow 0$ . The proof is completed. ■

## III. EXISTENCE OF COEXISTENT EQUILIBRIUM

Spatial diffusion and time delay do not change the number and location of the constant steady states, so systems (1)-(3) have the same equilibria, which are the roots of the following algebraic equations:

$$\begin{cases} rP \left(1 - \frac{P}{K}\right) - \alpha PZ = 0, \\ \beta PZ - \mu Z - \frac{\theta P^2 Z}{\gamma^2 + P^2} = 0. \end{cases} \quad (4)$$

Obviously, (4) always has the zero equilibrium  $E_0(0, 0)$  and boundary equilibrium  $E_1(K, 0)$ . Then we explore the existence of coexistent equilibrium. By solving the first equation of (4), we have

$$Z = \frac{r}{\alpha} \left(1 - \frac{P}{K}\right).$$

Similarly, from the second equation of (4), we have

$$\beta P - \mu - \frac{\theta P^2}{\gamma^2 + P^2} = 0,$$

which can be simplified as

$$\beta P^3 - (\mu + \theta)P^2 + \beta\gamma^2 P - \mu r^2 = 0. \quad (5)$$

According to the Descartes' rule of signs in [24], equation (5) has one or three positive real roots, and has no negative real root. Let

$$F(P) = \beta P^3 - (\mu + \theta)P^2 + \beta\gamma^2 P - \mu r^2,$$

then we have

$$F'(P) = 3\beta P^2 - 2(\mu + \theta)P + \beta\gamma,$$

and

$$\lim_{P \rightarrow -\infty} F(P) = -\infty, \quad \lim_{P \rightarrow +\infty} F(P) = +\infty.$$

Due to the properties of function  $f(P)$ , we can conclude that equation (5) has unique positive real root  $P^*$ . Further, (4) has the unique positive equilibrium  $E_2(P^*, Z^*)$  for  $P^* < K$ , which can be guaranteed when  $K$  is sufficiently large.

**Lemma 3.1.** If  $K$  is sufficiently large, then system (3) has the unique coexistent equilibrium  $E_2(P^*, Z^*)$ .

In what follows, we shall discuss the effect of parameter  $\theta$  on the monotonicity of  $E_2$ . Denote  $P^* = P^*(\theta)$  and differentiate both sides of  $F(P^*) = 0$  with respect to  $\theta$ , we have

$$3\beta P^{*2} \frac{dP^*}{d\theta} - P^{*2} - 2(\mu + \theta)P^* \frac{dP^*}{d\theta} + \beta\gamma^2 \frac{dP^*}{d\theta} = 0,$$

that is

$$\frac{dP^*}{d\theta} = \frac{P^{*2}}{3\beta P^{*2} - 2(\mu + \theta)P^* + \beta\gamma^2}.$$

Then we establish the sign of  $\frac{dP^*}{d\theta}$ . The above denominator can be transformed into

$$3\beta \left[ P^* - \frac{\mu + \theta}{3\beta} \right]^2 + \beta\gamma^2 - \frac{(\mu + \theta)^2}{3\beta},$$

we have

$$\begin{cases} \frac{dP^*}{d\theta} > 0, \frac{dZ^*}{d\theta} < 0, & \text{for } \sqrt{3}\beta\gamma > \mu + \theta, \\ \frac{dP^*}{d\theta} < 0, \frac{dZ^*}{d\theta} > 0, & \text{for } \sqrt{3}\beta\gamma < \mu + \theta. \end{cases}$$

Thus, we can find that the positive equilibrium state  $P^*$  is monotonically increasing and  $Z^*$  is monotonically decreasing as  $\theta$  increases when  $\sqrt{3}\beta\gamma > \mu + \theta$ , and vice versa.

#### IV. STABILITY OF NONNEGATIVE EQUILIBRIA IN THE ABSENCE OF TIME DELAY

In this section, we shall discuss the stability of three nonnegative equilibria  $E_0, E_1$  and  $E_2$  for non-delayed system (2). For convenience, we first introduce some concepts about linear stability for reaction-diffusion equations. Let  $0 = \mu_0 < \mu_1 < \mu_2 < \dots < \mu_n < \dots$  denote the eigenvalues of the operator  $-\Delta$  in  $\Omega$  under the homogeneous Neumann boundary condition and  $S(\mu_n)$  be the eigenfunction space corresponding to  $\mu_n$ .

**Theorem 4.1.** The equilibrium  $E_0$  is always unstable.

*Proof:* Linearizing system (2) at  $E_0$ , we can obtain the corresponding characteristic equation

$$(\lambda + d_1\mu_n - r)(\lambda + d_2\mu_n + \mu) = 0,$$

and  $\lambda_{1n} = -d_1\mu_n + r, \lambda_{2n} = -d_2\mu_n - \mu < 0$ . Thus,  $E_0$  is unstable for  $\lambda_{10} = r > 0$ . The proof is completed. ■

**Theorem 4.2.** We have the following conclusions about the stability conclusions for equilibrium  $E_1$ :

- (i) If  $K < P^*$ , then  $E_1$  is locally asymptotically stable.
- (ii) If  $\beta K < \mu + \theta$ , then  $E_1$  is globally asymptotically stable.
- (iii) If  $K > P^*$ , then  $E_1$  is unstable.

*Proof:* The characteristic equation of linearized system of (2) at  $E_1$  is

$$(\lambda + d_1\mu_n + r) \left[ \lambda + d_2\mu_n - \left( \beta K - \mu - \frac{\theta K^2}{\gamma^2 + K^2} \right) \right] = 0,$$

and

$$\lambda_{1n} = -d_1\mu_n - r < 0,$$

$$\lambda_{2n} = -d_2\mu_n + \left( \beta K - \mu - \frac{\theta K^2}{\gamma^2 + K^2} \right).$$

From the analyses in previous section, we have  $\beta K - \mu - \frac{\theta K^2}{\gamma^2 + K^2} < 0$  for  $K < P^*$  and  $E_1$  is locally asymptotically stable. On the other side,  $\lambda_{20} > 0$  for  $K > P^*$  and  $E_1$  is unstable.

Next we prove the global stability of  $E_1$ . Based on the nonnegativity of solutions and the proof of Lemma 2.2, we get

$$\lim_{t \rightarrow +\infty} Z(x, t) = 0.$$

Then there exists  $T_2 \in (0, +\infty)$  such that  $Z(x, t) \leq \varepsilon_1$  in  $\Omega \times [T_2, +\infty)$  for an arbitrary constant  $\varepsilon_1 > 0$ . We have from the first equation of (2)

$$\frac{\partial P(x, t)}{\partial t} - d_1 \Delta P(x, t) \geq P(x, t) \left[ r - \left( \frac{r}{K} + \varepsilon_1 \right) \right],$$

thus

$$\liminf_{t \rightarrow +\infty} P(x, t) \geq \frac{r}{\frac{r}{K} + \varepsilon_1},$$

which implies

$$\liminf_{t \rightarrow +\infty} P(x, t) \geq K$$

by the continuity as  $\varepsilon_1 \rightarrow 0$ . By combining with the proof of Lemma 2.2, we have  $\lim_{t \rightarrow +\infty} P(x, t) = K$ . Hence, we obtain the conclusion of this theorem. ■

**Remark 4.1.** From Theorem 4.2, we know that the boundary equilibrium  $E_1$  is always asymptotically stable when positive equilibrium  $E_2$  does not exist. Conversely,  $E_1$  is unstable when  $E_2$  exists.

**Theorem 4.3.** For system (2), the following statements are true:

- (i) If  $\beta(\gamma^2 + P^{*2})^2 > 2\theta\gamma^2 P^*$ , then  $E_2$  is locally asymptotically stable.
- (ii) If  $\beta(\gamma^2 + P^{*2})^2 < 2\theta\gamma^2 P^*$  and  $d_2$  is sufficiently small, then  $E_2$  is unstable.

*Proof:* Linearizing system (2) at  $E_2$  can lead to the following characteristic equation

$$\lambda^2 + T_n \lambda + D_n = 0, \tag{6}$$

where

$$T_n = (d_1 + d_2)\mu_n + \frac{r}{K} P^* > 0,$$

$$D_n = d_1 d_2 \mu_n + \frac{r}{K} P^* d_2 \mu_n + \alpha P^* Z^* \left( \beta - \frac{2\theta\gamma^2 P^*}{(\gamma^2 + P^{*2})^2} \right).$$

It is obvious that  $D_n > 0$  for any  $n \in \mathbb{N}$  when  $\beta(\gamma^2 + P^{*2})^2 > 2\theta\gamma^2 P^*$ . Then we can prove the first case of theorem.

On the other hand,  $E_2$  is unstable if there exists some  $n$  such that  $D_n < 0$ . For  $\beta(\gamma^2 + P^{*2})^2 < 2\theta\gamma^2 P^*$ , we solve  $D_n = 0$  and obtain

$$\mu_n^\pm = \frac{-\frac{rP^*d_2}{K} \pm \sqrt{\left(\frac{rP^*d_2}{K}\right)^2 - 4d_1d_2\alpha P^* Z^* \left(\beta - \frac{2\theta\gamma^2 P^*}{(\gamma^2 + P^{*2})^2}\right)}}{2d_1d_2}.$$

According to the property of a parabola, to guarantee the negativity of  $D_n$ , we only need the following condition

$$\mu_n^+ > \mu_1,$$

that is

$$\frac{\sqrt{\left(\frac{rP^*d_2}{K}\right)^2 - 4d_1d_2\alpha P^* Z^* \left(\beta - \frac{2\theta\gamma^2 P^*}{(\gamma^2 + P^{*2})^2}\right)} - \frac{rP^*d_2}{K}}{2d_1d_2} > \mu_1,$$

and

$$\sqrt{\left(\frac{rP^*}{2d_1K}\right)^2 - \frac{D_0}{d_1d_2} - \frac{rP^*}{2Kd_1}} \geq \mu_1.$$

Hence the above inequation holds only if  $D_0 < 0$  and  $d_1$  is sufficiently small. Therefore, the second case of theorem can be obtained. ■

V. DELAY-INDUCED HOPF BIFURCATION

Next, we mainly focus on the effect of time delay on the stability of positive equilibrium  $E_2$  when it is stable in the absence of time delay.

The characteristic equation of (3) at  $E_2$  can be expressed by

$$\lambda^2 + T_n\lambda + C_n + Be^{-\lambda\tau} = 0, \tag{7}$$

where

$$\begin{aligned} T_n &= (d_1 + d_2)\mu_n + \frac{r}{K}P^* > 0, \\ C_n &= d_1d_2\mu_n^2 + d_2\mu_n\frac{r}{K}P^* + \alpha\beta P^*Z^* > 0, \\ B &= -\frac{2\alpha\theta\gamma^2P^{*2}Z^*}{(\gamma^2 + P^{*2})^2} < 0, \end{aligned}$$

and

$$D_n = C_n + B.$$

And then, we investigate the effect of time delay on the stability of the positive equilibrium  $E_2$  when  $D_0 > 0$ . In fact, it is always true that  $D_n > 0$  for any  $n \in \mathbb{N}$  when  $D_0 > 0$ .

Assume that  $i\omega$  is a root of (7). We have

$$-\omega^2 + iT_n\omega + C_n + B(\cos \omega\tau - i \sin \omega\tau) = 0,$$

which can be reduced to

$$\begin{cases} \omega^2 - C_n = B \cos \omega\tau, \\ T_n\omega = B \sin \omega\tau. \end{cases} \tag{8}$$

Adding the squares of the both sides of (8) can result in

$$\omega^4 + (T_n^2 - 2C_n)\omega^2 + C_n^2 - B^2 = 0. \tag{9}$$

On the basis of  $D_0 > 0$ , we find that  $C_n^2 - B^2 > 0$ . In addition,  $T_n^2 - 2C_n = (d_1^2 + d_2^2)\mu_n^2 + d_1\mu_n\frac{2rP^*}{K} + \frac{r^2}{K^2}P^* - 2\alpha\beta P^*Z^*$ . If  $T_0^2 - 2C_0 = \frac{r^2}{K^2}P^* - 2\alpha\beta P^*Z^* < 0$ , then there exists  $N \in \mathbb{N}_+$  such that  $T_n^2 - 2C_n < 0$  for  $n \leq N$ . Hence, equation (9) has two positive roots only if the following assumption holds:

(H1)  $\Lambda = T_n^4 - 4C_nT_n^2 + 4B^2 > 0$  and  $T_n^2 - 2C_n < 0$  for  $n \in \mathbb{N}^* \subset \mathbb{N}$ .

The roots of equation (9) are in the form of

$$\omega_n^\pm = \sqrt{\frac{(2C_n - T_n^2) \pm \sqrt{\Lambda}}{2}}$$

with

$$\tau_{nj}^\pm = \frac{1}{\omega_n^\pm} \left\{ \arccos \frac{\omega_n^{\pm 2} - C_n}{B} + 2(j-1)\pi \right\}, j = 0, 1, 2, \dots,$$

where  $n \in \mathbb{N}^*$ .

Then we verify the transversality condition. Taking the derivative of (7) with respect to  $\tau$ , we have

$$2\lambda \frac{d\lambda}{d\tau} + T_n \frac{d\lambda}{d\tau} - Be^{-\lambda\tau} \left( \lambda + \tau \frac{d\lambda}{d\tau} \right) = 0,$$

which is equivalent to

$$\begin{aligned} & \operatorname{Re} \left[ \frac{d\lambda}{d\tau} \right]_{\tau=\tau_{nj}^\pm, \lambda=\pm i\omega_n^\pm}^{-1} \\ &= \operatorname{Re} \frac{(2i\omega_n^\pm + T_n) (\cos \omega_n^\pm \tau_{nj}^\pm + i \sin \omega_n^\pm \tau_{nj}^\pm)}{iB\omega_n^\pm} \\ &= \frac{2\omega_n^\pm \cos \omega_n^\pm \tau_{nj}^\pm + T_n \sin \omega_n^\pm \tau_{nj}^\pm}{B\omega_n^\pm} \\ &= \frac{2\omega_n^\pm (\omega_n^{\pm 2} - C_n) + T_n^2\omega_n^\pm}{B^2\omega_n^\pm} \\ &= \frac{2(\omega_n^{\pm 2} - C_n) + T_n^2}{B^2} \\ &= \frac{(2C_n - T_n^2) \pm \sqrt{\Lambda} + T_n^2 - 2C_n}{B^2} \\ &= \frac{\pm \sqrt{\Lambda}}{B^2}. \end{aligned}$$

Therefore

$$\operatorname{Re} \left( \frac{d\lambda}{d\tau} \right) \Big|_{\tau=\tau_{nj}^+} > 0, \quad \operatorname{Re} \left( \frac{d\lambda}{d\tau} \right) \Big|_{\tau=\tau_{nj}^-} < 0.$$

It is easy to show that  $\tau_{n0}^+ < \tau_{n0}^- < \tau_{n1}^+ < \tau_{n1}^- < \tau_{n2}^+ < \tau_{n2}^- < \dots < \tau_{nj}^+ < \tau_{nj}^- < \dots$  for the same  $n \in \mathbb{N}^*$ .

From above analyses, we can conclude the distribution of roots of characteristic equation (7).

**Lemma 5.1.** For equation (7), the following statements are true:

- (i) If  $T_n^2 - 2C_n > 0$  for any  $n \in \mathbb{N}$  and  $D_0 > 0$ , then all roots of (7) have negative real parts.
- (ii) If  $D_0 > 0$  and (H1) are satisfied, then all roots of (7) have negative real parts when  $\tau \in [0, \tau_{00}^+)$ , and equation (7) has a pair of conjugated complex roots with positive real parts when  $\tau \in (\tau_{00}^+, \tau_{00}^+ + \delta)$ , where  $\delta$  is an appropriately small constant.

Moreover, we get the stability of positive equilibrium  $E_2$  for time-delayed system (3).

**Theorem 5.1.** For system (3), the following statements are true:

- (i) If  $T_n^2 - 2C_n > 0$  for any  $n \in \mathbb{N}$  and  $D_0 > 0$ , then the positive equilibrium  $E_2$  is asymptotically stable for any  $\tau > 0$ .
- (ii) If  $D_0 > 0$  and (H1) are satisfied, then  $E_2$  is asymptotically stable when  $\tau \in [0, \tau_{00}^+)$ , and is unstable when  $\tau \in (\tau_{00}^+, \tau_{00}^+ + \delta)$ . Specially,  $\tau_{00}^+$  is the Hopf bifurcation value and the bifurcating periodic solutions are spatially homogeneous in this situation.

VI. NUMERICAL SIMULATIONS

In this section, we give some numerical simulations to validate the previous theoretical derivations. We consider systems (2) and (3) with  $d_1 = d_2 = 0.2$ .

**Example 6.1.** In non-delayed system (2), let  $(r, K, \alpha, \beta, \mu, \gamma, \theta) = (2, 4, 0.7, 0.01, 0.012, 5.7, 0.5)$ . Then the boundary equilibrium of system (2) is asymptotically stable. See Fig. 1.

**Example 6.2.** In non-delayed system (2), let  $(K, \alpha, \beta, \mu, \gamma, \theta) = (400, 0.7, 0.01, 0.012, 5.7, 0.5)$ . Then the positive equilibrium of system (2) is asymptotically

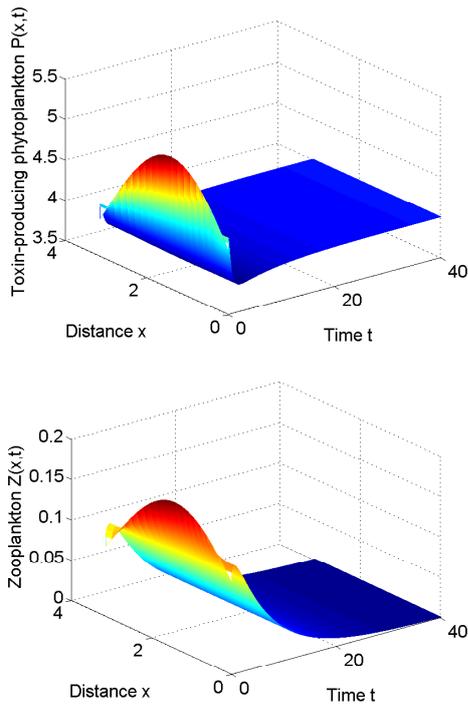


Fig. 1. The boundary equilibrium  $E_1$  of non-delayed system (2) is asymptotically stable.

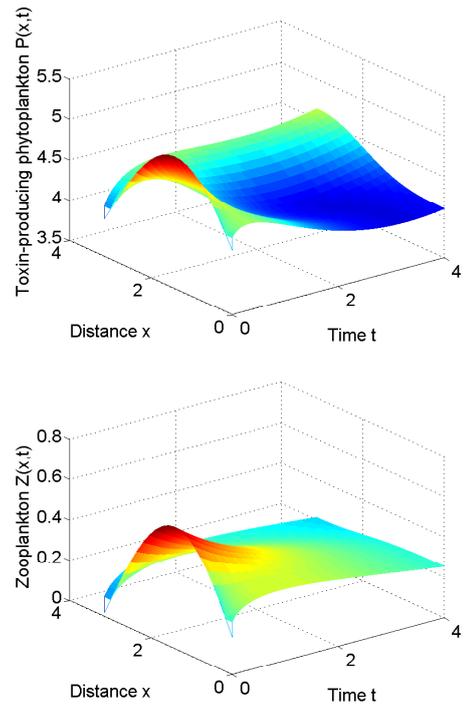


Fig. 3. The positive equilibrium  $E_2$  of non-delayed system (2) is unstable and nonconstant positive solution exists.

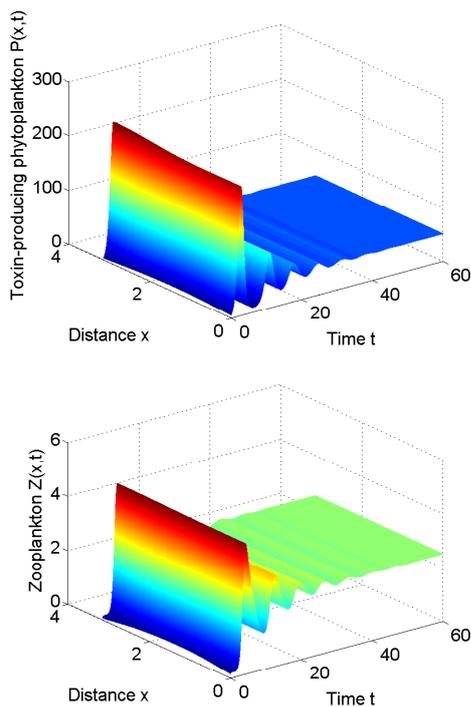


Fig. 2. The positive equilibrium  $E_2$  of non-delayed system (2) is asymptotically stable.

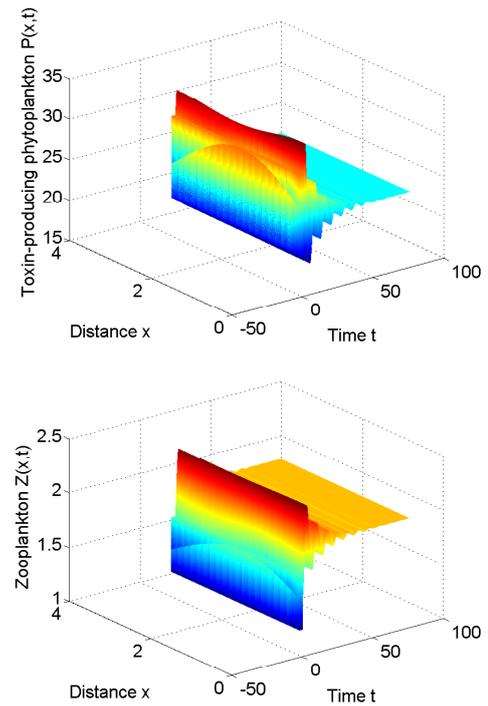


Fig. 4. The positive equilibrium  $E_2$  of time-delayed system (3) is asymptotically stable.

stable when  $r = 2$  and unstable when  $r = 0.2$ . See Figs. 2-3.

**Example 6.3.** In time-delayed system (3), let  $(r, K, \alpha, \beta, \mu, \gamma, \theta) = (1.5, 400, 0.7, 0.02, 0.012, 5.7, 0.5)$ . Then the positive equilibrium of system (3) is asymptotically stable when  $\tau = 2.5$  and unstable when  $\tau = 35$ . See Figs. 4-5.

## VII. CONCLUSIONS

Plankton is the foundation of marine ecosystem, can also be used as indicator species and as a marker for the exploration of undersea petroleum resources. It is of great significance to build the dynamic model to describe the interactions between phytoplankton and zooplankton. Therefore we revised the planktonic model proposed by means of ordinary differential equations in [7] with extensions, i.e., the

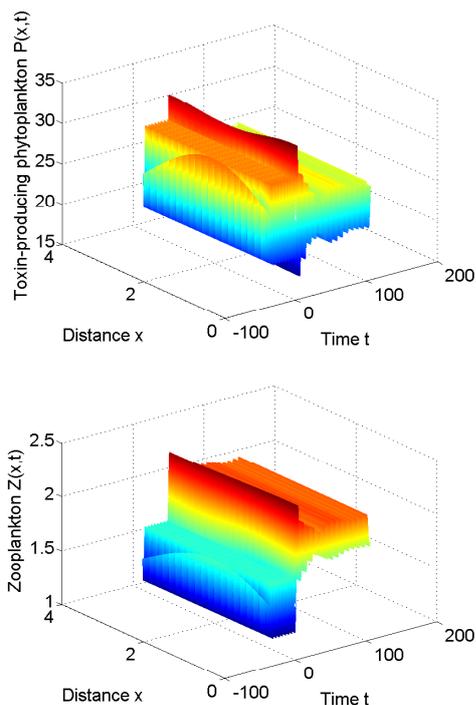


Fig. 5. The positive equilibrium  $E_2$  of time-delayed system (3) is unstable and periodic solution exists.

spatial diffusion of both phytoplankton and zooplankton and the action time delay of toxin.

In this paper, we are mainly concerned with the effects of spatial diffusion and time delay on the stability of constant nonnegative equilibrium solutions. By using meticulous analysis, the existence and asymptotic stability of constant nonnegative equilibrium solutions are investigated. It is found that time delay can destabilize the stability of equilibrium solution and generate time-periodic phenomenon of Hopf type. It is concluded that the phytoplankton-zooplankton system has the unique positive equilibrium if the environmental carrying capacity  $K$  of phytoplankton is sufficiently large, which means that the two planktonic populations may coexist.

From the viewpoint of mathematics, the boundary equilibrium  $E_1$  and positive equilibrium  $E_2$  are asymptotically stable under different conditions without considering time delay. In the non-delayed system (2), the intrinsic growth rate  $r$  of phytoplankton has vital impact on the stability of positive equilibrium. The two planktonic population quantities may approach constant values without time delay when the growth rate  $r$  is large. In stead, the spatial distribution of two populations would uneven when  $r$  is appropriately small.

When considering time delay, the stability of positive equilibrium depends on the delay parameter  $\tau$ . Small delay does not change the stability. However, periodic phenomenon can occur when time delay is larger than some critical value, which means the periodic outbreaks of phytoplankton and zooplankton.

Comparing with the study in [7], we not only extend the mathematical model through considering spatial diffusion and time delay, but also introduce the detailed mathematical deduction. In addition, phytoplankton-zooplankton oscilla-

tions are largely attributed to the time needed for zooplankton from ingesting toxic phytoplankton to dying.

Moreover, it is observed that the toxic phytoplankton can gathered in patches to reduce the zooplankton's grazing [27] and zooplankton can make appropriate response to the harmful phytoplankton [28]. The prey-taxis model can well describe this situation and we will further study the novel model in future.

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