Backstepping-based Sliding Mode Adaptive Control for Fractional-order System Considering Saturation Phenomenon

Xiaomin Tian, Guoshu Zhao, Zhong Yang, Jiaqi Ge, Hui Xie

Abstract—**This paper introduces a new backstepping-based sliding mode control strategy to realize the stabilization of a class of fractional-order system. The system is perturbed by unknown bounded unmodeled dynamics and external disturbances, meanwhile, the system parameters are unknown in advance. The effect of saturated input is considered to design the robust adaptive controller. For handling these unknown parameters, the proper sliding mode surface is constructed and some adaptive estimation laws can be assigned. The frequency distributed model is used so that the indirect Lyapunov theory is available in analyzing the stability of every subsystem. Finally, simulation results are presented to verify the effectiveness and feasibility of the proposed control strategy.**

Index Terms—**Backstepping-based sliding mode control, Fractional-order system, Input nonlinearity, Backstepping method.**

I. Introduction

THE history of fractional calculus is more than three centuries, which can be traced back to the contribution by Leibniz, Riemann, Liouville, Grunwald and Letnikov [1]. Fractional calculus particularly suitable for describing the viscoelastic system [2], and the memory and hereditary properties of various materials and processes. Recently, studying fractional-order system has become an active research area. There are two main reasons for this tendency, on one hand, many physical system cannot be described accurately without the introduction of fractional order calculus, on the other hand, it has been demonstrated by some previous researchers that fractional-order controllers have more potential superiorities than integer-order controllers [3].

In fractional-order nonlinear system, fractional-order chaotic system is a distinguished phenomenon that is characterized with some special features. Recently, control and synchronization of the fractional-order chaotic systems have attracted much attention from various scientific fields. Some methods have been proposed to achieve chaos synchronization in fractional-order chaotic systems. Such as nonlinear feedback control [4], a nonlinear state observer [5], active control [6], adaptive control [7-9], etc.

The backstepping method is a recursive approach for controller design, through designing virtual controllers and partial Lyapunov functions step by step, a common Lyapunov function of the whole system can be deduced from the above operations. This method can guarantee the global stability, tracking, and transient performance of nonlinear systems [10]. In view of the excellent performance of backstepping, an increasing number of researchers pay their attentions to this potential problem. Many preeminent literatures for the backstepping-based control or synchronization of fractional-order chaotic system have been reported. For example, Luo [11] researched the robust control and synchronization of fractional-order system by adding one power integrator. Shukla [12, 13] realized the stabilization and synchronization of fractional-order chaotic system by using backstepping method. Wei [14, 15] investigated the stability of fractional-order nonlinear system via adaptive backstepping technique.

However, in most of the above mentioned methods for control or stabilizing of fractional-order system, the system parameters are know in advance. In fact, the influence of unknown parameters maybe destroy the system's behavior and even cause unbounded outputs, so it is urgent to design an adaptive controller to deal with this problem. Meanwhile, when the system is in work, the external disturbance is unavoidable, so, it should be considered in design robust controller. Besides, nonlinearity in control input also is often encountered in various systems and can be a cause of instability. Thus, it is obvious that the effects of input nonlinearity must be taken into account when analyzing and implementing a control scheme.

Motivated by the above discussions, it is still very challenging and essential to research the stabilization for a class of fractional-order system with input nonlinearity by using adaptive backstepping-based sliding mode control technique. For compensation the nonlinear input, a fractional-order auxiliary system is constructed to generate necessary signal. Some appropriate estimation rules are given to deal with the system parameters and the unknown upper bound of uncertainties. The frequency distributed model is used to establish an indirect Lyapunov function to verify the stability and design virtual controller for every subsystem. Through design virtual controller step by step, a comprehensive actual

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controller is determined.

The remaining part of this paper is organized as follows: Section 2 introduces the relevant definitions, lemmas, and frequency distributed model. Main results are presented in Section 3. Some numerical simulations are provided in Section 4 to show the effectiveness of the proposed method. Finally, conclusions are given in Section 5.

II. PRELIMINARIES

The Caputo definition is the most commonly used definition of fractional calculus.

Definition 1. The Caputo fractional derivative of order α is defined as

$$ {}_{t_0}D_t^{\alpha} f(t)=\begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau, & m-1<\alpha<m \\ \frac{d^m}{dt^m}f(t), & \alpha=m \end{cases} \tag{1} $$

where m is the smallest integer number, larger than α. In the rest of this paper, we will use D^{α} instead of ${}_0D_t^{\alpha}$.

Lemma 1 (see [16]). Let $x=0$ be an equilibrium point for either Caputo or RL fractional nonautonomous system:

$$ D^q x(t) = f(x,t) \tag{2} $$

where $q \in (0,1)$ and $f(x,t)$ satisfies the Lipschitz condition with Lipschitz constant $l>0$. Assume that there exists a Lyapunov function $V(t,x(t))$ satisfying

$$ \begin{cases} \alpha_1||x||^a \le V(t,x(t)) \le \alpha_2||x|| \\ \dot{V}(t,x(t)) \le -\alpha_3||x|| \end{cases} \tag{3} $$

where α_1, α_2, α_3 and a are positive constants and $||\cdot||$ denotes an arbitrary norm. Then the equilibrium point of system (2) is asymptotically stable.

Lemma 2 (see [17]). Consider a nonlinear fractional-order system

$$ D^{\alpha}x(t) = f(x(t)) \tag{4} $$

where $\alpha \in (0,1)$. Then the system can be equivalently converted to the following continuous frequency distributed model

$$ \begin{cases} \frac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + f(x(t)) \\ x(t) = \int_0^{\infty}\mu_{\alpha}(\omega)z(\omega,t)d\omega \end{cases} \tag{5} $$

where $\mu_{\alpha}(\omega) = \frac{sin(\alpha\pi)}{\pi\omega^{\alpha}}$, and $z(\omega,t)$ is the true state of the system.

III. MAIN RESULTS

Backstepping technique is suitable for research strict feedback system, which can be described as follows

$$ \begin{aligned} D^{\alpha}x_1 &= g_1(x_1,t)x_2+\delta_1^T F_1(x_1,t)+f_1(x_1,t) \\ D^{\alpha}x_2 &= g_2(x_1,x_2,t)x_3+\delta_2^T F_2(x_1,x_2,t)+f_2(x_1,x_2,t) \\ &\vdots \\ D^{\alpha}x_{n-1} &= g_{n-1}(x_1,x_2,...,x_{n-1},t)x_n+\delta_{n-1}^T F_{n-1}(x_1, \\ &\quad x_2,...,x_{n-1},t)+f_{n-1}(x_1,x_2,...,x_{n-1},t) \\ D^{\alpha}x_n &= g_n(x_1,x_2,...,x_n,t)u+\delta_n^T F_n(x_1,x_2,...,x_n,t) \\ &\quad +f_n(x_1,x_2,...,x_n,t) \end{aligned} \tag{6} $$

where δ_i is the system parameters vector of the i-th state equation, $g_i(\cdot)$, $F_i(\cdot)$, $f_i(\cdot)$ for $i=1,2,...,n$ are known, smooth nonlinear functions. This paper investigates a class of typical fractional-order strict feedback system, it has the following form

$$ \begin{aligned} D^{\alpha}x_1 &= x_2+\delta_1^T F_1(x_1)+f_1(x_1)+\Delta f_1(X)+d_1(t) \\ D^{\alpha}x_2 &= x_3+\delta_2^T F_2(x_1,x_2)+f_2(x_1,x_2)+\Delta f_2(X)+d_2(t) \\ &\vdots \\ D^{\alpha}x_n &= k\Psi(u(t))+\delta_n^T F_n(X)+f_n(X)+\Delta f_n(X)+d_n(t) \end{aligned} \tag{7} $$

where $\alpha \in (0,1)$, $X=[x_1,x_2,...,x_n]^T$ is state variables vector, k is non-zero constant, δ_i is unknown system parameters vector, $F_i(\cdot)$ and $f_i(\cdot)$ are system nonlinear parts, $\Delta f_i(X)$ and $d_i(t)$ for $i=1,2,...,n$ respectively are unmodeled dynamics and external disturbance. $\Psi(u(t))$ is saturated input.

Assumption 1. The nonlinear saturation function is defined as follows:

$$ \Psi(u(t)) = \begin{cases} u_H, & u(t) \ge u^h \\ \theta u(t), & u^l \le u(t) \le u^h \\ u_L, & u(t) \le u^l \end{cases} \tag{8} $$

where u_H, $u^h \in R^+$, and u_L, $u^l \in R^-$ are bounds of the saturation function and $\theta \in R$ is the saturation slope. Subsequently, the above saturation function can be rewritten as

$$ \Psi(u(t)) = u(t) + \Delta u(t) \tag{9} $$

and $\Delta u(t)$ is satisfied as

$$ \Delta u(t) = \begin{cases} u_H - u(t), & u(t) \ge u^h \\ (\theta-1)u(t), & u^l \le u(t) \le u^h \\ u_L - u(t), & u(t) \le u^l \end{cases} \tag{10} $$

A typical nonlinear saturation function is described in Figure 1.

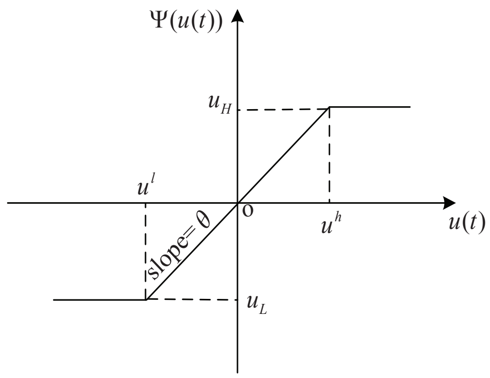

Figure 1. A typical nonlinear saturation function

Remark 1. Many fractional-order systems can be described as equation (7), such as, fractional-order gyro system, fractional-order Genesio-Tesi system, fractional-order Arneodo system, and so on. For using backstepping-based sliding mode control strategy, transformation variables are firstly assigned as

$$ \begin{cases} \xi_1 = x_1 - \sigma_1 \\ \xi_i = x_i - \vartheta_{i-1} - \sigma_i, \quad i=2,3,...,n. \end{cases} \tag{11} $$

where $\vartheta_j (j=1,2,...,n-1)$ is virtual controller to be determined later. $\sigma_j (j=1,2,...,n)$ is the virtual signal generated by the following auxiliary fractional-order system to compensate the nonlinear input

$$\left\{\begin{array}{c} D^{\alpha}\sigma_i = \sigma_{i+1} - c_i|\sigma_i|^{\gamma}, \quad i = 1, 2, ..., n-1. \\ D^{\alpha}\sigma_n = k\Delta u(t) - c_n|\sigma_n|^{\gamma} \end{array}\right. \tag{12}$$

where $\gamma \in (0, 1)$, $c_i > 0$, $c_n > 0$.

For handling the unknown bounded uncertainties, the following assumption is given.

Assumption 2. The unmodeled dynamics and external disturbance are unknown bounded, which satisfy the following conditions

$$\left\{\begin{array}{c} |\Delta f_i(X)| \leq \beta_{i1}|\xi_i| \\ |d_i(t)| \leq \beta_{i2} \end{array}\right. \tag{13}$$

where β_{i1} and β_{i2} are unknown positive constants. In this paper, the sliding mode surface can be constructed as

$$\begin{aligned} s_p &= D^{\alpha-1}\xi_p + \int_0^t \big[-\xi_{p+1} + h_{p1}\xi_p + h_{p2}|\xi_p|^{\gamma} sgn(\xi_p)\big] d\tau \\ s_n &= D^{\alpha-1}\xi_n + \int_0^t \big[h_{n1}\xi_n + h_{n2}|\xi_n|^{\gamma} sgn(\xi_n)\big] d\tau \end{aligned} \tag{14}$$

in which $p = 1, 2, ..., n-1$. $h_{11} > \frac{1}{2}$, $h_{n1} > \frac{1}{2}$, $h_{q1} > 1$ ($q = 2, 3, ..., n-1$), $h_{p2} > 0$, $h_{n2} > 0$. Taking the derivative of s with respect time, we have

$$\left\{\begin{array}{c} \dot{s}_p = D^{\alpha}\xi_p - \xi_{p+1} + \big[h_{p1}\xi_p + h_{p2}|\xi_p|^{\gamma} sgn(\xi_p)\big] \\ \dot{s}_n = D^{\alpha}\xi_n + \big[h_{n1}\xi_n + h_{n2}|\xi_n|^{\gamma} sgn(\xi_n)\big] \end{array}\right. \tag{15}$$

when system trajectories arrived at the sliding mode surface, we have $\dot{s} = 0$, that is

$$\left\{\begin{array}{c} D^{\alpha}\xi_p = \xi_{p+1} - \big[h_{p1}\xi_p + h_{p2}|\xi_p|^{\gamma} sgn(\xi_p)\big] \\ D^{\alpha}\xi_n = -\big[h_{n1}\xi_n + h_{n2}|\xi_n|^{\gamma} sgn(\xi_n)\big] \end{array}\right. \tag{16}$$

according the sliding mode dynamics, the virtual controller can be determined as

$$\begin{aligned} \vartheta_1 &= -m_1 s_1 - \hat{\delta}_1^T F_1 - f_1 - (\hat{\beta}_{11}|\xi_1| + \hat{\beta}_{12}) sgn(s_1) \\ &\quad -c_1|\sigma_1|^{\gamma} - \big[h_{11}\xi_1 + h_{12}|\xi_1|^{\gamma} sgn(\xi_1)\big] \\ \vartheta_j &= -m_j s_j - \hat{\delta}_j^T F_j - f_j - (\hat{\beta}_{j1}|\xi_j| + \hat{\beta}_{j2}) sgn(s_j) \\ &\quad + D^{\alpha}\vartheta_{j-1} - c_j|\sigma_j|^{\gamma} - \big[h_{j1}\xi_j + h_{j2}|\xi_j|^{\gamma} sgn(\xi_j)\big] \end{aligned} \tag{17}$$

where $j = 2, 3, ..., n-1$. $m_i > 0$, F_i and f_i are the abbreviations of $F_i(\cdot)$ and $f_i(\cdot)$. $\hat{\delta}_i$, $\hat{\beta}_{i1}$ and $\hat{\beta}_{i2}$ are estimations of δ_i, β_{i1} and β_{i2} for $i = 1, 2, ..., n$, respectively. Denote $\widetilde{\delta}_i = \hat{\delta}_i - \delta_i$, $\widetilde{\beta}_{i1} = \hat{\beta}_{i1} - \beta_{i1}$, $\widetilde{\beta}_{i2} = \hat{\beta}_{i2} - \beta_{i2}$ as parameters estimation errors, which adaptive update laws are designed as

$$\left\{\begin{array}{c} D^{\alpha}\widetilde{\delta}_i = D^{\alpha}\hat{\delta}_i = F_i s_i \\ D^{\alpha}\widetilde{\beta}_{i1} = D^{\alpha}\hat{\beta}_{i1} = \eta_{i1}|\xi_i||s_i|, \quad \eta_{i1} > 0 \\ D^{\alpha}\widetilde{\beta}_{i2} = D^{\alpha}\hat{\beta}_{i2} = \eta_{i2}|s_i|, \qquad \eta_{i2} > 0 \end{array}\right. \tag{18}$$

Theorem 1. Consider the system (7) with saturated nonlinear input, if the system is controlled by the following controller

$$\begin{aligned} u(t) = \frac{1}{k}\Big\{ &-m_n s_n - \hat{\delta}_n^T F_n - f_n - (\hat{\beta}_{n1}|\xi_n| + \hat{\beta}_{n2}) sgn(s_n) \\ &+ D^{\alpha}\vartheta_{n-1} - c_n|\sigma_n|^{\gamma} - \big[h_{n1}\xi_n + h_{n2}|\xi_n|^{\gamma} sgn(\xi_n)\big]\Big\} \end{aligned} \tag{19}$$

then the system trajectories can converge to the sliding surface $s_i (i = 1, 2, ..., n) = 0$ asymptotically.

Proof. Step 1: The first new subsystem can be obtain according to equations (7), (11) and (12)

$$\begin{aligned} D^{\alpha}\xi_1 &= D^{\alpha}x_1 - D^{\alpha}\sigma_1 \\ &= x_2 + \delta_1^T F_1 + f_1 + \Delta f_1(X) + d_1(t) - \sigma_2 + c_1|\sigma_1|^{\gamma} \\ &= \xi_2 + \vartheta_1 + \delta_1^T F_1 + f_1 + \Delta f_1(X) + d_1(t) + c_1|\sigma_1|^{\gamma} \end{aligned} \tag{20}$$

according to Lemma 2, the parameters adaptation laws (18) can transform into the frequency distributed model, that is

$$\left\{\begin{array}{c} \frac{\partial z_{\widetilde{\delta}_1}(\omega,t)}{\partial t} = -\omega z_{\widetilde{\delta}_1}(\omega, t) + F_1 s_1 \\ \widetilde{\delta}_1 = \int_0^{\infty} \mu_{\alpha}(\omega) z_{\widetilde{\delta}_1}(\omega, t) d\omega \\ \frac{\partial z_{\widetilde{\beta}_{11}}(\omega,t)}{\partial t} = -\omega z_{\widetilde{\beta}_{11}}(\omega, t) + \eta_{11}|\xi_1||s_1| \\ \widetilde{\beta}_{11} = \int_0^{\infty} \mu_{\alpha}(\omega) z_{\widetilde{\beta}_{11}}(\omega, t) d\omega \\ \frac{\partial z_{\widetilde{\beta}_{12}}(\omega,t)}{\partial t} = -\omega z_{\widetilde{\beta}_{12}}(\omega, t) + \eta_{12}|s_1| \\ \widetilde{\beta}_{12} = \int_0^{\infty} \mu_{\alpha}(\omega) z_{\widetilde{\beta}_{12}}(\omega, t) d\omega \end{array}\right. \tag{21}$$

selecting the Lyapunov function as

$$\begin{aligned} V_1 &= \frac{1}{2}s_1^2 + \frac{1}{2}\int_0^{\infty} \mu_{\alpha}(\omega) z_{\widetilde{\delta}_1}^T(\omega, t) z_{\widetilde{\delta}_1}(\omega, t) d\omega \\ &\quad + \frac{1}{2\eta_{11}}\int_0^{\infty} \mu_{\alpha}(\omega) z_{\widetilde{\beta}_{11}}^2(\omega, t) d\omega \\ &\quad + \frac{1}{2\eta_{12}}\int_0^{\infty} \mu_{\alpha}(\omega) z_{\widetilde{\beta}_{12}}^2(\omega, t) d\omega \end{aligned} \tag{22}$$

taking the derivative of V_1 with respect to time, it yields

$$\begin{aligned} \dot{V}_1 &= s_1\dot{s}_1 - \int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\delta}_1}^T(\omega, t) z_{\widetilde{\delta}_1}(\omega, t) d\omega + \widetilde{\delta}_1^T F_1 s_1 \\ &\quad - \frac{1}{\eta_{11}}\int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\beta}_{11}}^2(\omega, t) d\omega + \widetilde{\beta}_{11}|\xi_1||s_1| \\ &\quad - \frac{1}{\eta_{12}}\int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\beta}_{12}}^2(\omega, t) d\omega + \widetilde{\beta}_{12}|s_1| \end{aligned} \tag{23}$$

substituting $\dot{s}_1$ from equation (15) into equation (23), one has

$$\begin{aligned} \dot{V}_1 &= s_1\Big\{D^{\alpha}\xi_1 - \xi_2 + \big[h_{11}\xi_1 + h_{12}|\xi_1|^{\gamma} sgn(\xi_1)\big]\Big\} \\ &\quad - \int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\delta}_1}^T(\omega, t) z_{\widetilde{\delta}_1}(\omega, t) d\omega + \widetilde{\delta}_1^T F_1 s_1 \\ &\quad - \frac{1}{\eta_{11}}\int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\beta}_{11}}^2(\omega, t) d\omega + \widetilde{\beta}_{11}|\xi_1||s_1| \\ &\quad - \frac{1}{\eta_{12}}\int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\beta}_{12}}^2(\omega, t) d\omega + \widetilde{\beta}_{12}|s_1| \end{aligned} \tag{24}$$

substituting the new subsystem (20) into the above equation

$$\begin{aligned} \dot{V}_1 &= s_1\Big\{\xi_2 + \vartheta_1 + \delta_1^T F_1 + f_1 + \Delta f_1(X) + d_1(t) \\ &\quad + c_1|\sigma_1|^{\gamma} - \xi_2 + \big[h_{11}\xi_1 + h_{12}|\xi_1|^{\gamma} sgn(\xi_1)\big]\Big\} \\ &\quad - \int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\delta}_1}^T(\omega, t) z_{\widetilde{\delta}_1}(\omega, t) d\omega + \widetilde{\delta}_1^T F_1 s_1 \\ &\quad - \frac{1}{\eta_{11}}\int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\beta}_{11}}^2(\omega, t) d\omega + \widetilde{\beta}_{11}|\xi_1||s_1| \\ &\quad - \frac{1}{\eta_{12}}\int_0^{\infty} \omega\mu_{\alpha}(\omega) z_{\widetilde{\beta}_{12}}^2(\omega, t) d\omega + \widetilde{\beta}_{12}|s_1| \end{aligned} \tag{25}$$

replacing ϑ_1 from equation (17) into equation (25) and using Assumption 2, we have

$$\begin{aligned}\dot{V}_1 \;&\leq\; s_1\big[-m_1 s_1-\widetilde{\delta}_1^T F_1-(\hat{\beta}_{11}|\xi_1|+\hat{\beta}_{12})sgn(s_1)\big]+\beta_{11}\times\\ &\quad |\xi_1||s_1|+\beta_{12}|s_1|-\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_1}^T(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega\\ &\quad +\widetilde{\delta}_1^T F_1 s_1-\frac{1}{\eta_{11}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{11}}^2(\omega,t)d\omega+\widetilde{\beta}_{11}\times\\ &\quad |\xi_1||s_1|-\frac{1}{\eta_{12}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{12}}^2(\omega,t)d\omega+\widetilde{\beta}_{12}|s_1|\\ &= -\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_1}^T(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega\\ &\quad -\frac{1}{\eta_{11}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{11}}^2(\omega,t)d\omega\\ &\quad -\frac{1}{\eta_{12}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{12}}^2(\omega,t)d\omega-m_1 s_1^2<0 \qquad (26)\end{aligned}$$

because of $\dot{V}_1<0$, then s_1, $\widetilde{\delta}_1$, $\widetilde{\beta}_{11}$, $\widetilde{\beta}_{12}$ are all asymptotically converge to zero.

Step 2: The second subsystem about ξ_2 can be established as

$$\begin{aligned}D^\alpha\xi_2 &= D^\alpha x_2-D^\alpha\vartheta_1-D^\alpha\sigma_2\\ &= x_3+\delta_2^T F_2+f_2+\Delta f_2(X)+d_2(t)-D^\alpha\vartheta_1\\ &\quad -\sigma_3+c_2|\sigma_2|^\gamma\\ &= \xi_3+\vartheta_2+\delta_2^T F_2+f_2+\Delta f_2(X)+d_2(t)\\ &\quad -D^\alpha\vartheta_1+c_2|\sigma_2|^\gamma \qquad (27)\end{aligned}$$

similar to the step 1, the frequency distributed model of adaptive estimation laws can be constructed as

$$\begin{cases}\frac{\partial z_{\widetilde{\delta}_2}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\delta}_2}(\omega,t)+F_2 s_2\\ \widetilde{\delta}_2=\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\delta}_2}(\omega,t)d\omega\\ \frac{\partial z_{\widetilde{\beta}_{21}}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\beta}_{21}}(\omega,t)+\eta_{21}|\xi_2||s_2|\\ \widetilde{\beta}_{21}=\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{21}}(\omega,t)d\omega\\ \frac{\partial z_{\widetilde{\beta}_{22}}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\beta}_{22}}(\omega,t)+\eta_{22}|s_2|\\ \widetilde{\beta}_{22}=\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{22}}(\omega,t)d\omega\end{cases} \qquad (28)$$

selecting the Lyapunov function as

$$\begin{aligned}V_2 &= V_1+\frac{1}{2}s_2^2+\frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\delta}_2}^T(\omega,t)z_{\widetilde{\delta}_2}(\omega,t)d\omega\\ &\quad +\frac{1}{2\eta_{21}}\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{21}}^2(\omega,t)d\omega\\ &\quad +\frac{1}{2\eta_{22}}\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{22}}^2(\omega,t)d\omega \qquad (29)\end{aligned}$$

thus its derivative can be described as

$$\begin{aligned}\dot{V}_2 \;&\leq\; -\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega-m_1 s_1^2\\ &\quad +s_2\dot{s}_2+\widetilde{\delta}_2^T F_2 s_2+\widetilde{\beta}_{21}|\xi_2||s_2|+\widetilde{\beta}_{22}|s_2| \qquad (30)\end{aligned}$$

substituting $\dot{s}_2$ from equation (15) into (30) and according to equation (27), one obtains

$$\begin{aligned}\dot{V}_2 \;&\leq\; -\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega-m_1 s_1^2\\ &\quad +s_2\Big\{D^\alpha\xi_2-\xi_3+\big[h_{21}\xi_2+h_{22}|\xi_2|^\gamma sgn(\xi_2)\big]\Big\}\\ &\quad +\widetilde{\delta}_2^T F_2 s_2+\widetilde{\beta}_{21}|\xi_2||s_2|+\widetilde{\beta}_{22}|s_2|\\ &= -\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega-m_1 s_1^2\\ &\quad +s_2\Big\{\xi_3+\vartheta_2+\delta_2^T F_2+f_2+\Delta f_2(X)+d_2(t)\\ &\quad -D^\alpha\vartheta_1+c_2|\sigma_2|^\gamma-\xi_3+\big[h_{21}\xi_2+h_{22}|\xi_2|^\gamma sgn(\xi_2)\big]\Big\}\\ &\quad +\widetilde{\delta}_2^T F_2 s_2+\widetilde{\beta}_{21}|\xi_2||s_2|+\widetilde{\beta}_{22}|s_2| \qquad (31)\end{aligned}$$

replace ϑ_2 from equation (17) into the above equation, we have

$$\begin{aligned}\dot{V}_2 \;&\leq\; -\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega-m_1 s_1^2\\ &\quad +s_2\Big[-m_2 s_2-\widetilde{\delta}_2^T F_2-(\hat{\beta}_{21}|\xi_2|+\hat{\beta}_{22})sgn(s_2)\Big]\\ &\quad +\beta_{21}|\xi_2||s_2|+\beta_{22}|s_2|+\widetilde{\delta}_2^T F_2 s_2+\widetilde{\beta}_{21}|\xi_2||s_2|+\widetilde{\beta}_{22}|s_2|\\ &\leq -\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega\\ &\quad -\sum_{j=1}^{2}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega-m_1 s_1^2-m_2 s_2^2 \qquad (32)\end{aligned}$$

since $\dot{V}_2<0$, then s_2, $\widetilde{\delta}_2$, $\widetilde{\beta}_{21}$, $\widetilde{\beta}_{22}$ are all asymptotically converge to zero.

Step i: We continue to investigate the i-th new subsystem with transformation variables, that is

$$\begin{aligned}D^\alpha\xi_i &= D^\alpha x_i-D^\alpha\vartheta_{i-1}-D^\alpha\sigma_i\\ &= x_{i+1}+\delta_i^T F_i+f_i+\Delta f_i(X)+d_i(t)-D^\alpha\vartheta_{i-1}\\ &\quad -\sigma_{i+1}+c_i|\sigma_i|^\gamma\\ &= \xi_{i+1}+\vartheta_i+\delta_i^T F_i+f_i+\Delta f_i(X)+d_i(t)\end{aligned}$$

$$-D^{\alpha}\vartheta_{i-1}+c_i|\sigma_i|^{\gamma} \tag{33}$$

similar to the above steps, the frequency distributed model of adaptive estimation laws can be constructed as

$$\begin{cases} \frac{\partial z_{\widetilde{\delta}_i}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\delta}_i}(\omega,t)+F_i s_i \\ \widetilde{\delta}_i=\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\delta}_i}(\omega,t)d\omega \\ \frac{\partial z_{\widetilde{\beta}_{i1}}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\beta}_{i1}}(\omega,t)+\eta_{i1}|\xi_i||s_i| \\ \widetilde{\beta}_{i1}=\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{i1}}(\omega,t)d\omega \\ \frac{\partial z_{\widetilde{\beta}_{i2}}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\beta}_{i2}}(\omega,t)+\eta_{i2}|s_i| \\ \widetilde{\beta}_{i2}=\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{i2}}(\omega,t)d\omega \end{cases} \tag{34}$$

selecting the Lyapunov function as

$$\begin{aligned} V_i &= V_{i-1}+\frac{1}{2}s_i^2+\frac{1}{2}\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\delta}_i}^T(\omega,t)z_{\widetilde{\delta}_i}(\omega,t)d\omega \\ &\quad +\frac{1}{2\eta_{i1}}\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{i1}}^2(\omega,t)d\omega \\ &\quad +\frac{1}{2\eta_{i2}}\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{i2}}^2(\omega,t)d\omega \end{aligned} \tag{35}$$

taking the derivative of V_i, and using the deduce results of the above steps, one has

$$\begin{aligned} \dot{V}_i &\leq -\sum_{j=1}^{i}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i-1}m_j s_j^2+s_i\dot{s}_i+\widetilde{\delta}_i^T F_i s_i+\widetilde{\beta}_{i1}|\xi_i||s_i|+\widetilde{\beta}_{i2}|s_i| \end{aligned} \tag{36}$$

substituting $\dot{s}_i$ from equation (15) into (36), and considering equations (13), (17) and (33), it yields

$$\begin{aligned} \dot{V}_i &\leq -\sum_{j=1}^{i}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i-1}m_j s_j^2+s_i\Big\{D^{\alpha}\xi_i-\xi_{i+1}+\big[h_{i1}\xi_i+h_{i2}\times \\ &\quad |\xi_i|^{\gamma}sgn(\xi_i)\big]\Big\}+\widetilde{\delta}_i^T F_i s_i+\widetilde{\beta}_{i1}|\xi_i||s_i|+\widetilde{\beta}_{i2}|s_i| \\ &\leq -\sum_{j=1}^{i}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i-1}m_j s_j^2+s_i\big[-m_i s_i-\widetilde{\delta}_i^T F_i-(\hat{\beta}_{i1}|\xi_i| \\ &\quad +\hat{\beta}_{i2})sgn(s_i)\big]+\beta_{i1}|\xi_i||s_i|+\beta_{i2}|s_i|+\widetilde{\delta}_i^T F_i s_i \\ &\quad +\widetilde{\beta}_{i1}|\xi_i||s_i|+\widetilde{\beta}_{i2}|s_i| \\ &\leq -\sum_{j=1}^{i}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega-\sum_{j=1}^{i}m_j s_j^2 \end{aligned} \tag{37}$$

because of $\dot{V}_i<0$, then s_i, $\widetilde{\delta}_i$, $\widetilde{\beta}_{i1}$, $\widetilde{\beta}_{i2}$ can converge to zero asymptotically.

Step n: In the last step, the actual controller is designed. Similar to the above steps, the last subsystem with transformation variable ξ_n is determined as

$$\begin{aligned} D^{\alpha}\xi_n &= D^{\alpha}x_n-D^{\alpha}\vartheta_{n-1}-D^{\alpha}\sigma_n \\ &= k\Psi(u(t))+\delta_n^T F_n+f_n+\Delta f_n(X)+d_n(t) \\ &\quad -D^{\alpha}\vartheta_{n-1}-k\Delta u(t)+c_n|\sigma_n|^{\gamma} \\ &= ku(t)+\delta_n^T F_n+f_n+\Delta f_n(X)+d_n(t) \\ &\quad -D^{\alpha}\vartheta_{n-1}+c_n|\sigma_n|^{\gamma} \end{aligned} \tag{38}$$

the frequency distributed model of adaptive parameters estimation law is

$$\begin{cases} \frac{\partial z_{\widetilde{\delta}_n}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\delta}_n}(\omega,t)+F_n s_n \\ \widetilde{\delta}_n=\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\delta}_n}(\omega,t)d\omega \\ \frac{\partial z_{\widetilde{\beta}_{n1}}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\beta}_{n1}}(\omega,t)+\eta_{n1}|\xi_n||s_n| \\ \widetilde{\beta}_{n1}=\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{n1}}(\omega,t)d\omega \\ \frac{\partial z_{\widetilde{\beta}_{n2}}(\omega,t)}{\partial t}=-\omega z_{\widetilde{\beta}_{n2}}(\omega,t)+\eta_{n2}|s_n| \\ \widetilde{\beta}_{n2}=\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{n2}}(\omega,t)d\omega \end{cases} \tag{39}$$

selecting the Lyapunov function as

$$\begin{aligned} V_n &= V_{n-1}+\frac{1}{2}s_n^2+\frac{1}{2}\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\delta}_n}^T(\omega,t)z_{\widetilde{\delta}_n}(\omega,t)d\omega \\ &\quad +\frac{1}{2\eta_{n1}}\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{n1}}^2(\omega,t)d\omega \\ &\quad +\frac{1}{2\eta_{n2}}\int_0^{\infty}\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{n2}}^2(\omega,t)d\omega \end{aligned} \tag{40}$$

taking the derivative of V_n with respect time, and using the deduce results of the above steps, one has

$$\begin{aligned} \dot{V}_n &\leq -\sum_{j=1}^{n}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\delta}_j}^T(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{n}\frac{1}{\eta_{j1}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{n}\frac{1}{\eta_{j2}}\int_0^{\infty}\omega\mu_{\alpha}(\omega)z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega \\ &\quad -\sum_{j=1}^{n-1}m_j s_j^2+s_n\big[D^{\alpha}\xi_n+(h_{n1}\xi_n+h_{n2}|\xi_n|^{\gamma}\times \end{aligned}$$

$$
\begin{aligned}
& sgn(\xi_n))\big] + \widetilde{\delta}_n^T F_n s_n + \widetilde{\beta}_{n1}|\xi_n||s_n| + \widetilde{\beta}_{n2}|s_n| \\
= & -\sum_{j=1}^{n}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\delta}_j}^T(\omega,t) z_{\widetilde{\delta}_j}(\omega,t)d\omega \\
& -\sum_{j=1}^{n}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\
& -\sum_{j=1}^{n}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega - \sum_{j=1}^{n-1} m_j s_j^2 \\
& + s_n\Big\{ku(t) + \delta_n^T F_n + f_n + \Delta f_n(X) + d_n(t) \\
& - D^\alpha \vartheta_{n-1} + c_n|\sigma_n|^\gamma + \big[h_{n1}\xi_n + h_{n2}|\xi_n|^\gamma \times \\
& sgn(\xi_n)\big]\Big\} + \widetilde{\delta}_n^T F_n s_n + \widetilde{\beta}_{n1}|\xi_n||s_n| + \widetilde{\beta}_{n2}|s_n| \quad (41)
\end{aligned}
$$

substituting $u(t)$ from equation (19) into equation (41), we have

$$
\begin{aligned}
\dot{V}_n \le & -\sum_{j=1}^{n}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\delta}_j}^T(\omega,t) z_{\widetilde{\delta}_j}(\omega,t)d\omega \\
& -\sum_{j=1}^{n}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\
& -\sum_{j=1}^{n}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega - \sum_{j=1}^{n-1} m_j s_j^2 \\
& + s_n\Big[- m_n s_n - \widetilde{\delta}_n^T F_n - (\hat{\beta}_{n1}|\xi_n| \\
& + \hat{\beta}_{n2}) sgn(s_n)\Big] + \beta_{n1}|\xi_n||s_n| + \beta_{n2}|s_n| + \widetilde{\delta}_n^T F_n s_n \\
& + \widetilde{\beta}_{n1}|\xi_n||s_n| + \widetilde{\beta}_{n2}|s_n| \\
\le & -\sum_{j=1}^{n}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\delta}_j}^T(\omega,t) z_{\widetilde{\delta}_j}(\omega,t)d\omega \\
& -\sum_{j=1}^{n}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\beta}_{j1}}^2(\omega,t)d\omega \\
& -\sum_{j=1}^{n}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega) z_{\widetilde{\beta}_{j2}}^2(\omega,t)d\omega - \sum_{j=1}^{n} m_j s_j^2 \quad (42)
\end{aligned}
$$

because of $\dot{V}_n < 0$, so s_n, $\widetilde{\delta}_n$, $\widetilde{\beta}_{n1}$, $\widetilde{\beta}_{n2}$ will converge to zero asymptotically, that is the system trajectories can reach to sliding mode surface gradually, therefore the proof is completed.

Theorem 2. Consider the sliding mode dynamics (16), the system is stable and its state trajectories converge to zero asymptotically.

Proof. Step 1: For the first sliding mode dynamic, the corresponding frequency distributed model is

$$
\begin{aligned}
\frac{\partial z_{\xi_1}(\omega,t)}{\partial t} &= -\omega z_{\xi_1}(\omega,t) + \xi_2 - \big[h_{11}\xi_1 + h_{12}|\xi_1|^\gamma sgn(\xi_1)\big] \\
\xi_1 &= \int_0^\infty \mu_\alpha(\omega) z_{\xi_1}(\omega,t)d\omega \quad (43)
\end{aligned}
$$

selecting the following Lyapunov function

$$
W_1 = \frac{1}{2}\int_0^\infty \mu_\alpha(\omega) z_{\xi_1}^2(\omega,t)d\omega \quad (44)
$$

taking the derivation of W_1 with respect time, one has

$$
\begin{aligned}
\dot{W}_1 &= -\int_0^\infty \omega\mu_\alpha(\omega) z_{\xi_1}^2(\omega,t)d\omega + \xi_1\Big\{\xi_2 - \big[h_{11}\xi_1 \\
&\quad + h_{12}|\xi_1|^\gamma sgn(\xi_1)\big]\Big\} \\
&= -\int_0^\infty \omega\mu_\alpha(\omega) z_{\xi_1}^2(\omega,t)d\omega + \xi_1\xi_2 - h_{11}\xi_1^2 \\
&\quad - h_{12}|\xi_1|^{\gamma+1} \quad (45)
\end{aligned}
$$

according to $\xi_1\xi_2 \le \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2$, then the above equation can be rewritten as

$$
\begin{aligned}
\dot{W}_1 &\le -\int_0^\infty \omega\mu_\alpha(\omega) z_{\xi_1}^2(\omega,t)d\omega - (h_{11} - \frac{1}{2})\xi_1^2 \\
&\quad + \frac{1}{2}\xi_2^2 - h_{12}|\xi_1|^{\gamma+1} \quad (46)
\end{aligned}
$$

obviously, if $\xi_2 = 0$, then $\dot{W}_1 < 0$, according to Lemma 1, the first sliding mode dynamics is asymptotical stable, that is $\xi_1 \to 0$ as $t \to \infty$.

Step 2: According to the second sliding mode dynamics in equation (16), its frequency distributed model can be written as

$$
\begin{aligned}
\frac{\partial z_{\xi_2}(\omega,t)}{\partial t} &= -\omega z_{\xi_2}(\omega,t) + \xi_3 - \big[h_{21}\xi_2 + h_{22}|\xi_2|^\gamma sgn(\xi_2)\big] \\
\xi_2 &= \int_0^\infty \mu_\alpha(\omega) z_{\xi_2}(\omega,t)d\omega \quad (47)
\end{aligned}
$$

selecting the following Lyapunov candidate function for

$$
W_2 = W_1 + \frac{1}{2}\int_0^\infty \mu_\alpha(\omega) z_{\xi_2}^2(\omega,t)d\omega \quad (48)
$$

taking the derivation of W_2, and according to the above deduced results, we have

$$
\begin{aligned}
\dot{W}_2 &\le -\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega) z_{\xi_j}^2(\omega,t)d\omega - (h_{11} - \frac{1}{2})\xi_1^2 \\
&\quad - (h_{21} - 1)\xi_2^2 - \sum_{j=1}^{2} h_{j2}|\xi_j|^{\gamma+1} + \frac{1}{2}\xi_3^2 \quad (49)
\end{aligned}
$$

similarly, if $\xi_3 = 0$, then $\dot{W}_2 < 0$, according to Lemma 1, the second sliding mode dynamics is asymptotical stable, that is $\xi_2 \to 0$ as $t \to \infty$.

Step i: We continue to investigate the stability of the i-th sliding mode dynamics, which frequency distributed model is

$$
\begin{aligned}
\frac{\partial z_{\xi_i}(\omega,t)}{\partial t} &= -\omega z_{\xi_i}(\omega,t) + \xi_{i+1} - \big[h_{i1}\xi_i + h_{i2}|\xi_i|^\gamma sgn(\xi_i)\big] \\
\xi_i &= \int_0^\infty \mu_\alpha(\omega) z_{\xi_i}(\omega,t)d\omega \quad (50)
\end{aligned}
$$

choosing the following form Lyapunov function

$$
W_i = W_{i-1} + \frac{1}{2}\int_0^\infty \mu_\alpha(\omega) z_{\xi_i}^2(\omega,t)d\omega \quad (51)
$$

taking the time derivation of W_i, one has

$$
\begin{aligned}
\dot{W}_i &\le -\sum_{j=1}^{i}\int_0^\infty \omega\mu_\alpha(\omega) z_{\xi_j}^2(\omega,t)d\omega - (h_{11} - \frac{1}{2})\xi_1^2 \\
&\quad - \sum_{j=2}^{i}(h_{j1} - 1)\xi_j^2 - \sum_{j=1}^{i} h_{j2}|\xi_j|^{\gamma+1} + \frac{1}{2}\xi_{i+1}^2 \quad (52)
\end{aligned}
$$

indeed, when $\xi_{i+1}=0$, then ξ_i is asymptotically converge to zero, which returns to step i-1.

Step n: In the last step, the stability of the whole sliding mode dynamics is demonstrated, the corresponding frequency distributed model is

$$\begin{aligned}\frac{\partial z_{\xi_n}(\omega,t)}{\partial t} &= -\omega z_{\xi_n}(\omega,t) - \left[h_{n1}\xi_n + h_{n2}|\xi_n|^{\gamma} sgn(\xi_n)\right] \\ \xi_n &= \int_0^\infty \mu_\alpha(\omega) z_{\xi_n}(\omega,t)d\omega \end{aligned} \tag{53}$$

we select the following Lyapunov function

$$W_n = W_{n-1} + \frac{1}{2}\int_0^\infty \mu_\alpha(\omega) z_{\xi_n}^2(\omega,t)d\omega \tag{54}$$

its fractional order derivation is satisfying

$$\begin{aligned}\dot{W}_n \le & -\sum_{j=1}^{n}\int_0^\infty \omega\mu_\alpha(\omega) z_{\xi_j}^2(\omega,t)d\omega - (h_{11}-\frac{1}{2})\xi_1^2 \\ & -(h_{n1}-\frac{1}{2})\xi_n^2 - \sum_{j=2}^{n-1}(h_{j1}-1)\xi_j^2 - \sum_{j=1}^{n} h_{j2}|\xi_j|^{\gamma+1} \end{aligned} \tag{55}$$

therefore $\dot{W}_n < 0$, the whole sliding mode dynamics (16) is asymptotically stale. This completes the proof.

IV. Simulation Results

In this section, simulation results are given to demonstrate the effectiveness and feasibility of the proposed control strategy. Consider the fractional-order Arneodo system with nonlinear input, which is described as

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= x_3 \\ D^\alpha x_3 &= \Psi(u(t)) + a_1x_1 + a_2x_2 + a_3x_3 \\ &\quad + x_1^3 + \Delta f(X) + d(t) \end{aligned} \tag{56}$$

where $a_1=-5.5$, $a_2=-3.5$, $a_3=-1$, $\delta_3=[a_1,a_2,a_3]^T$, $F_3=[x_1,x_2,x_3]^T$, $f_3=x_1^3$, $\Delta f(X)=-0.02cos(2t)x_3$ and $d(t)=0.015sin(3t)$ are unmodeled dynamics and external disturbance, respectively. Considering $\Psi(u(t))$ as saturated input in this example

$$\Psi(u(t)) = \begin{cases} 4, & u(t)\ge 2 \\ 2u(t), & -2\le u(t)\le 2 \\ -4, & u(t)\le -2 \end{cases} \tag{57}$$

the parameters $m_1=m_2=m_3=15$, $c_1=c_2=c_3=2$, $\eta_{31}=10$, $\eta_{32}=5$, $\gamma=0.2$, $h_{11}=h_{21}=h_{31}=5$, $h_{12}=h_{22}=h_{32}=2$, the initial conditions are chosen as $x_1(0)=0.3$, $x_2(0)=0.5$, $x_3(0)=0.1$, $\hat{\delta}_3(0)=[0.1,0.1,0.1]^T$, $\hat{\beta}_{31}(0)=0$, $\hat{\beta}_{32}(0)=0$. When $\alpha=0.98$, the uncontrolled system (56) can behave chaotically, the strange attractors are shown in Figure 2.

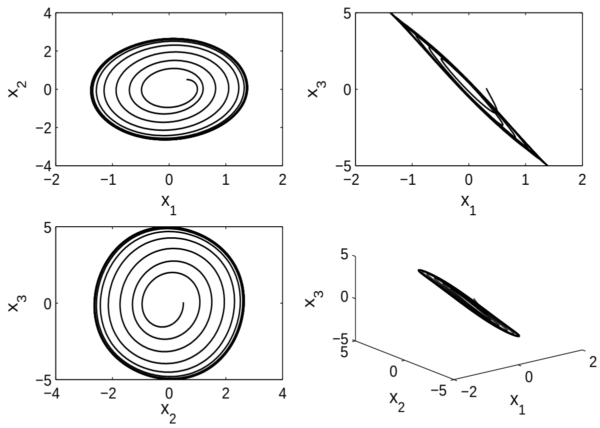

Figure 2. Strange attractors of fractional-order Arneodo system (56)

When activated the controller $u(t)$, the time response of subsystem with transformation variables are displayed in Figure 3, it is obviously that all trajectories converge to zero asymptotically, which implies that under the control of the proposed control strategy, the adaptive stabilization of the controlled system with saturated nonlinear input is realized.

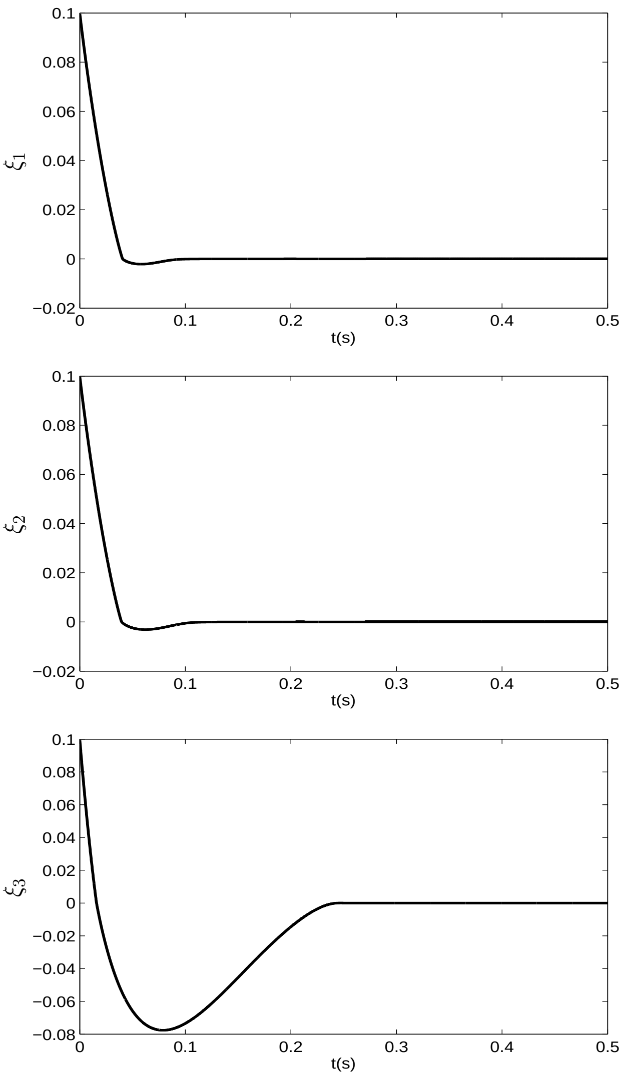

Figure 3. Time response of transformation system

All above simulation results sufficiently demonstrate that the proposed control scheme is effective in stabilizing this kinds of uncertain fractional-order nonlinear systems with unknown parameters and nonlinear inputs.

V. Conclusions

In this paper, a backstepping-based sliding mode control scheme for adaptive stabilization of a class of fractional-order system is investigated. The system is perturbed by unknown bounded external disturbance, and system parameters are unknown in advance. The effect of saturated nonlinear input is considered in the design of actual controller. For compensation the influence of nonlinear input, an auxiliary fractional-order system is construct to generate the necessary virtual signal. To deal with the unknown parameters and unknown uncertainties, a proper sliding mode surface is established to determine the adaptive estimation laws. In order to verify the stability of the controlled system, the frequency distributed model is used so that indirect Lyapunov function can be applied. Simulation results demonstrated the feasibility and effectiveness of the proposed control scheme.

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