The K-(2,1)-Total Choosability of 1-Planar Graphs without Adjacent Short Cycles

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***Abstract*—A list assignment of a graph G is a function $L : V(G) \cup E(G) \to 2^N$. A graph G is k-(2,1)-Total choosable if and only if for every list assignment L provided that $|L(x)| = k, x \in V(G) \cup E(G)$, there exists a function c that $c(x) \in L(x)$, and for all $x \in V(G) \cup E(G)$, $|c(u) - c(v)| \geq 1$ if $uv \in E(G)$, $|c(e_1) - c(e_2)| \geq 1$ if the edges e_1 and e_2 are adjacent, and $|c(u) - c(e)| \geq 2$ if the vertex u is incident to the edge e. Denote by $C^T_{(2,1)}$ the minimum k such that G is k-(2,1)-Total choosable. We use (k, k)-cycle to denote that k-cycle is adjacent to k-cycle. In this paper, we prove that if G is a 1-planar graph with $\Delta(G) \geq 12$ and without (k, k)-cycle, where $k \in \{3, 4\}$, then $C^T_{(2,1)}(G) \leq \Delta + 4$.**

***Index Terms*—L-(2,1)-total labeling, k-(2,1)-total choosable, 1-planar graph.**

I. INTRODUCTION

IN this paper, G is a finite simple graph. By $V(G)$, $E(G)$, $F(G)$, $\triangle(G)$, $\delta(G)$, we denote, respectively, the vertex set, the edge set, the face set, the maximum degree, and the minimum degree of G. Call u a k-vertex, a k^+-vertex, or a k^+-vertex, if $d(u) = k$, $d(u) \geq k$, or $d(u) \leq k$, respectively. Similarly a k-face, a k^+-face, and a k^--face are also defined. A k-cycle is a cycle of length k. We say that two cycles (or faces) are adjacent if they share at least one edge. Especially, we use (k, k)-cycle to denote that k-cycle is adjacent to k-cycle.

A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one another edge. Such a drawing that the number of crossings is as small as possible is called a 1-plane graph. Undefined notations are referred to [1].

The $(p, 1)$-Total labeling problem of graph G was proposed by Havet and Yu[4]. A graph G is said to be k - $(p, 1)$-Total labeling if and only if there is a function c from $V(G) \bigcup E(G)$ to $\{0, 1, 2, \ldots, k\}$ so that $|c(u)-c(v)| \geq 1$ if $uv \in E(G)$, $|c(e_1) - c(e_2)| \geq 1$ if the edges e_1 and e_2 are adjacent, and $|c(u) - c(e)| \geq p$ if the vertex u is incident to the edge e. The $(p, 1)$-Total labeling number of G, denoted by $\lambda^T_p(G)$, is the minimum k such that G is k-$(p, 1)$-Total labeling. Readers can refer to [3], [6], [7], [9], [10], [14] for further research.

Suppose a list assignment of a graph G is a function $L : V(G) \bigcup E(G) \to 2^N$. We say G is L-$(p, 1)$-Total labeling if there exists a $(p, 1)$-Total labeling c that $c(x) \in L(x)$ for all $x \in V(G) \bigcup E(G)$. If L is any list assignment of G such that $|L(x)| = k$ for all $x \in V(G) \bigcup E(G)$, then the function c is called a k-$(p, 1)$-Total choosable function of G with respect to L. The $(p, 1)$-Total choice number of G, denoted by $C^T_{p,1}(G)$, is the minimum k such that G has a k-$(p, 1)$-Total choosable function c. Clearly, L-$(1, 1)$-Total labeling problem of graph is the list total coloring problem of graph. It is known that there is a List Total Coloring Conjecture $\chi^{''}_l(G) = \chi^{''}(G)$, we may conjecture $C^T_{p,1}(G) = \lambda^T_p(G)+1$. Unfortunately, we found some graphs satisfying $C^T_{p,1}(G) > \lambda^T_p(G) + 1$ in[11]. So, Y. Yu[11] proposed the following "Week List $(p, 1)$-Total Labeling Conjecture".

Conjecture 1.1 ([11]) If G is a simple graph with maximum degree Δ, then $C^T_{p,1}(G) \leq \Delta + 2p$.

Y. Yu[11] showed the conjecture to be true for tree and path. Y. Yu[11] also proved the following results. (1) If G is a star graph $K_{1,n}$, where $n \geq 3$ and $p \geq 2$, then $C^T_{p,1}(G) \leq \Delta+2p-1$ (2) If G is a outerplanar graph with $\Delta(G) \geq p+3$, then $C^T_{p,1}(G) \leq \Delta + 2p - 1$. (3) If G is a graph embedded in surface with Euler characteristic ε and $\Delta(G)$ big enough, then $C^T_{p,1}(G) \leq \Delta + 2p$.

Especially, for the $(1, 1)$-Total choice number, J. Hou et al.[5] proved that if G is a planar graph with $\Delta(G) \geq 9$, then $C^T_{1,1}(G) \leq \Delta + 2$. O. Borodin et al.[2] proved that if G is a planar graph with $\Delta(G) \geq 12$, then $C^T_{1,1}(G) \leq \Delta + 1$. X. Zhang.[12] proved that if G is a 1-planar graph with $\Delta(G) \geq 21$, then $C^T_{1,1}(G) \leq \Delta+1$. For the $(2, 1)$-Total choice number of a planar graph, Y. Song and L. Sun [8] proved that (1) if G is a planar graph with $\Delta(G) \geq 7$ and 3-cycle is not adjacent to k-cycle, $k \in \{3, 4\}$, then $C^T_{2,1}(G) \leq \Delta + 4$. (2) if G is a planar graph with $\Delta(G) \geq 8$ and i-cycle is not adjacent to j-cycle, where $i, j \in \{3, 4, 5\}$, then $C^T_{2,1}(G) \leq \Delta + 3$.

In this paper, we mainly studies the $(2, 1)$-Total choice number of 1-planar graph. For Conjecture 1.1, we give some positive answers. We prove the following theorem.

Theorem 1.2 If G is a 1-planar graph with $\Delta(G) \geq 12$ and without (k, k)-cycle, where $k \in \{3, 4\}$, then $C^T_{2,1}(G) \leq \Delta + 4$.

II. PRELIMINARIES

The associated plane graph $G^\times$ of a 1-plane graph G is a new plane graph obtained by replacing all crossings of G with new 4-vertices. A vertex u of $G^\times$ is a false vertex if $u \in V(G^\times) \setminus V(G)$, and a true vertex otherwise. Any face $f \in F(G^\times)$ is false if it is incident with at least one false vertex, and true otherwise.

Lemma 2.1[13] Let G be a 1-plane graph without adjacent triangles and let $G^\times$ be its associated plane graph. For every vertex $v \in V(G)$, if $d_G(v) \geq 5$, then v is incident with at most $\lfloor \frac{4}{5} d_G(v) \rfloor$ 3-faces in $G^\times$.

Manuscript received December 3, 2021; revised February 7, 2022. This work was supported by the National Natural Science Foundation of China (Grant No.12071265) and the Natural Science Foundation of Shandong Province (Grant No. ZR2019MA032) of China.

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Lemma 2.2[13] Let G be a 1-plane graph and let $G^{\times}$ be its associated plane graph. Then the following hold:

(1) For any two false vertices u and v in $G^{\times}$, $uv \notin E(G^{\times})$.

(2) If there is a 3-face $uvwu$ in $G^{\times}$ such that $d_G(v) = 2$, then u and w are both true vertices.

(3) If $d_G(u) = 3$ and v is a false vertex in $G^{\times}$, then either $uv \notin E(G^{\times})$ or uv is not incident with two 3-faces.

(4) If a 3-vertex v in G is incident with two 3-faces and adjacent to two false vertices in $G^{\times}$, then v must also be incident with a 5^+-face.

(5) For any 4-vertex u in G, u is incident with at most three false 3-faces.

III. Structural Properties

We will give some properties of G as follows. For convenience, let $\Theta(x) \in L(x)$, where $x \in V(G) \bigcup E(G)$, be a partially $(2,1)$-Total choosable function of graph G, and the function satisfies the definition of L-$(2,1)$-Total labeling in the following sections. We denote the set of available colors of x for $x \in V(G) \bigcup E(G)$ under the partially $(2,1)$-Total choosable function $\Theta(x)$ by $A_\Theta(x)$.

Property 3.1: $\delta(G) \geq 3$.

Proof: It is similar to the proof of Property 3.1 of [8]. ■

Property 3.2: Every 3-vertex in G is adjacent to 12^+-vertex.

Proof: It is similar to the proof of Property 3.2 of [8]. ■

Property 3.3: Every 4-vertex in G is adjacent to 10^+-vertex.

Proof: Suppose that a 4-vertex u is adjacent to a 9^--vertex v. By the minimality of G, the graph $G - uv$ has a Δ+4-$(2,1)$-Total choosable function Θ. We first erase the color of the vertex u. Since $|A_\Theta(uv)| \geq \Delta + 4 - (3 + 8 + 3) \geq 2$ and $|A_\Theta(u)| \geq \Delta + 4 - (4 + 3 \times 3) \geq 3$. Let $\alpha \in A_\Theta(uv)$. If $A_\Theta(u) \neq \{\alpha-1, \alpha, \alpha+1\}$, then let $\Theta(u) \in A_\Theta(u) \setminus \{\alpha - 1, \alpha, \alpha + 1\}$ and $\Theta(uv) = \alpha$. If $A_\Theta(u) = \{\alpha - 1, \alpha, \alpha + 1\}$, then let $\Theta(u) = \beta \in A_\Theta(u) \setminus \{\alpha\}$ and $\Theta(uv) \in A_\Theta(uv) \setminus \{\beta-1, \beta, \beta+1\}$. We can recolor the vertex v and the edge vv_1, easily. Therefore, G is Δ+4-$(2,1)$-Total choosable, a contradiction. ■

Property 3.4: If a 5-vertex v in G is adjacent to a 5-vertex, then v is adjacent to four 9^+-vertices.

Proof: It is similar to the proof of Property 3.3. ■

Property 3.5: If a 5-vertex v in G is adjacent to a 5-vertex and a 6-vertex, then v is adjacent to three 9^+-vertices.

Proof: It is similar to the proof of Property 3.3. ■

IV. Proof of Theorem 1

In this section, we give the proof of our main results by discharging method.

According to Euler's formula, we get:

$$\sum_{v \in V(G^{\times})} (d_{G^{\times}}(v) - 4) + \sum_{f \in F(G^{\times})} (d_{G^{\times}}(f) - 4) = -8$$

Then, we define an initial charge ω on $V(G^{\times}) \bigcup E(G^{\times})$ by setting $\omega(x) = d_{G^{\times}}(x) - 4$ for all $x \in V(G^{\times}) \bigcup F(G^{\times})$. So, we have $\sum_{x \in V(G^{\times}) \bigcup F(G^{\times})} \omega(x) = -8$. Our aim is to obtain a new nonnegative charge $\omega'(x)$ for all $x \in V(G^{\times}) \bigcup E(G^{\times})$ by designing discharging rules and redistributing the charges, then we can get a contradiction:

$$0 \leq \sum_{x \in V(G^{\times}) \bigcup F(G^{\times})} \omega'(x) = \sum_{x \in V(G^{\times}) \bigcup F(G^{\times})} \omega(x) = -8$$

This contradiction proves the non-existence of G and completes the proof. For convenience, let $\tau(a_1 \to a_2)$ be the charges transferred from a_1 to a_2. Let $\tau(a_1 \to a_2, a_3)$ be the charges transferred from element a_1 to each of element a_2 and a_3. And, $\tau^*(a_1 \to a_2, a_3)$ be the charges transferred from element a_1 through a false vertex v to each of element a_2 and a_3.

So, we design discharging rules as follows.

$R1$. If $d_{G^{\times}}(v) \geq 8$ and f be a face that is incident with v in $G^{\times}$, then $\tau(v \to f) = \frac{d_{G^{\times}}(v) - 4}{d_{G^{\times}}(v)}$.

$R2$. If $d_{G^{\times}}(v) = 7$ and f_1, f_2 be a 3-face and a 4^+-face that is incident with v in $G^{\times}$, respectively, then $\tau(v \to f_1) = \frac{1}{2}$ and $\tau(v \to f_2) = \frac{1}{4}$.

$R3$. If $d_{G^{\times}}(v) = 6$ and f be a 3-face that is incident with v in $G^{\times}$, then $\tau(v \to f) = \frac{1}{2}$.

$R4$. If $d_{G^{\times}}(v) = 5$ and f_1 be a $(5, 9^+, F)$-face that is incident with v, and f_2 be the other 3-face that is incident with v in $G^{\times}$, then $\tau(v \to f_1) = \frac{4}{9}$ and $\tau(v \to f_2) = \frac{1}{2}$.

$R5$. If v is a true 4-vertex and f be a 3-face that is incident with v in $G^{\times}$, then $\tau(v \to f) = \frac{1}{5}$.

$R6$. Let v be a false vertex of $G^{\times}$ such that v_1v_3 crossed v_2v_4 in G at v, and let f_i with $1 \leq i \leq 4$ be the face that is incident with vv_i and vv_{i+1} in $G^{\times}$ (here v_5 is recognized as v_1).

$R6.1$ Suppose that $min\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_2)\} \geq 12$.

$R6.1.1$ Let f_1 be a 3-face. If $v_2v_3 \in E(G^{\times})$, then $\tau^*(f_1 \to f_2) = \frac{1}{3}$. If $v_1v_4 \in E(G^{\times})$, then $\tau^*(f_1 \to f_4) = \frac{1}{3}$.

$R6.1.2$ Let f_1 be a 4^+-face. If both $v_2v_3 \in E(G^{\times})$ and $v_1v_4 \in E(G^{\times})$, then $\tau^*(f_1 \to f_2, f_4, v_3, v_4) = \frac{1}{3}$. If $v_2v_3 \in E(G^{\times})$, then $\tau^*(f_1 \to f_2, v_3) = \frac{1}{3}$. If $v_1v_4 \in E(G^{\times})$, then $\tau^*(f_1 \to f_4, v_4) = \frac{1}{3}$.

$R6.2$ Suppose that $10 \leq min\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_2)\} \leq 11$.

$R6.2.1$ Let f_1 be a 3-face. If $v_2v_3 \in E(G^{\times})$, then $\tau^*(f_1 \to f_2) = \frac{1}{5}$. If $v_1v_4 \in E(G^{\times})$, then $\tau^*(f_1 \to f_4) = \frac{1}{5}$.

$R6.2.2$ Suppose f_1 is a 4^+-face, then $\tau^*(f_1 \to v_3, v_4) = \frac{1}{5}$. Especially, if both $v_2v_3 \in E(G^{\times})$ and $v_1v_4 \in E(G^{\times})$, then $\tau^*(f_1 \to f_2, f_4, v_3, v_4) = \frac{1}{5}$. If $v_2v_3 \in E(G^{\times})$, then $\tau^*(f_1 \to f_2, v_3, v_4) = \frac{1}{5}$. If $v_1v_4 \in E(G^{\times})$, then $\tau^*(f_1 \to f_4, v_3, v_4) = \frac{1}{5}$.

$R6.3$ Suppose that $min\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_2)\} = 9$.

$R6.3.1$ Let f_1 be a 3-face. If $v_2v_3 \in E(G^{\times})$, then $\tau^*(f_1 \to f_2) = \frac{1}{9}$. If $v_1v_4 \in E(G^{\times})$, then $\tau^*(f_1 \to f_4) = \frac{1}{9}$.

$R6.3.2$ Let f_1 is a 4^+-face, then $\tau^*(f_1 \to v_3, v_4) = \frac{2}{9}$.

$R6.4$ Suppose that $min\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_2)\} = 8$, and f_1 is a 4^+-face, then $\tau^*(f_1 \to v_3, v_4) = \frac{1}{4}$.

$R6.5$ Suppose that $min\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_2)\} = 7$, and f_1 is a 4^+-face. If $7 \leq max\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_2)\} \leq 11$, then $\tau^*(f_1 \to v_3, v_4) = \frac{1}{8}$. If $max\{d_{G^{\times}}(v_1), d_{G^{\times}}(v_2)\} \geq 12$ then $\tau^*(f_1 \to v_3, v_4) = \frac{5}{24}$.

$R6.6$ Let $5 \leq d_{G^{\times}}(v_1) \leq 6, d_{G^{\times}}(v_2) = 12^+, d_{G^{\times}}(v_4) = 3$, and f_1 is a 4^+-face, then $\tau^*(f_1 \to v_4) = \frac{1}{3}$.

$R6.7$ Let $5 \leq d_{G^{\times}}(v_1) \leq 6, d_{G^{\times}}(v_2) \geq 10, d_{G^{\times}}(v_4) = 4$, and f_1 is a 4^+-face, then $\tau^*(f_1 \to v_4) = \frac{1}{5}$.

$R7$ Every 3^+-face redistributes its remaining charge after applying the previous rules equitably to each of its incident true 5^--vertices.

Suppose that the vertex v on $f \in F(G^\times)$ is a false vertex. Let the false vertex v through which the face f transfers out charges in $R6$ be a transitive false vertex of the face f. Then, a transitive false vertex v on $f \in F(G^\times)$ is a false vertex such that its two neighbors u, w on f both have degrees of at least 5. If f sends out charges via a false vertex, then this false vertex must be transitive by R6. And let v^* denote a true 5^--vertex on f. The following will discuss the weight of each 3^+-face to the incidented true 5^--vertices after discharging rules.

Claim 4.1: If f is a 6^+-face and is incident with at least one 3-vertex in $G^\times$, then f sends at least $\frac{2}{3}$ to each of its incident true 5^--vertices.

Proof: Suppose $f = v_1v_2\cdots v_kv_1$ and $d_{G^\times}(v_1) = 3$. Then v_2 and v_k are neither transitive false vertex nor true 5^--vertex. Let f be incident with at most s true 5^--vertices, and t transitive false vertices, then $s + t \leq d_{G^\times}(f) - 2$. Suppose v_i is a transitive false vertex. Let $\rho^+(v_i)$ be the amount of charges that f gets from v_{i-1} and v_{i+1}. Let $\rho^-(v_i)$ be the amount of charges that f sends out via v_i. By $R6$, we have $\rho^+(v_i) - \rho^-(v_i) \geq 0$, and the worst case is $min\{d_{G^\times}(v_1), d_{G^\times}(v_2)\} = 12$. Then, $\tau(f \to v^*) \geq \frac{d(f)-4-\frac{4t}{3}+\frac{2t}{3}}{s} \geq \frac{d(f)-4-\frac{2(d(f)-2-s)}{3}}{s} \geq \frac{\frac{d(f)}{3}-\frac{8}{3}}{s} + \frac{2}{3} \geq \frac{2}{3}$, where $d_{G^\times}(f) \geq 8$.

If $d_{G^\times}(f) = 6$, then $t \leq 2$. Suppose $t = 2$, then $1 \leq s \leq 2$. So v_3 and v_5 are transitive false vertices. By $R1, R6$ and $R7$, we have $\tau(f \to v^*) \geq min\{\frac{6-4-\frac{1}{3}\times 2+\frac{2}{3}\times 2}{2}, 6 - 4 - \frac{4}{3} \times 2 + \frac{2}{3} \times 3\} > \frac{2}{3}$. Suppose $t \leq 1$, then $s \leq 3$. By $R1, R6$ and $R7$, we have $\tau(f \to v^*) \geq min\{\frac{6-4+\rho^+(v_t)-\rho^-(v_t)}{3}, \frac{6-4}{3}\} \geq \frac{2}{3}$, where v_t is a transitive false vertex. If $d_{G^\times}(f) = 7$, then the proof is similar to the $d_{G^\times}(f) = 6$. ■

Claim 4.2: If f is a 6^+-face and is incident with at least one true 4-vertex in $G^\times$, then f sends at least $\frac{2}{3}$ to each of its incident true 5^--vertices.

Proof: It is similar to the proof of Claim 4.1. ■

Claim 4.3: If f is a 7^+-face and is incident with at least one 5-vertex in $G^\times$, then f sends at least $\frac{2}{3}$ to each of its incident true 5^--vertices.

Proof: Suppose $f = v_1v_2\cdots v_kv_1$ and $d_{G^\times}(v_1) = 5$. Let f be incident with at most s true 5^--vertices, and t transitive false vertices. Case 1: If both v_2 and v_k are transitive false vertices, then $s + t \leq d_{G^\times}(f) - 2$. By $R1, R6$ and $R7$, we have $\tau(f \to v^*) \geq \frac{d_{G^\times}(f)-4+(\frac{2}{3}-\frac{1}{3})\times 2-\frac{4(t-2)}{3}+\frac{2(t-3)}{3}}{s} \geq \frac{d_{G^\times}(f)-4-\frac{2(d_{G^\times}(f)-4-s)}{3}}{s} = \frac{\frac{d_{G^\times}(f)}{3}-\frac{3}{3}}{s} + \frac{2}{3} \geq \frac{2}{3}$, where $d_{G^\times}(f) \geq 3$.

Case 2: If there is only one transitive false vertex in v_2 and v_k, say v_2, then v_3 is a 10^+-vertex and $s + t \leq d_{G^\times}(f) - 1$. By $R1, R6$ and $R7$, we have $\tau(f \to v^*) \geq \frac{d_{G^\times}(f)-4-\frac{1}{3}+\frac{2}{3}-\frac{4(t-1)}{3}+\frac{2(t-2)}{3}}{s} \geq \frac{d_{G^\times}(f)-\frac{11}{3}-\frac{2(d(f)-1-s)}{3}}{s} = \frac{\frac{d_{G^\times}(f)}{3}-\frac{9}{3}}{s} + \frac{2}{3} \geq \frac{2}{3}$, where $d_{G^\times}(f) \geq 9$. If $7 \leq d_{G^\times}(f) \leq 8$, then the proof is similar to the Claim 4.1 of $d_{G^\times}(f) = 6$.

Case 3: If neither v_2 nor v_k is transitive false vertex, then at most one of v_2 and v_k is 5-vertex by Property 3.4. Without loss of generality, we can assume v_2 is a 5-vertex. If v_3 is a transitive false vertex, then v_4 is a 12^+-vertex and $s + t \leq d_{G^\times}(f) - 2$. By $R1, R6$ and $R7$, we have $\tau(f \to v^*) \geq \frac{d_{G^\times}(f)-4-\frac{1}{3}+\frac{2}{3}-\frac{4(t-1)}{3}+\frac{2(t-2)}{3}}{s} \geq \frac{d_{G^\times}(f)-\frac{11}{3}-\frac{2(d_{G^\times}(f)-2-s)}{3}}{s} = \frac{\frac{d_{G^\times}(f)}{3}-\frac{7}{3}}{s} + \frac{2}{3} \geq \frac{2}{3}$, where $d_{G^\times}(f) \geq 7$. Otherwise, v_3 is neither transitive false vertex nor true 5^--vertex and $s + t \leq d_{G^\times}(f) - 2$. Then, by claim 4.1, $\tau(f \to v^*) \geq \frac{2}{3}$.

Case 4: If v_2 and v_k are neither transitive false vertex nor true 5^--vertex, then $s + t \leq d_{G^\times}(f) - 2$. By claim 4.1, $\tau(f \to v^*) \geq \frac{2}{3}$. ■

Claim 4.4: If f is a 6-face and is incident with at least one 5-vertex in $G^\times$, then f sends at least $\frac{1}{2}$ to each of its incident true 5^--vertices.

Proof: It is similar to the proof of Claim 4.3. ■

Claim 4.5: If f is a 5-face and is incident with at least one true 3-vertex(or 4-vertex) in $G^\times$, then f sends at least $\frac{1}{3}$ to each of its incident true 5^--vertices. Especially, if f is incident with at least two 12^+-vertex in $G^\times$, then f sends at least $\frac{2}{3}$ to each of its incident true 5^--vertices.

Proof: It is similar to the proof of Claim 4.3. ■

Claim 4.6: If $f = v_1v_2v_3v_4v_5v_1$ is a 5-face and is incident with at least one 5-vertex in $G^\times$, then f sends at least $\frac{1}{3}$ to each of its incident true 5^--vertices. Especially, if $d_{G^\times}(v_1) = 5, d_{G^\times}(v_2) = 7^+$, and v_5 is a false vertex, then f sends at least $\frac{1}{2}$ to each of its incident true 5^--vertices.

Proof: It is similar to the proof of Claim 4.3. ■

Claim 4.7: If $f = v_1v_2v_3\cdots v_kv_1$ is a 6^+-face in $G^\times$, $d_{G^\times}(v_1) = 3, d_{G^\times}(v_3) = 10^+$ and $d_{G^\times}(v_k) = 12^+$, then f sends at least 1 to each of its incident true 5^--vertices.

Proof: Suppose that $f = v_1v_2v_3\cdots v_kv_1$, $d_{G^\times}(v_1) = 3$, $d_{G^\times}(v_3) = 10^+$ and $d_{G^\times}(v_k) = 12^+$, then $s+t \leq d_{G^\times}(f) - 3$. Case 1: If $t = 0$, then $s \leq d_{G^\times}(f) - 3$. By $R1$ and $R7$, we have $\tau(f \to v^*) \geq \frac{d_{G^\times}(f)-4+\frac{2}{3}+\frac{3}{5}}{s} \geq \frac{d_{G^\times}(f)-4+\frac{2}{3}+\frac{3}{5}}{d_{G^\times}(f)-3} > 1$.

Case 2: Suppose $t = 1$, then $s \leq d_{G^\times}(f) - 4$. By $R1, R6$ and $R7$, we have $\tau(f \to v^*) \geq \frac{d_{G^\times}(f)-4}{s} \geq \frac{d_{G^\times}(f)-4}{d_{G^\times}(f)-4} = 1$.

Case 3: Suppose $t = 2$, then $s \leq d_{G^\times}(f) - 5$. Suppose that v_i, v_j, v_k and v_h, where $i \leq j \leq k \leq h$, be the neighbors of two transitive false vertices on the face f, and $\xi(f)$ be the residual charge of f after R1-R6. Let $min\{d_{G^\times}(v_i), d_{G^\times}(v_j), d_{G^\times}(v_k), d_{G^\times}(v_h)\} = q$. If $q \geq 12$, then $\xi(f) \geq d_{G^\times}(f) - 4 - \frac{4}{3} \times 2 + \frac{2}{3} \times 3 = d_{G^\times}(f) - \frac{14}{3}$ by $R1, R6.1$ and $R7$. Similarly, if $10 \leq q \leq 11$, $q = 9$, $q = 8$, $q = 7$ and $5 \leq q \leq 6$, then $\xi(f) \geq d_{G^\times}(f) - \frac{14}{3}$ by $R1 - R7$. So, $\tau(f \to v^*) \geq \frac{d_{G^\times}(f)-\frac{14}{3}}{s} \geq \frac{d(f)-\frac{14}{3}}{d_{G^\times}(f)-5} > 1$.

Case 4: Suppose that $t \geq 3$, then $s \leq d_{G^\times}(f) - 6$. By $R1, R6$ and $R7$, we have $\tau(f \to v^*) \geq \frac{d_{G^\times}(f)-4-\frac{4t}{3}+\frac{2t}{3}}{s} \geq \frac{d_{G^\times}(f)-4-\frac{2(d_{G^\times}(f)-3-s)}{3}}{s} \geq \frac{d_{G^\times}(f)-6}{3(d_{G^\times}(f)-6)} + \frac{2}{3} \geq 1$. ■

Checking $\omega'(x) \geq 0$ for $x \in V(G) \bigcup F(G)$. Firstly, we check all the vertices in $V(G)$. Among the neighbors of true k-vertex v of G, the neighbor with the smallest degree is $v_{1'}$. Then denote by $v_{1'}, v_{2'}, \cdots, v_{k'}$ the neighbors of v in G that lie consecutively around v. Similarly, we denote by $v_1, v_2, \cdots, v_k$ the neighbors of v in $G^\times$ that lie consecutively around v, where $d_{G^\times}(v_i) = 4$ or $d_{G^\times}(v_i) = d_G(v_{i'})$ for $i = 1, 2, \cdots, k$. And denote by f_i the face that is incident with vv_i and vv_{i+1} in $G^\times$. If f_i is a false 3-face that is incident with v_iv_{i+1}, then the face adjacent to v_iv_{i+1} in $G^\times$ that is different from f_i is denoted by h_i.(the subscript is taken by modular k). These notations will be used in the proof of the next propositions without explaining their meanings again.

(1)$d_{G^\times}(v)=3$.

By Lemma 2.2, v is incident with at most two 3-faces.

Case 1: Suppose that v is not incident with any 3-faces.

Case 1.1: Suppose v is incident with at least one 6^+-face and one 5^+-face in $G^\times$, then $\omega'(v)\geq -1+\frac{2}{3}+\frac{1}{3}=0$ by claim 4.1 and claim 4.5.

Let v be incident with one 6^+-face, say f_1, and two 4-faces $f_2=vv_2u_2v_3$ and $f_3=vv_3u_3v_1$. If v_1 or v_2 is true, say v_1, then $\tau(f_3\rightarrow v)\geq min\{\frac{2}{3}+\frac{2}{3}-\frac{2}{3},\frac{\frac{2}{3}}{2}\}=\frac{1}{3}$ by R1, R6 and R7. If v_3 is true vertex, then $\tau(f_2\rightarrow v)\geq\frac{1}{3}$ and $\tau(f_3\rightarrow v)\geq\frac{1}{3}$ by R1, R6 and R7. Thus, $\omega'(v)\geq -1+\frac{1}{3}+\frac{2}{3}=0$. Otherwise, v_1,v_2 and v_3 are all false vertices. If u_2 or u_3 is a true 8^+-vertex, say u_2, $\tau(f_2\rightarrow v)\geq\frac{1}{2}$ by R1 and R7. Thus, $\omega'(v)\geq -1+\frac{1}{2}+\frac{2}{3}>0$. Otherwise, $5\leq d_{G^\times}(u_2),d_{G^\times}(u_3)\leq 7$ by property 3.1, property 3.2 and property 3.3. The face incident to $v_2u_2(u_2v_3)$ in $G^\times$ that is different from f_2 is denoted by $k_1(k_2)$. The face incident to $v_1u_3(u_3v_3)$ in $G^\times$ that is different from f_3 is denoted by $k_3(k_4)$. Since G doesn't have $(4,4)$-cycle, so at least one of k_1, k_2, k_3 and k_4 is a 4^+-face. We can assume k_1 is a 4^+-face. If $d_{G^\times}(u_2)=7$, then $\tau^*(k_1\rightarrow v)\geq\frac{5}{24}$ by R6.5, and $\tau(f_2\rightarrow v)\geq\frac{1}{4}$ by R2 and R7. Thus, $\omega'(v)\geq -1+\frac{5}{24}+\frac{1}{4}+\frac{2}{3}>0$. If $d_{G^\times}(u_2)\neq 7$, then $\tau^*(k_1\rightarrow v)\geq\frac{1}{3}$ by R6.6. Thus, $\omega'(v)\geq -1+\frac{1}{3}+\frac{2}{3}=0$.

Case 1.2: Suppose v is not incident with 6^+-face and is at least incident with one 5-face.

Case 1.2.1: If v is incident with three 5-faces, then $\omega'(v)\geq -1+\frac{1}{3}\times 3=0$ by claim 4.5.

Case 1.2.2: If v is incident with two 5-faces, then we can assume $f_1=vv_1w_1w_2v_2v$, $f_2=vv_2u_1u_2v_3v$ are 5-faces and $f_3=vv_3z_3v_1v$ is a 4-face. Suppose there is at least one true vertex in v_1, v_2 and v_3. If v_1 or v_3 is true, then by the symmetry, assume that v_1 is true. Since G doesn't have $(4,4)$-cycle, so $\tau(f_3\rightarrow v)\geq min\{\frac{2}{3}+\frac{2}{3}-\frac{2}{3},\frac{\frac{2}{3}}{2}\}=\frac{1}{3}$. By Claim 4.5, $\tau(f_2\rightarrow v)\geq\frac{1}{3}$ and $\tau(f_2\rightarrow v)\geq\frac{1}{3}$. Thus, $\omega'(v)\geq -1+\frac{1}{3}\times 3=0$. If v_2 is true, then $\tau(f_1\rightarrow v)\geq min\{\frac{1+\frac{2}{3}-\frac{1}{3}}{2},\frac{1+\frac{2}{3}}{3}\}=\frac{5}{9}$ by R1, R6 and R7. Similarly, $\tau(f_2\rightarrow v)\geq\frac{5}{9}$. Thus, $\omega'(v)\geq -1+\frac{5}{9}+\frac{5}{9}>0$.

Otherwise, v_1, v_2 and v_3 are all false vertices. Then there are at most three true 5^--vertices in w_1,w_2,u_1 and u_2 by Property 3.2, Property 3.3 and Property 3.4. Suppose there are three true 5^--vertices in w_1,w_2,u_1 and u_2, without loss of generality, then we can assume w_2 is not a true 5^--vertex and both u_1 and u_2 are 5-vertices. So, z_3 is a 9^+-vertex by Property 3.4. Then, $\tau(f_1\rightarrow v)\geq\frac{1}{2}$ and $\tau(f_3\rightarrow v)\geq\frac{5}{9}$ by R1 and R7. Thus, $\omega'(v)\geq -1+\frac{1}{2}+\frac{5}{9}>0$. Suppose there are at most two true 5^--vertices in w_1,w_2,u_1 and u_2. We only consider w_1,w_2, w_1,u_1 or w_1,u_2 are true 5^--vertices by the symmetry. If w_1,w_2 are true 5^--vertices, then $\tau(f_2\rightarrow v)\geq 1$ by R7. Thus, $\omega'(v)\geq -1+1=0$. If w_1,u_1 (or w_1,u_2) are true 5^--vertices, then $\tau(f_1\rightarrow v)\geq\frac{1+\frac{2}{3}-\frac{1}{3}}{2}=\frac{2}{3}$ and $\tau(f_2\rightarrow v)\geq\frac{1}{3}$. Thus, $\omega'(v)\geq -1+\frac{2}{3}+\frac{1}{3}=0$.

Case 1.2.3: If v is only incident with one 5-face, then we can assume $f_1=vv_1w_1w_2v_2v$ is a 5-face, $f_2=vv_2z_2v_3v$ and $f_3=vv_3z_3v_1v$ are 4-faces. Suppose there is at least one true vertex in v_1, v_2 and v_3. If v_3 is true, then $\tau(f_3\rightarrow v)\geq min\{\frac{2}{3}+\frac{2}{3}-\frac{2}{3},\frac{\frac{2}{3}}{2}\}=\frac{1}{3},\tau(f_2\rightarrow v)\geq\frac{1}{3}$ and $\tau(f_1\rightarrow v)\geq\frac{1}{3}$ by R1, R6, R7 and Claim 4.5. Thus, $\omega'(v)\geq -1+\frac{1}{3}\times 3=0$.

If v_1 or v_2 is true, say v_1, then f_3 is a $(3,F,3^+,12^+)$-face. By $R1$ and $R7$, $\tau(f_3\rightarrow v)\geq\frac{1}{3}$. Suppose both w_1 and w_2 are true 5^--vertices. By $R1$ and $R7$, $\tau(f_1\rightarrow v)\geq\frac{1}{3}$. If v_2 is true, then $\tau(f_2\rightarrow v)\geq\frac{1}{3}$ by $R1$ and $R7$. If v_2 is false, then z_2 is a 9^+-vertex by Property 3.4. By $R1$ and $R7$, $\tau(f_2\rightarrow v)\geq\frac{5}{9}$. Thus, $\omega'(v)\geq -1+\frac{1}{3}\times 3=0$. Suppose there is at most one true 5^--vertex in w_1 and w_2. If f_1 is incident with transitive false vertex, then $\tau(f_1\rightarrow v)\geq min\{1+\frac{2}{3}\times 2-\frac{4}{3},\frac{1+\frac{2}{3}-\frac{1}{3}}{2}\}=\frac{2}{3}$ by $R1,R6.2,R6,6$ and $R7$. If f_1 is not incident with transitive false vertex, then $\tau(f_1\rightarrow v)\geq\frac{1+\frac{2}{3}}{2}>\frac{2}{3}$ by $R1,R7$. By $R1,R6$ and $R7$, $\tau(f_3\rightarrow v)\geq\frac{1}{3}$. Thus, $\omega'(v)\geq -1+\frac{2}{3}+\frac{1}{3}=0$.

Otherwise, v_1, v_2 and v_3 are all false. If w_1 and w_2 are true 5^--vertices, then z_2 and z_3 are 9^+-vertices by Property 3.4. By R1 and R7, $\tau(f_2\rightarrow v)\geq\frac{5}{9}$, $\tau(f_3\rightarrow v)\geq\frac{5}{9}$. Thus, $\omega'(v)\geq -1+\frac{5}{9}+\frac{5}{9}>0$. If w_1 and w_2 are 6^+-vertices, then $\tau(f_1\rightarrow v)\geq 1$ by R1 and R7. Otherwise, we can assume w_2 is a 6^+-vertices and w_1 is a true 5^--vertex by the symmetry. If w_2 is a 6-vertex, then z_3 is a 9^+-vertices by Property 3.5, then $\tau(f_3\rightarrow v)\geq\frac{5}{9}$. And $\tau(f_1\rightarrow v)\geq\frac{1+\frac{2}{3}}{2}=\frac{5}{6}$ by R1 and R7. Thus, $\omega'(v)\geq -1+\frac{5}{6}+\frac{5}{9}>0$. If w_2 is a 7^+-vertex, then $\tau(f_1\rightarrow v)\geq\frac{1+\frac{2}{3}+\frac{1}{4}}{2}=\frac{23}{24}$ by R1 and R7. Since $z_2z_3\in E(G)$, $w_1z_3\in E(G)$, and w_1 is a true 5^--vertex, then there is at least one 7^+-vertex in z_2 and z_3. Thus, $\omega'(v)\geq -1+\frac{23}{24}+\frac{1}{4}>0$ by R1 and R7.

Case 1.3: Suppose $f_1=vv_1u_1v_2v$, $f_2=vv_2u_2v_3v$ and $f_3=vv_3u_3v_1v$ are all 4-faces.

Case 1.3.1: Suppose there is at least two true vertices in v_1, v_2 and v_3, say v_1 and v_2, then $\tau(f_1\rightarrow v)\geq\frac{2}{3}$ and $\tau(f_2\rightarrow v)\geq\frac{1}{3}$ by R1 and R7. Thus, $\omega'(v)\geq -1+\frac{2}{3}+\frac{1}{3}=0$.

Case 1.3.2: If there is only one true vertex in v_1, v_2 and v_3, say v_1, then u_1, u_2 and u_3 are all true vertices. Suppose there is at most one true 5^--vertex in u_1 and u_3, say u_1, then $\tau(f_3\rightarrow v)\geq\frac{2}{3}$ and $\tau(f_1\rightarrow v)\geq\frac{1}{3}$ by R1, R6 and R7. Thus, $\omega'(v)\geq -1+\frac{2}{3}+\frac{1}{3}=0$. Suppose there is at most one true 4^--vertex in u_1 and u_3, say u_1, then u_2 is a 10^+-vertex. By $R1$ and $R7$, $\tau(f_2\rightarrow v)\geq\frac{3}{5}$, $\tau(f_1\rightarrow v)\geq\frac{1}{3}$ and $\tau(f_3\rightarrow v)\geq\frac{1}{3}$. Thus, $\omega'(v)\geq -1+\frac{3}{5}+\frac{1}{3}\times 2>0$. Otherwise, both u_1 and u_3 are 5-vertices. The face incident to v_2u_1 in $G^\times$ that is different from f_1 is denoted by k_1. Since G doesn't have $(4,4)$-cycle, so k_1 is a 4^+-face. By $R6.6$, $\tau^*(k_1\rightarrow v)\geq\frac{1}{3}$. By $R1$ and $R6$, $\tau(f_1\rightarrow v)\geq\frac{1}{3}$ and $\tau(f_3\rightarrow v)\geq\frac{1}{3}$. Thus, $\omega'(v)\geq -1+\frac{1}{3}\times 3=0$.

Case 1.3.3: Suppose v_1, v_2 and v_3 are all false vertices, then u_1, u_2 and u_3 are all true vertices. If there is at least one true 4^--vertex in u_1,u_2 and u_3, then $\omega'(v)\geq -1+\frac{3}{5}+\frac{3}{5}>0$ by Property 3.1.2, Property 3.1.3, R1 and R7.

Otherwise, u_1,u_2 and u_3 are all 5^+-vertices. Suppose that u_1,u_2 and u_3 are all 5-vertex or 6-vertex. The face incident to u_1u_3 in $G^\times$ that is different from $f_1(f_3)$ is denoted by $k_1(k_2)$. The face incident to u_3u_2 in $G^\times$ that is different from $f_3(f_2)$ is denoted by $k_3(k_4)$. The face incident to u_1u_2 in $G^\times$ that is different from $f_2(f_1)$ is denoted by $k_5(k_6)$. Since G doesn't have $(3,3)$-cycle, so at least three 4^+-faces in k_1, k_2, k_3, k_4, k_5 and k_6. By R6.6, $\tau^*(k_i\rightarrow v)\geq\frac{1}{3}$, where k_i is a 4^+-face. Then, $\omega'(v)\geq -1+\frac{1}{3}\times 3=0$.

Suppose there is at least two 7^+-vertex in u_1,u_2 and u_3, say u_1 and u_2, then u_3 is a 5-vertex or 6-vertex. If u_1 and u_2 are all 12^+-vertices, then $\tau(f_1\rightarrow v)\geq\frac{2}{3}$ and $\tau(f_2\rightarrow v)\geq\frac{2}{3}$ by $R1,R2$ and $R7$. If u_1 or u_2 is a 12^+-vertex, say u_1, then $\tau(f_1\rightarrow v)\geq\frac{2}{3}$ and $\tau(f_2\rightarrow v)\geq\frac{1}{4}$ by

$R1, R2$ and $R7$. Since G doesn't have $(3,3)$-cycle, so at least one 4^+-face in k_3 and k_4, say k_3. By $R6.2-R6.6$, $\tau^*(k_3 \to v) \geq \frac{1}{5}$. Thus, $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{5}+\frac{1}{4} > 0$. If $7 \leq d_{G^\times}(u_1) \leq 11, 7 \leq d_{G^\times}(u_2) \leq 11$, then $\tau^*(k_i \to v) \geq \frac{1}{5}$ by $R6.2-R6.6$, where $i = 1,2,3,4,5,6$ and k_i is a 4^+-face. By $R1, R2$ and $R7$, $\tau(f_1 \to v) \geq \frac{1}{4}$ and $\tau(f_2 \to v) \geq \frac{1}{4}$. Thus, $\omega'(v) \geq -1+\frac{1}{5}\times 3+\frac{1}{4}\times 2 > 0$.

Suppose there is only one 7^+-vertex in u_1,u_2 and u_3, say u_1, then u_2 and u_3 are 5-vertices or 6-vertices. If $d_{G^\times}(u_1)$ ≥ 12, then $\tau(f_1 \to v) \geq \frac{2}{3}$ by $R1$ and $R7$. Since G doesn't have $(3,3)$-cycle, so at least one 4^+-face in k_3 and k_4, say k_3. By $R6.6$, $\tau^*(k_3 \to v) \geq \frac{1}{3}$. Thus, $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{3} = 0$. If $10 \leq d_{G^\times}(u_1) \leq 11$, then $\tau(f_1 \to v) \geq \frac{3}{5}$ by $R1, R7$. By $R6.2$ and $R6.6$, $\tau^*(k_i \to v) \geq \frac{1}{5}$, where k_i is a 4^+-face. Thus, $\omega'(v) \geq -1+\frac{3}{5}+\frac{1}{5}\times 3 = 0$. If $7 \leq d_{G^\times}(u_1) \leq 9$, then $\tau(f_1 \to v) \geq \frac{1}{4}$ by $R2$ and $R7$. By $R6.3-R6.6$, $\tau^*(k_i \to v) \geq \frac{5}{24}$, where $i = 1,2,5,6$ and k_i is a 4^+-face. By $R6.6$, $\tau^*(k_i \to v) \geq \frac{1}{3}$, where $i = 3,4$ and k_i is a 4^+-face. Thus, $\omega' \geq -1+\frac{5}{25}\times 2+\frac{1}{4}+\frac{1}{3} = 0$.

Case 2: Suppose v is incident with one 3-face, say f_1.

Case 2.1: Suppose that f_1 is a true 3-face. $\tau(f_1 \to v) \geq \frac{1}{3}$ by R1,R7. If $d_{G^\times}(f_2) = 4$, then $\tau(f_2 \to v) \geq min\{\frac{2}{3}+\frac{2}{3}-\frac{2}{3}, \frac{\frac{2}{3}}{2}\} = \frac{1}{3}$ by R1, R6 and R7. If $d_{G^\times}(f_2) \geq 5$, then $\tau(f_2 \to v) \geq \frac{1}{3}$. f_3 is similar to f_2. Thus, $\omega'(v) \geq -1+\frac{1}{3}\times 3 = 0$.

Case 2.2: Suppose that f_1 is a false 3-face, then by the symmetry, assume that v_1 is false and v_2 is true.

Case 2.2.1: Suppose v_3 is a true vertex, then $\tau(f_2 \to v) \geq \frac{2}{3}$ and $\tau(f_3 \to v) \geq \frac{1}{3}$ by Claim 4.1, Claim 4.5, R1, R6 and R7. Thus, $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{3} = 0$.

Case 2.2.2: Suppose v_3 is a false vertex.

(a)Suppose $d_{G^\times}(f_2) \geq 6$, then $\tau(f_2 \to v) \geq \frac{2}{3}$ by Claim 4.1. If $d_{G^\times}(f_3) \geq 5$, then $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{3} = 0$ by Claim 4.1 and Claim 4.5. If $d_{G^\times}(f_3) = 4$, then let $f_3 = vv_3u_3v_1v$. Suppose $d_{G^\times}(u_3) \leq 4$, then $\tau(f_2 \to v) \geq 1$ by Claim 4.7. Suppose $5 \leq d_{G^\times}(u_3) \leq 6$, then the face incident to v_2u_3 in $G^\times$ that is different from $f_1(f_2)$ is denoted by $h_1(k_1)$. Since G doesn't have $(3,3)$-cycle, so at least one 4^+-face in h_1 and k_1. By R6.1 and R6.6, $\tau^*(h_1 \to v) \geq \frac{1}{3}$ or $\tau^*(k_1 \to v) \geq \frac{1}{3}$. Thus, $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{3} = 0$ by Claim 4.1. Suppose $d_{G^\times}(u_3) = 7$, then $\tau^*(h_1 \to v) \geq \frac{1}{3}$ or $\tau^*(k_1 \to v) \geq \frac{5}{24}$ by R6.5 and R6.6. Thus, $\omega'(v) \geq -1+\frac{2}{3}+\frac{5}{24}+\frac{1}{4} > 0$ by Claim 4.1, R1 and R7. Suppose $d_{G^\times}(u_3) \geq 8$, then $\tau(f_3 \to v) \geq \frac{1}{2}$ by R1 and R7. Thus, $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{2} > 0$.

(b)Let $f_2 = vv_2w_1w_2v_3v$. If f_2 is at most incident with two true 5^--vertices, then $\tau(f_2 \to v) \geq \frac{1+\frac{2}{3}-\frac{1}{3}}{2} = \frac{2}{3}$ by R1, R6 and R7. Thus, $\omega'(v) \geq 0$ by Case 2.2.2(a). If f_2 is incident with three true 5^--vertices, then $d_{G^\times}(w_1) = d_{G^\times}(w_2) = 5$. And f_3 is incident with at least one 9^+-vertex. By R1, R6 and R7, $\tau(f_2 \to v) \geq \frac{1+\frac{2}{3}}{3} = \frac{5}{9}$ and $\tau(f_3 \to v) \geq min\{\frac{2}{3}, \frac{1+\frac{5}{9}}{2}, \frac{5}{9}\} = \frac{5}{9}$. Thus, $\omega'(v) \geq -1+\frac{5}{9}+\frac{5}{9} > 0$.

(c)If $f_2 = vv_2u_2v_3v$ is a 4-face, then $\tau(f_2 \to v) \geq \frac{\frac{2}{3}}{2} = \frac{1}{3}$. If f_3 is a 6^+-face, then $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{3} = 0$. If f_3 is a 5-face, then let $f_3 = vv_3z_1z_2v_1v$ and the face incident to u_2z_1 in $G^\times$ that is different from f_2 is denoted by k_2. Since G doesn't have $(4,4)$-cycle, so k_2 is a 4^+-face in $G^\times$. Suppose $d_{G^\times}(u_2) \leq 4$, then $d(z_1) \geq 10$. By R1, R6 and R7, $\tau(f_3 \to v) \geq \frac{1+\frac{3}{5}}{2} = \frac{4}{5}$. Thus, $\omega'(v) \geq -1+\frac{4}{5}+\frac{1}{3} > 0$. Suppose $5 \leq d_{G^\times}(u_2) \leq 6$, then $\tau^*(k_2 \to v) \geq \frac{1}{3}$ by R6.6. By Claim 4.5, R1 and R7, $\tau(f_3 \to v) \geq \frac{1}{3}$ and $\tau(f_2 \to v) \geq \frac{1}{3}$. Thus, $\omega'(v) \geq -1+\frac{1}{3}\times 3 = 0$. Suppose $d_{G^\times}(u_2) \geq 7$, then $\tau(f_2 \to v) \geq \frac{2}{3}+\frac{1}{4}$ by R1 and R7. Thus, $\omega'(v) \geq -1+\frac{1}{3}+\frac{2}{3}+\frac{1}{4} > 0$.

If f_3 is a 4-face, then let $f_3 = vv_3u_3v_1v$. Similarly, the face incident to u_2u_3 in $G^\times$ that is different from $f_2(f_3)$ is denoted by $k_2(k_3)$. Suppose there is at least one 8^+-vertex in u_2 and u_3. If $d_{G^\times}(u_2) \geq 8$, then $\omega'(v) \geq -1+\frac{1}{2}+\frac{2}{3} > 0$. If $d_{G^\times}(u_3) \geq 8$, then $\tau^*(h_1 \to v) \geq \frac{1}{3}$ or $\tau^*(k_1 \to v) \geq \frac{1}{5}$ by R6, where h_1 or k_1 is a 4^+-face. Thus, $\omega'(v) \geq -1+\frac{1}{3}+\frac{1}{2}+\frac{1}{5} > 0$. Otherwise, $5 \leq d_{G^\times}(u_2) \leq 7$ and $5 \leq d_{G^\times}(u_3) \leq 7$. If $d_{G^\times}(u_2) = 7$, then $\omega'(v) \geq -1+\frac{2}{3}+\frac{1}{4}+\frac{5}{24}\times 2 > 0$. Since there are at least two 4^+-faces in k_1, k_2, k_3 and h_1, then each of 4^+-face sends at least $\frac{5}{24}$ to v by R6.1 and R6.6. If $5 \leq d_{G^\times}(u_2) \leq 6$ and $5 \leq d_{G^\times}(u_3) \leq 6$, then $\omega'(v) \geq -1+\frac{1}{3}+\frac{1}{3}\times 2 = 0$. If $5 \leq d_{G^\times}(u_2) \leq 6$ and $d_{G^\times}(u_3) = 7$, then $\omega'(v) \geq -1+\frac{1}{3}+\frac{1}{4}+\frac{5}{24}\times 2 = 0$ by R6.1 and R6.6.

Case 3: Suppose that v is incident with two 3-faces, then we can assume f_1 and f_2 are 3-faces.

Case 3.1: If f_1 or f_2 is true, say f_1, then f_2 is false 3-face and $\tau(f_1 \to v) \geq \frac{1}{3}$ by R1 and R7. Since G doesn't have $(3,3)$-cycle, so f_3 is a 5^+-face and h_2 is a 4^+-face. By R6.1, Claim 4.1 and Claim 4.5, $\tau^*(h_2 \to v) \geq \frac{1}{3}$ and $\tau(f_3 \to v) \geq \frac{1}{3}$. Thus, $\omega'(v) \geq -1+\frac{1}{3}\times 3 = 0$.

Case 3.2: If both f_1 and f_2 are all false, then f_3 is a 5^+-face and v is incident with two false vertices by Lemma 2.2(3)(4). Without loss of generality, we can assume that v_1 and v_3 are false. Since G doesn't have $(3,3)$-cycle, so there is at least one 4^+-face in h_1 and h_2. By the symmetry, assume that h_1 is a 4^+-face. Then, $\tau^*(h_1 \to v) \geq \frac{1}{3}$ by R6.1. If f_3 is a 6^+-face, then $\omega'(v) \geq -1+\frac{1}{3}+\frac{2}{3} = 0$ by Claim 4.1. Otherwise, $f_3 = vv_3z_1z_2v_1v$ is a 5-face. The face incident to $v_3z_1(v_1z_2)$ in $G^\times$ that is different from f_3 is denoted by $k_1(k_2)$. Since G doesn't have $(4,4)$-cycle, so there is at least one 4^+-face in k_1 and k_2. Without loss of generality, we can assume that k_1 is a 4^+-face. If $d_{G^\times}(z_1) \leq 4$, then $\tau(f_3 \to v) \geq \frac{1+\frac{3}{5}}{2} = \frac{4}{5}$ by R1, R7 and Property 3.1.3. Thus, $\omega'(v) \geq -1+\frac{1}{3}+\frac{4}{5} > 0$. If $5 \leq d_{G^\times}(z_1) \leq 6$, then $\tau(f_3 \to v) \geq \frac{1}{3}$ and $\tau^*(k_1 \to v) \geq \frac{1}{3}$ by R6.6 and Claim 4.5. Thus, $\omega'(v) \geq -1+\frac{1}{3}\times 3 = 0$. If $7 \leq d_{G^\times}(z_1) \leq 11$, then $\tau^*(k_1 \to v) \geq \frac{1}{5}$ and $\tau(f_3 \to v) \geq \frac{1+\frac{1}{4}}{2} = \frac{5}{8}$ by R6, R1 and R7. Thus, $\omega'(v) \geq -1+\frac{1}{3}+\frac{1}{5}+\frac{5}{8} > 0$. If $d_{G^\times}(z_1) \geq 12$, then $\tau(f_3 \to v) \geq \frac{1+\frac{2}{3}}{2} = \frac{5}{6}$ by R1 and R7. Thus, $\omega'(v) \geq -1+\frac{1}{3}+\frac{5}{6} > 0$.

(2) $d_{G^\times}(v) = 4$.

If v is a false vertex or is not incident with any 3-face, then $\omega'(v) \geq 0$ by discharging rules. So v is a true vertex and is incident with at most three 3-faces by Lemma 2.2.

Case 1: Suppose that v is only incident with one 3-face, say f_1. If f_1 is a true 3-face, then $\tau(f_1 \to v) \geq -1+\frac{3}{5}+\frac{3}{5}+\frac{1}{5} = \frac{2}{5}$ by R1 and R7. If f_1 is a false 3-face, say v_1 is false vertex and v_2 is true vertex, then $\tau(f_2 \to v) \geq min\{\frac{1}{3}, \frac{\frac{3}{5}}{2}, \frac{3}{5}\times 2-\frac{3}{5}\} = \frac{3}{10}$ by Claim 4.2, Claim 4.5, R1, R6.2 and R7. Thus, $\omega'(v) \geq 0+\frac{3}{10}-\frac{1}{5} > 0$ by R4.

Case 2: Suppose that v is incident with two 3-faces.

Case 2.1: If v is incident with at least one true 3-face, then $\omega'(v) \geq 0+\frac{2}{5}-\frac{1}{5}\times 2 = 0$ by R1, R5 and R7.

Case 2.2: If v is incident with two false 3-faces.

Case 2.2.1: Suppose the two false 3-faces are adjacent, say f_1 and f_2. If v_2 is false, then both h_1 and h_2 are 4^+-face. Thus, $\omega'(v) \geq 0+\frac{1}{5}\times 2-\frac{1}{5}\times 2 = 0$ by R6.2 and

R5. If v_2 is true, then h_1 or h_2 is a 4^+-face, say h_1. By R6.2, $\tau^*(h_1 \to v) \geq \frac{1}{5}$. Suppose v_4 is true. Since G doesn't have $(4,4)$-cycle, so f_3 or f_4 is a 5^+-face. Thus, $\omega'(v) \geq 0 + \frac{1}{5} + \frac{1}{3} - \frac{1}{5} \times 2 > 0$ by Claim 4.2, Claim 4.5 and R5. Suppose v_4 is false. If f_3 or f_4 is a 5^+-face, say f_3, then $\tau(f_3 \to v) \geq \frac{1}{3}$ by Claim 4.2 and Claim 4.5. Thus, $\omega'(v) \geq 0 + \frac{1}{5} + \frac{1}{3} - \frac{1}{5} \times 2 > 0$. If both f_3 and f_4 are 4-faces, then let $f_3 = vv_3u_3v_4v$ and $f_4 = vv_4u_4v_1v$. If there is at least one 7^+-vertex in u_3 and u_4, say u_3, then $\tau(f_3 \to v) \geq \frac{1}{4}$ by R2 and R7. Thus, $\omega'(v) \geq 0 + \frac{1}{5} + \frac{1}{4} - \frac{1}{5} \times 2 > 0$. Otherwise, $5 \leq d(u_3) \leq 6$ and $5 \leq d(u_4) \leq 6$. The face incident to v_3u_3 (v_1u_4) in $G^\times$ that is different from $f_3(f_4)$ is denoted by $k_3(k_4)$. Since G doesn't have $(4,4)$-cycle, so at least three 4^+-faces in k_3 and k_4. Without loss of generality, we can assume that k_3 is 4^+-face, then $\tau^*(k_3 \to v) \geq \frac{1}{5}$ by R6.7. Thus, $\omega'(v) \geq 0 + \frac{1}{5} \times 2 - \frac{1}{5} \times 2 = 0$.

Case 2.2.2: Suppose the two false 3-faces are not adjacent, say f_1 and f_3. If v_1 and v_3 are false, then $\tau(f_2 \to v) \geq min\{\frac{1}{3}, \frac{\frac{3}{5}}{2}, \frac{3}{5} \times 2 - \frac{3}{5}\} = \frac{3}{10}$ and $\tau(f_4 \to v) \geq \frac{3}{10}$ by Claim 4.2, Claim 4.5, R1, R6.2 and R7. Thus, $\omega'(v) \geq 0 + \frac{3}{10} \times 2 - \frac{1}{5} \times 2 > 0$. If v_1 and v_4 are false, then $\tau(f_2 \to v) \geq min\{\frac{2}{3}, \frac{1+\frac{3}{5}\times 2}{3}, \frac{1+\frac{3}{5}\times 2 - \frac{1}{5}}{2}\} = \frac{11}{15}$ by R1, R6, R7 and Claim 4.2, where f_2 is a 5^+-face. If f_2 is a 4-face, then $\tau(f_2 \to v) \geq min\{\frac{\frac{3}{5}\times 2}{2}, \frac{3}{5} \times 2 - \frac{3}{5}\} = \frac{3}{5}$ by R1, R6 and R7. Thus, $\omega'(v) \geq \frac{3}{5} - \frac{1}{5} \times 2 > 0$.

Case 3: If v is incident with three false 3-faces, then v is incident with at most one true 3-face, say f_1. Without loss of generality, we can assume that f_i and f_j are false 3-faces, where $i, j \in \{2,3,4\}$ Since G doesn't have $(4,4)$-cycle, so h_i and h_j are 4^+-faces. Thus, $\omega'(v) \geq \frac{2}{5} + \frac{1}{5} \times 2 - \frac{1}{5} \times 3 > 0$ by R6,R5. Otherwise, v is incident with three false 3-faces, say f_1, f_2 and f_3, then h_1, h_2 and h_3 are all 4^+-faces. Thus, $\omega'(v) \geq \frac{1}{5} \times 3 - \frac{1}{5} \times 3 = 0$ by R6 and R5.

(3) $d_{G^\times}(v) = 5$.

By Lemma 2.2, v is incident with at most four 3-faces.

Case 1: Suppose that v is incident with at most two 3-faces, then $\omega'(v) \geq 1 - \frac{1}{2} \times 2 = 0$ by R4.

Case 2: Suppose that v is incident with three 3-faces.

Case 2.1: If the neighbors of v in G are 5(6)-vertex and 9^+-vertex, then let $d_{G^\times}(v_{1'}) = 5(6)$ and $d_{G^\times}(v_{i'}) = 9^+$, where $i = 2,3,4,5$.

Case 2.1.1: Suppose v is incident with at last one true 3-face, say f_i. If f_i is a $(5, 9^+, 9^+)$-face, then $\omega'(v) \geq 1 - \frac{1}{2} \times 3 + \frac{11}{18} > 0$ by R1, R4 and R7. If f_i is a $(5, 5(6), 9^+)$-face, then $\tau(f_i \to v) \geq \frac{5}{18}$ and $i = 1$. If $d_{G^\times}(f_2) \geq 5$, then $\omega'(v) \geq 1 - \frac{1}{2} \times 3 + \frac{5}{18} + \frac{1}{3} > 0$ by Claim 4.3, Claim 4.4 and Claim 4.6. If $d_{G^\times}(f_2) = 4$, then $\tau(f_2 \to v) \geq \frac{5}{18}$ by $R4, R6$ and $R7$. Thus, $\omega'(v) \geq 1 - \frac{1}{2} \times 3 + \frac{5}{18} + \frac{5}{18} > 0$. If $d_{G^\times}(f_2) = 3$, then f_2 is a false 3-face. Since G doesn't have $(3,3)$-cycle, so h_2 is a 4^+-face. By $R6$, $\tau(h_2 \to v) \geq \frac{1}{5}$. Thus, $\omega'(v) \geq 1 - \frac{1}{2} \times 2 - \frac{4}{9} + \frac{5}{18} + \frac{1}{5} > 0$.

Case 2.1.2: If v is incident with three false 3-faces, then there must be two adjacent false 3-faces. Suppose there are only two adjacent false 3-faces.

(a) If f_1 and f_2 are two adjacent false 3-faces, then f_3 and f_5 are 4^+-faces. If v_2 is a true vertex in $G^\times$, then f_3 or f_5 is a 5^+-face. Thus, $\omega'(v) \geq 1 + \frac{1}{3} - \frac{4}{9} \times 3 = 0$ by Claim 4.3, Claim 4.4, Claim 4.6 and R4. If v_2 is a false vertex in $G^\times$, then h_2 is a 4^+-face. By R6, $\tau^*(h_2 \to v) \geq \frac{1}{5}$. Suppose v_4 is a false vertex in $G^\times$. If f_3 is a 5^+-face, then $\omega'(v) \geq 1 + \frac{1}{3} + \frac{1}{5} - \frac{4}{9} \times 2 - \frac{1}{2} > 0$ by Claim 4.3, Claim 4.4, Claim 4.6 and R4. If $f_3 = vv_3u_3v_4v$ is a 4-face, then both v_3 and u_3 are true vertices. Thus, $\omega'(v) \geq 1 + \frac{5}{18} + \frac{1}{5} - \frac{4}{9} \times 2 - \frac{1}{2} > 0$ by R1, R4 and R7. Suppose v_5 is a false vertex in $G^\times$. If f_3 or f_5 is a 5^+-face, then $\omega'(v) \geq 1 + \frac{1}{3} + \frac{1}{5} - \frac{4}{9} \times 2 - \frac{1}{2} > 0$ by Claim 4.3, Claim 4.4, Claim 4.6 and R4. If both f_3 and f_5 are 4-faces, then f_3 is a $(5, 9^+, F, 9^+)$-face. Since G doesn't have $(4,4)$-cycle, so $\tau(f_3 \to v) \geq \frac{5}{9}$ by R1, R6 and R7. Thus, $\omega'(v) \geq 1 + \frac{5}{9} + \frac{1}{5} - \frac{4}{9} \times 2 - \frac{1}{2} > 0$.

(b) If f_2 and f_3 are two adjacent false 3-faces, then f_1 and f_4 are 4^+-faces. If v_3 is a true vertex in $G^\times$, then f_3 or f_5 is a 5^+-face, say f_3. By Claim 4.3, Claim 4.4 and Claim 4.6, $\tau(f_3 \to v) \geq \frac{1}{3}$. Since G doesn't have $(3,3)$-cycle, so h_2 or h_3 is a 4^+-face, say h_2. By R6, $\tau^*(h_2 \to v) \geq \frac{1}{5}$. Thus, $\omega'(v) \geq 1 + \frac{1}{3} + \frac{1}{5} - \frac{4}{9} \times 2 - \frac{1}{2} > 0$. If v_3 is a false vertex in $G^\times$, then both h_2 and h_3 are 4^+-faces. Thus, $\omega'(v) \geq 1 + \frac{1}{5} \times 2 - \frac{4}{9} \times 2 - \frac{1}{2} > 0$ by R4 and R6.

(c) If f_3 and f_4 are two adjacent false 3-faces, then by the symmetry, it is similar to (a).

(d) If f_4 and f_5 are two adjacent false 3-faces, then by the symmetry, it is similar to (b).

(e) If f_5 and f_1 are two adjacent false 3-faces, then f_2 and f_4 are 4^+-faces. Suppose v_1 is a false vertex in $G^\times$. If f_2 or f_4 is a 5^+-face, then $\omega'(v) \geq 1 + \frac{1}{3} - \frac{4}{9} \times 3 = 0$. If both f_2 and f_4 are 4-face, then f_2 or f_4 is a $(5, 9^+, F, 9^+)$-face or $(5, 9^+, 3^+, 9^+)$-face, say f_2. By R6.3.2 and R7, $\tau(f_2 \to v) \geq \frac{5}{9}$. Thus, $\omega'(v) \geq 1 + \frac{5}{9} - \frac{4}{9} \times 3 > 0$.

Suppose v_1 is a true vertex in $G^\times$, then f_2 or f_4 is a 5^+-face, say f_2. By Claim 4.3, Claim 4.4 and Claim 4.6, $\tau(f_2 \to v) \geq \frac{1}{3}$. If h_3 is a 3-face, then $\tau^*(h_3 \to f_3) \geq \frac{1}{9}$ by R6.3. Then, $\tau(f_3 \to v) \geq \frac{1}{9}$ by R7. So, $\omega'(v) \geq 1 - \frac{4}{9} - \frac{1}{2} \times 2 + \frac{1}{3} + \frac{1}{9} > 0$. If h_3 is a 4^+-face, then $\tau^*(h_3 \to f_3) \geq \frac{1}{5}$ by R6. Thus, $\omega'(v) \geq 1 - \frac{4}{9} - \frac{1}{2} \times 2 + \frac{1}{3} + \frac{1}{5} > 0$.

Case 2.1.3: Suppose there are three adjacent false 3-faces.

(a) If f_1, f_2 and f_3 are three adjacent false 3-faces, then h_1, h_2 and h_3 are 4^+-faces. By R6, $\tau^*(h_2 \to v) \geq \frac{1}{5}$ and $\tau^*(h_3 \to v) \geq \frac{1}{5}$. Thus, $\omega'(v) \geq 1 + \frac{1}{5} \times 2 - \frac{4}{9} \times 2 - \frac{1}{2} > 0$.

(b) If f_2, f_3 and f_4 are three adjacent false 3-faces, then h_2, h_3 and h_4 are 4^+-faces. So, $\omega'(v) \geq 1 + \frac{1}{5} \times 3 - \frac{4}{9} \times 3 > 0$.

(c) If f_3, f_4 and f_5 are three adjacent false 3-faces, then by the symmetry, it is similar to (a).

(d) If f_4, f_5 and f_1 are three adjacent false 3-faces, then $\tau^*(h_4 \to v) \geq \frac{1}{5}$. If f_2 or f_3 is 5^+-face, say f_2, then $\tau(f_2 \to v) \geq \frac{1}{3}$ by Claim 4.3, Claim 4.4 and Claim 4.6. Thus, $\omega'(v) \geq 1 + \frac{1}{5} + \frac{1}{3} - \frac{4}{9} - \frac{1}{2} \times 2 > 0$. If both f_2 and f_3 are 4-faces, then f_2 or f_3 is $(5, F, 3^+, 9^+)$-face, say f_2. By R1 and R6, $\tau(f_2 \to v) \geq \frac{5}{18}$. Thus, $\omega'(v) \geq 1 + \frac{1}{5} + \frac{1}{18} - \frac{4}{9} - \frac{1}{2} \times 2 > 0$.

(e) If f_5, f_1 and f_2 are three adjacent false 3-faces, then by the symmetry, it is similar to (d).

Case 2.2: If the neighbors of v in G are all 7^+-vertices.

Case 2.2.1: If v is incident with at last one true 3-face, then $\omega'(v) \geq 1 + \frac{1}{2} - \frac{1}{2} \times 3 = 0$ by R1, R2, R4 and R7.

Case 2.2.2: If v is incident with three false 3-faces, then there must be two adjacent false 3-faces. Suppose there are only two adjacent false 3-faces, then by the symmetry, assume that f_1, f_2 and f_4 are false 3-faces, and v_5 is a false vertex. If v_2 is true, then h_1 or h_2 is a 4^+-face, say h_1. By R6, $\tau^*(h_1 \to v) \geq \frac{1}{8}$. Suppose f_3 is a 5^+-face, then $\omega'(v) \geq 1 + \frac{1}{8} + \frac{1}{2} - \frac{1}{2} \times 3 > 0$. Suppose f_3 is a 4-face, then f_5 is a 5^+-face. By Claim 4.3, Claim 4.4, Claim 4.6, R6 and R7, $\tau(f_5 \to v) \geq \frac{1}{3}$ and $\tau(f_3 \to v) \geq \frac{1}{8}$. Thus, $\omega'(v) \geq 1 + \frac{1}{8} + \frac{1}{8} + \frac{1}{3} - \frac{1}{2} \times 3 > 0$.

If v_2 is false, then both h_1 and h_2 are 4^+-face. Since v_3 and v_4 are true, then $\tau(f_3 \to v) \geq \frac{1}{4}$ by Claim 4.3, Claim 4.4, Claim 4.6, R6 and R7. Thus, $\omega'(v) \geq 1+\frac{1}{8}\times 2+\frac{1}{4}-\frac{1}{2}\times 3 = 0$.

Case 2.2.3: Suppose there are three adjacent false 3-faces, then by the symmetry, assume that f_1, f_2 and f_3 are false 3-faces, and v_2 and v_4 are false vertices. By R6, $\tau^*(h_i \to v) \geq \frac{1}{8}$, where $i = 1, 2, 3$. Since f_5 is a 4^+-face, then $\tau(f_5 \to v) \geq \frac{1}{8}$ Claim 4.3, Claim 4.4, Claim 4.6, R6 and R7. Thus, $\omega'(v) \geq 1 + \frac{1}{8} \times 3 + \frac{1}{8} - \frac{1}{2} \times 3 = 0$.

Case 3: Suppose that v is incident with four 3-faces, then they are all false 3-faces.

Case 3.1: If the neighbors of v in G are 5(6)-vertex and 9^+-vertex, then let $d_{G^\times}(v_{1'}) = 5(6)$ and $d_{G^\times}(v_{i'}) = 9^+$, where $i = 2, 3, 4, 5$.

(a) If f_i is a false 3-face, then h_i is a 4^+-face, where $i = 1, 2, 3, 4$. By R6, $\tau^*(h_i \to v) \geq \frac{1}{5}$, where $i = 2, 3, 4$. Since G doesn't have $(3, 3)$-cycle, so f_5 is a 5^+-face. Thus, $\omega'(v) \geq 1 + \frac{1}{5} \times 3 + \frac{1}{3} - \frac{4}{9} \times 3 - \frac{1}{2} > 0$.

(b) If f_i is a false 3-face, where $i = 2, 3, 4, 5$, then by the symmetry, it is similar to (a).

(c) If f_i is a false 3-face, where $i = 1, 3, 4, 5$, then $\tau^*(h_3 \to v) \geq \frac{1}{5}$, $\tau^*(h_4 \to v) \geq \frac{1}{5}$, and f_2 is a 5^+-face. If f_2 is a 6^+-face, then $\tau(f_2 \to v) \geq \frac{1}{2}$ by Claim 4.3 and Claim 4.4. If f_2 is a 5-face, then f_2 is incident with at most two true 5^--vertices by Property 3.4. Then, $\tau(f_2 \to v) \geq \frac{1}{2}$. Thus, $\omega'(v) \geq 1 + \frac{1}{5} \times 2 + \frac{1}{2} - \frac{4}{9} \times 2 - \frac{1}{2} \times 2 > 0$.

(d) If f_i is a false 3-face, then h_i is a 4^+-face, where $i = 1, 2, 4, 5$. Let $min\{d_{G^\times}(v_{2'}), d_{G^\times}(v_{3'})\} = p$, $min\{d_{G^\times}(v_{4'}), d_{G^\times}(v_{5'})\} = q$. If $p = q = 9$, then $\tau^*(h_2 \to v) \geq \frac{2}{9}$ and $\tau^*(h_4 \to v) \geq \frac{2}{9}$ by R6.3.2. Thus, $\omega'(v) \geq 1+\frac{2}{9}\times 2+\frac{1}{3}-\frac{4}{9}\times 4 = 0$. If $p = 9$ and $10 \leq q \leq 11$, then $\tau^*(h_2 \to v) \geq \frac{2}{9}$, $\tau^*(h_4 \to v) \geq \frac{1}{5}$, $\tau(f_4 \to v) \geq \frac{2}{45}$, and $\tau(f_5 \to v) \geq \frac{2}{45}$ by R1, R4, R6.2, R6.3 and R7. Thus, $\omega'(v) \geq 1 + \frac{2}{9} + \frac{1}{5} + \frac{1}{3} + \frac{2}{45} \times 2 - \frac{4}{9} \times 4 > 0$. If $q = 9$ and $10 \leq p \leq 11$, then $\omega'(v) \geq 0$, similarly. If $10 \leq p$ and $q \leq 11$, then $\tau(f_i \to v) \geq \frac{2}{45}$, where $i = 1, 2, 4, 5$. Thus, $\omega'(v) \geq 1+\frac{1}{5}\times 2+\frac{1}{3}+\frac{2}{45}\times 4-\frac{4}{9}\times 4 > 0$. If $p \geq 12$ or $q \geq 12$, say $p \geq 12$, then $q \geq 10$. By $R6.1.2$, $\tau^*(h_2 \to v) \geq \frac{1}{3}$ and $\tau^*(h_4 \to v) \geq \frac{1}{5}$. Thus, $\omega'(v) \geq 1 - \frac{4}{9} \times 4 + \frac{1}{3} \times 2 + \frac{1}{5} > 0$.

(e) If f_i are four adjacent false 3-faces, where $i = 1, 2, 3, 5$, then by the symmetry, it is similar to (c).

Case 3.2: If the neighbors of v in G are all 7^+-vertices, then by the symmetry, assume that f_i is 3-face, where $i = 1, 2, 3, 4$. Since G doesn't have $(3, 3)$-cycle, so f_5 is a 5^+-face and h_i is a 4^+-face, where $i = 1, 2, 3, 4$. If f_5 is a 6^+-face or a 5-face that is incident with at most two true 5^--vertices, then $\omega'(v) \geq 1 + \frac{1}{8} \times 4 + \frac{1}{2} - \frac{1}{2} \times 4 = 0$ by Claim 4.3, Claim 4.4 and Claim 4.6. If f_5 is a 5-face that is incident with three true 5^--vertices, then v_2 and v_4 are 9^+-vertices. Thus, $\omega'(v) \geq 1 + \frac{1}{8} \times 4 + \frac{1}{3} - \frac{4}{9} \times 4 > 0$.

(4) $d_{G^\times}(v) = 6$.

By Lemma 2.2, v is incident with at most four 3-faces. By $R3$, we have $\omega'(v) \geq 6 - 4 - \frac{1}{2} \times 4 = 0$.

(5) $d_{G^\times}(v) = 7$.

By Lemma 2.2, v is incident with at most five 3-faces. By $R2$, we have $\omega'(v) \geq 7 - 4 - \frac{1}{2} \times 5 - \frac{1}{4} \times 2 = 0$.

(6) $d_{G^\times}(v) \geq 8$.

By $R1$, $\omega'(v) \geq d_{G^\times}(v) - 4 - \frac{d_{G^\times}(v)-4}{d_{G^\times}(v)} \times d_{G^\times}(v) = 0$.

Next, we consider the discharge of the faces in G.

(1) $d_{G^\times}(f) = 3$.

Case 1: Suppose $f = v_1v_2v_3$ is true, where $d_{G^\times}(v_1) \geq d_{G^\times}(v_2) \geq d_{G^\times}(v_3)$. If $d_{G^\times}(v_1) = 3$ or 4, then $\omega'(f) \geq -1 + \frac{3}{5} \times 2 > 0$ by property 3.2, property 3.1.3 and R1. If $d_{G^\times}(v_1) \geq 5$, then $\omega'(f) \geq -1 + \frac{1}{2} \times 3 > 0$ by $R1 - R4$.

Case 2: If $f = vv_1v_2$ is false, where $d_{G^\times}(v_1) \leq d_{G^\times}(v_2)$ and v be a false vertex of $G^\times$ such that v_1v_3 crossed v_2v_4 in G at v. If $d_{G^\times}(v_1) = 3$, then $d_{G^\times}(v_2), d_{G^\times}(v_3) \geq 12$ by Property 3.2. By $R1, R5$ and $R6.1$, we have $\tau(f_2 \to v) \geq \frac{1}{3}$ and $\tau(v_2 \to v) \geq \frac{2}{3}$. Thus, $\omega'(f) \geq -1 + \frac{2}{3} + \frac{1}{3} = 0$. If $d_{G^\times}(v_1) = 4$, then $\omega'(f) \geq -1+\frac{3}{5}+\frac{1}{5}+\frac{1}{5} = 0$ by $R1, R5$ and $R6$. If $d_{G^\times}(v_1) = 5$ and $d_{G^\times}(v_2) = 9^+$, then $\omega'(f) \geq -1 + \frac{4}{9} + \frac{5}{9} = 0$ by $R1, R2, R3$ and $R4$. If $d_{G^\times}(v_1) = 5$ and $d_{G^\times}(v_2) \neq 9^+$, then $\omega'(f) \geq -1 + \frac{1}{2} \times 2 = 0$ by $R4$. If $d_{G^\times}(v_1) = 6^+$, then $\omega'(f) \geq -1 + \frac{1}{2} \times 2 = 0$ by $R1$ and $R2$.

(2) $d_{G^\times}(f) = 4$.

Case 1: Suppose f is not incident with any transitive false vertex, then $\omega'(f) \geq d_{G^\times}(f) - 4 \geq 0$ by $R6$ and $R7$.

Case 2: Suppose $f = v_1v_2v_3v_4$ is incident with two transitive false vertices, say v_1 and v_3, then let $min\{d_{G^\times}(v_2), d_{G^\times}(v_4)\} = p$, and $max\{d_{G^\times}(v_2), d_{G^\times}(v_4)\} = q$. If $5 \leq p \leq 6$ and $q \geq 12$, then $\omega'(f) \geq 0 + \frac{2}{3} - \frac{1}{3} \times 2 = 0$ by $R1, R2$ and $R6.6$. If $5 \leq p \leq 6$ and $10 \leq q \leq 11$, then $\omega'(f) \geq 0 + \frac{3}{5} - \frac{1}{5} \times 2 > 0$ by $R1, R2$ and $R6.7$. If $7 \leq p \leq 9$, then $\omega'(f) \geq 0$ by $R1, R6.4.1$ and $R6.3$, similarly. If $10 \leq p \leq 11$. Since G doesn't have $(4, 4)$-cycle, f sends out at most $\frac{3}{5} \times 2$ $R1$ and $R6.2$. Thus, $\omega'(f) \geq 0 - \frac{3}{5} \times 2 + \frac{3}{5} \times 2 = 0$. If $p \geq 12$, then $\omega'(f) \geq 0 - \frac{4}{3} + \frac{2}{3} \times 2 = 0$ by $R1$ and $R6.1$, similarly.

Case 3: Suppose f is only incident with one transitive false vertex, then it is similar to the proof of Case 2.

(3) $d_{G^\times}(f) = 5$.

If $d_{G^\times}(f) = 5$, then f is incident with at most two transitive false vertices. Similar to the proof of $d_{G^\times}(f) = 4$, we can get $\omega'(f) \geq d_{G^\times}(f) - 4 \geq 0$.

(4) $d_{G^\times}(f) \geq 6$.

Suppose f is incident with at least t transitive false vertices, then $t \leq \lfloor \frac{d_{G^\times}(f)}{2} \rfloor$. The worst case is that the neighbors of transitive false vertices on f are 12^+-vertices, then $\omega'(f) \geq d_{G^\times}(f) - 4 - \frac{4t}{3} + \frac{2t}{3} \geq d_{G^\times}(f) - 4 - \frac{d_{G^\times}(f)}{3} = \frac{2d_{G^\times}(f)}{3} - 4 \geq 0$ by R1, R6 and R7.

The proof of Theorem 1.2 is complete.

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