I. INTRODUCTION

MOBILE sensors are used for data monitoring and communication for various purposes, such as oceanographic research [15], tropical air analysis [4], robotics [17], and security monitoring [14], among others.

One of the main research topics in this area is the determination of an optimal allocation of the sensors in order to generate good coverage at a minimum cost [3].

Through mobile sensor technology, good coverage can be achieved by placing the sensors in the desired positions. However, mobile sensors are generally equipped with a battery and the energy expenditure is much greater during the movement of the sensor than during its detection function. Therefore, it is important to minimize the movements of the sensor to increase its useful life and maintain the reliability of the network to which it belongs.

There are two approaches to studying the minimum expected cost of transport: the sum or maximum of the movements of the sensors from their initial positions to the destination. With respect to the sum, Huesmann and Sturm [10] give a mathematical approach to the optimal transport from Lebesgue to Poisson process. An empirical approach to the cost of optimal incomplete transportation can be found in [6]. For an unified approach of a series of papers about behaviour asymptotic of a binomial and a Poisson sum which arose as (average) displacement costs when moving randomly placed sensors to anchor positions, see [8].

Ajtai and Komlós [1] considered 2n sensors, n blue $X_1, X_2, \ldots, X_n$ and n red $Y_1, Y_2, \ldots, Y_n$, distributed independently and uniformly in a unit square, and proved that the expected minimum cost of transportation, denoted by $T_n$ and defined by $T_n := \min \sum_{i=1}^{n} d(X_{\pi(i)}, Y_{i})$, belong $\Theta(\sqrt{n \log n})$. Kranakis [13], when assuming that the sensors move randomly on a line according to two independent and identically distributed Poisson processes with arrival rate $\lambda$ and respective arrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ determined an interval for the expected minimum cost of transport, defined by $C_T = \sum_{k=1}^{n} E[|X_k - Y_k|]$. Kapelko [12] generalized the result of Kranakis [13]. He considered the same hypotheses as [13] and determined an asymptotic expression for the expected minimum cost at power $a > 0$, $C_T^a = \sum_{k=1}^{n} E[|X_k - Y_k|^a]$. Recently, Kapelko [11], when considering two identical and independent general random processes, determined asymptotic expressions for the expected minimum transport cost at power $b > 0$, $C_T^b$. A more general transportation cost problem than that addressed in the articles cited above occurs when it is assumed that the sensors move according to two independent stochastic processes, that do not necessarily have the same distribution. In this paper, we study this more general problem. Our results generalize Kranakis’s results [13].

We obtain the transport cost $C_{opt} = C_T$ for a particular case and an exact interval for $C_{opt}$, by considering a network of two sensors $\{X_1, Y_1\}$, where $X_1, X_2, \ldots$ are blue and $Y_1, Y_2, \ldots$ are red, which are initially randomly allocated according to a Poisson process with arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Note that $\lambda_1$ can be different from $\lambda_2$, so the sensors $\{X_i\}$ and $\{Y_i\}$ follow a different law. In addition to obtaining an interval for the expected transport cost, here we carry out a statistical inference study and verify that the sample transport cost is a consistent estimator of the theoretical transport cost found.

Kranakis [13], Kapelko [12] and Kapelko [11] based their results on combinatorial theory, but for the proof of our results we also use results of the following special functions: gamma function, upper and lower incomplete gamma functions, beta function, and incomplete beta function. These functions are defined, respectively, by:

$$\Gamma(a) := \int_{0}^{\infty} t^{a-1} e^{-t} dt , \quad (1)$$

$$\Gamma(a, x) := \int_{x}^{\infty} t^{a-1} e^{-t} dt , \quad (2)$$

$$\gamma(a, x) := \int_{0}^{x} t^{a-1} e^{-t} dt , \quad (3)$$
\[ B(a, b) := \int_0^1 t^{a-1}(1-t)^{b-1} dt, \quad \text{and} \]
\[ B_2(a, b) := \int_0^x t^{a-1}(1-t)^{b-1} dt. \]

The following identities (see [9]) are also used:
\[ \Gamma(a) = \gamma(a, x) + \Gamma(a, x), \]
\[ \Gamma(n+1, x) = n! e^{-x} \sum_{r=0}^n \frac{x^r}{r!}, \]
\[ \gamma(n+1, x) = n!(1-e^{-x} \sum_{r=0}^n \frac{x^r}{r!}), \]
and
\[ \frac{\Gamma(x+h)}{\Gamma(x)} = x^h : \quad \text{Pochhammer polynomial} \]
\[ = x(x+1)(x+2) \cdots (x+h-1), \quad \text{if } h \geq 1. \]

The rest of the paper is organized as follows: Section 2 describes our main results; Section 3, presents the statistical inference results about transport cost and illustrations of the results generated from Monte Carlo simulation experiments; and Section 4 concludes.

II. Expected Distance

In this section we present Theorem 1 in which we determine a closed analytic expression for \( E\left[|X_k - Y_k|\right] \). Let \( X_i \) and \( Y_k \) be random variables that represent the \( i \)-th and \( k \)-th arrival times of two independent Poisson processes with rates \( \lambda_1 \) and \( \lambda_2 \). Then, \( X_i \) and \( Y_k \) have gamma distribution. With the notation
\[ X_i \sim \text{Gamma}(i, \lambda_1) \quad \text{and} \quad Y_k \sim \text{Gamma}(k, \lambda_2), \]
the random variables \( X_i \) and \( Y_k \) have probability density functions (pdf’s)
\[ f_{X_i}(x) := f_i(x) = \frac{\lambda_1^i}{\Gamma(i)} x^{i-1} e^{-\lambda_1 x} \quad x > 0 \] 
\[ \text{and} \]
\[ f_{Y_k}(y) := f_k(y) = \frac{\lambda_2^k}{\Gamma(k)} y^{k-1} e^{-\lambda_2 y} \quad y > 0, \]
respectively. The shape parameters are \( i \) and \( k \) and the scale parameters are \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \). The particular cases of our results are in Corollaries 1 and 2. These results correspond to the main results of [13].

Theorem 1: Consider two independent Poisson processes with arrival times \( X_1, X_2, \ldots \) and \( Y_1, Y_2, \ldots \) and arrival rates \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), respectively; \( \lambda_1 \neq \lambda_2 \) or \( \lambda_1 = \lambda_2 \). Then
\[ E[X_{i+k} - Y_k] = \frac{k+r}{\lambda_1} - \frac{k}{\lambda_2} + 2k(k+r) \left( \frac{2k+r}{k} \right) \Delta B_p, \]
for non-negative integers \( r \geq 0 \) and \( k \geq 1 \), where
\[ \Delta B_p = \frac{B_p(k+r, k+1)}{\lambda_2} - \frac{B_p(k+r+1, k)}{\lambda_1}. \]

Proof: By using the conditional expectation property, we have:
\[ E[X_i - Y_k] = E\left[ E\left[|X_i - Y_k| \mid Y_k\right]\right] \]
\[ = \int_0^\infty E[X_i - y] f_2(y) dy. \]
To find the the expected value of [13], consider:
\[ E[X_i - y] = I_1 + I_2, \quad \text{with} \]
\[ I_1 = \int_0^y -(x-y) f_1(x) dx \quad \text{and} \quad I_2 = \int_y^\infty (x-y) f_1(x) dx. \]

By combining (10), \( I_1 \) and \( I_2 \), we deduce that:
\[ I_1 = \frac{\lambda_1}{\Gamma(i)} \int_y^\infty x^i e^{-\lambda_1 x} dx - \frac{i}{\lambda_1} + \frac{y \lambda_1^i}{\Gamma(i)} \int_0^y x^{i-1} e^{-\lambda_1 x} dx. \]
\[ \text{and} \]
\[ I_2 = \frac{\lambda_2^i}{\Gamma(i)} \int_y^\infty x^i e^{-\lambda_2 x} dx - \frac{y \lambda_2^i}{\Gamma(i)} \int_0^\infty x^{i-1} e^{-\lambda_2 x} dx. \]

Now (15) and (16) are replaced in equation (14). The expected value result in terms of the incomplete gamma functions (3) and (2) is:
\[ E[X_i - Y_k] = -\frac{i}{\lambda_1} + \frac{2}{\lambda_1 \Gamma(i)} J_1 + \frac{1}{\Gamma(i)} J_2 - \frac{1}{\Gamma(i)} J_3, \]
where
\[ J_1 := \int_0^\infty \Gamma(i+1, \lambda_1 y) f_2(y) dy, \]
\[ J_2 := \int_0^\infty y \Gamma(i, \lambda_1 y) f_2(y) dy \]
\[ \text{and} \]
\[ J_3 := \int_0^\infty \Gamma(i, \lambda_1 y) f_2(y) dy. \]

These integrals are calculated using the series representation of the incomplete gamma functions (17), (8) and the density (11). After algebraic manipulations, we deduce that:
\[ J_1 = \frac{\Gamma(i+1)}{\Gamma(k)} \sum_{s=0}^{i} \frac{\Gamma(s+k)}{s!} \]
\[ \left( \frac{\lambda_2^k}{\lambda_1^k} \right)^s. \]
\[ J_2 = \frac{\Gamma(i)k}{\lambda_2} - \frac{\Gamma(i)}{\lambda_2 \Gamma(k)} q^{k+1} \sum_{s=1}^{i-1} \left[ \sum_{r=0}^{s} \frac{\Gamma(s+k+1)}{s!} \right] \]  
(20)

and

\[ J_3 = \frac{\Gamma(i)}{\lambda_2 \Gamma(k)} q^{k+1} \sum_{s=1}^{i-1} \left[ \sum_{r=0}^{s} \frac{\Gamma(s+k+1)}{s!} \right]. \]  
(21)

with \( p = \lambda_1/(\lambda_1 + \lambda_2) \) and \( q = 1 - p. \)

By combining integrals (19), (20) and (21) in (18), we get:

\[ E[X_i - Y_k] = \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i q^k}{\lambda_1 \Gamma(k)} \sum_{r=0}^{i} \frac{\Gamma(s+k)}{s!} \]

\[ - \frac{2k q^{k+1}}{\lambda_2} \sum_{s=0}^{i-1} \left( \sum_{r=0}^{s} \frac{\Gamma(s+k+1)}{s!} \right) p^s, \]

(22)

where \( p = \lambda_1/(\lambda_1 + \lambda_2) \) and \( q = 1 - p. \)

Finally, we update equation (22), to obtain:

\[ E[X_i - Y_k] = \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i q^k}{\lambda_1 \Gamma(k)} \left[ 1 - \frac{(i+k)}{k} B_p(i+1,k) \right] \]

\[ - \frac{2k q^{k+1}}{\lambda_2} \left[ 1 - \frac{i+k}{k} B_p(i+1,k) \right] \]

\[ = \frac{i}{\lambda_1} - \frac{k}{\lambda_2} + \frac{2i}{k} \frac{B_p(i+1,k+1)}{\lambda_2} - \frac{B_p(i+1,k)}{\lambda_1}, \]  
(23)

by using the identity (see [27]),

\[ \sum_{s=0}^{\infty} \binom{n+s}{s} p^s = \frac{1 - (L+1) \binom{L+n+1}{n} B_p(L+1,n+1)}{1 - (1-p)^{n+1}}, \]  
(24)

Replacing \( i \) by \( k+r \) in (23) finishes the proof.

An expression equivalent to (12), in terms of the regularized incomplete beta function, is provided below. To obtain this result, just replace the expressions

\[ B(k+r,k+1) = \frac{1}{(k+r) \binom{2k+r}{k}} \]

and

\[ B(k+r+1,k) = \frac{1}{k \binom{2k+r}{k}} \]

in the regularized incomplete beta function

\[ I_p(a,b) := \frac{B_p(a,b)}{B(a,b)}. \]

Then

\[ E[X_{k+r} - Y_k] = \frac{k + r - k}{\lambda_1} + \frac{2k I_p(k+r,k+1)}{\lambda_2} - \frac{2(k+r) I_p(k+r+1,k)}{\lambda_1} \]

(27)

is equivalent to (12).

**Corollary 1:** Consider two independent Poisson processes with arrival times \( X_1, X_2, \cdots \) and \( Y_1, Y_2, \cdots \) and arrival rates \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), respectively. If \( r = 0, k \in \mathbb{Z}_{\geq 1} \) and \( \lambda_1 = \lambda_2 = \lambda > 0 \), then

\[ E[X_{k+r} - Y_k] = \frac{k^2 - 2k - 1}{\lambda} \binom{2k}{k} \]

(28)

**Proof:** From Theorem [1] for \( r = 0 \) and \( p = \lambda_1/(\lambda_1 + \lambda_2) = 1/2 \), we have

\[ E[X_{k+r} - Y_k] = \frac{2k^2}{\lambda} \left[ B_{\frac{1}{2}}(k+1,k+1) - B_{\frac{1}{2}}(k+1,k) \right]. \]

(29)

The identity

\[ B_{\frac{1}{2}}(a;n+1-a) = B(a;n+1-a) \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j}, \]

(30)

(see [7]), allows rewriting the difference in equation (29) as:

\[ B_{\frac{1}{2}}(k+1,k+1) - B_{\frac{1}{2}}(k+1,k) = \frac{B(k+1)}{2k} \binom{2k}{k} = \frac{1}{k} \binom{2k}{k}. \]

(31)

by using the identity

\[ B(k+1) = \frac{1}{2k} \binom{2k}{k} \]

(32)

The result (28) is obtained by replacing (31) by (29). ■

**Corollary 2:** Consider two independent Poisson processes with arrival times \( X_1, X_2, \cdots \) and \( Y_1, Y_2, \cdots \) and arrival rates \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), respectively. If \( r > 0, k \in \mathbb{Z}_{\geq 1} \) and \( \lambda_1 = \lambda_2 = \lambda > 0 \), then

\[ S_{k,r} = \frac{k^2 - 2k + 1}{\lambda} \binom{2k}{k} \left[ 1 + \sum_{s=0}^{r-1} \frac{r-s}{(2k+s) 2s (k+1)^s} \right], \]

(33)

where \( S_{k,r} = E[X_{k+r} - Y_k]. \)

**Proof:** For \( \lambda_1 = \lambda_2 = \lambda \) and \( r > 0 \), from Theorem [1] we have:

\[ S_{k,r} = \frac{r + 2k(k+r)}{\lambda} \binom{2k}{k} Q(k,r). \]

(34)

with \( Q(k,r) = \left[ B_{\frac{1}{2}}(k+r,k+1) - B_{\frac{1}{2}}(k+r+1,k) \right]. \)

Equation (34) is updated by rewriting \( B_{\frac{1}{2}}(k+r+1,k) \)
and $B^2(k + r + 1, k)$ with identity $[30]$ as

$$S_{k,r} = \frac{-r}{\lambda} + \frac{2(k + r)}{\lambda^2 k} \sum_{s=0}^{k+r} \binom{s + k - 1}{s} \frac{1}{2^s}$$

$$- \frac{2k}{\lambda^2} \sum_{s=0}^{k+r} \binom{s + k}{s} \frac{1}{2^s}$$

$$= \frac{-r}{\lambda} + \frac{2(k + r)}{2k\lambda\Gamma(k)} \sum_{j=0}^{1} H_j,$$

where

$$H_0 = \sum_{s=0}^{k+r} \frac{(s + k - 1)!}{2^s} \binom{s + k - 1}{s}$$

$$\Gamma(k) \left[ 2^{k-1} + 2^{-k} \sum_{t=0}^{r} \binom{t + 2k - 1}{k - 1} 2^{-t} \right]$$

and

$$H_1 = \sum_{s=0}^{k+r} \binom{s + k}{s} \frac{k!}{\Gamma(2k + r + 1)} \left[ 2^{-k} \sum_{t=0}^{r} \binom{2k + t - 1}{k - 1} 2^{-t} \right].$$

By replacing $H_0$ and $H_1$ in (35), we get:

$$S_{k,r} = \frac{-r}{\lambda} + \frac{2(k + r)}{\lambda^2 k} \sum_{s=0}^{k+r} \binom{s + k - 1}{s} \frac{1}{2^s}$$

$$+ \frac{2k}{\lambda^2} \sum_{s=0}^{k+r} \binom{s + k}{s} \frac{1}{2^s}$$

$$\Gamma(k) \left[ 2^{k-1} + 2^{-k} \sum_{t=0}^{r} \binom{t + 2k - 1}{k - 1} 2^{-t} \right].$$

Now, the identity

$$\sum_{t=0}^{r} \binom{2k + t - 1}{k - 1} 2^{-t} = 2kC_{2k-1}^{2k+r-1} = 2^{-r}(2k+r)C_{2k+r-1}^{k-1},$$

valid for $k \geq 1$, is applied in (38), so:

$$S_{k,r} = \frac{2^{2k} - 2k}{\lambda^2} \binom{2k}{k} + \frac{2k}{\lambda^2} \sum_{s=0}^{k+r} \binom{s + k}{s} \frac{1}{2^s}$$

$$\sum_{t=0}^{r} \binom{t + 2k - 1}{k - 1} 2^{-t}.$$ (38)

Finally, by replacing the binomial identity

$$\binom{2k + s - 1}{k - 1} = k \binom{2k}{k} \frac{1}{2k + s} \binom{2k + 1}{s},$$

in (39), the result (33) is obtained.

**III. Minimum Expected Transport Cost**

In this section, we present an interval for the expected transport cost of a pair $\{X_i, Y_k\}$ of sensors placed randomly in the interval $[0, \infty)$. The position of the $i$-th sensor (blue) and the $k$-th (red) are determined by the arrival times $X_i$ and $Y_k$, according to two Poisson process with arrival rates $\lambda_1$ and $\lambda_2$, respectively. This expected transport cost corresponds to:

$$C_{opt}(\lambda_1, \lambda_2, n) = \sum_{k=1}^{n} E[|X_k - Y_k|].$$

In Theorem 2, we obtain a closed formula in terms of incomplete gamma function for $C_{opt}(\lambda_1, \lambda_2, n)$ and in Theorem 3, we obtain an exact interval for $C_{opt}(\lambda_1, \lambda_2, n)$.

**Theorem 2:** Consider two independent Poisson processes with arrival rate $\lambda_1 > 0$ and $\lambda_2 > 0$ and respectively arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$. If $k \geq 1$ is integer and $\lambda_2 > \lambda_1$, then:

$$C_{opt}(\lambda_1, \lambda_2, n) = \frac{2\lambda_1 e^{\lambda_1 \lambda_2/(\lambda_1 + \lambda_2)}}{\lambda_2 (\lambda_1 + \lambda_2)} \Gamma(n, \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}),$$

(41)

**Lemma 1:** Let $f_1(\cdot)$ and $f_2(\cdot)$ be densities of the random variables $X_i \sim \text{Gamma}(i, \lambda_1)$ and $Y_k \sim \text{Gamma}(k, \lambda_2)$, respectively, with $\lambda_1 > 0$ and $\lambda_2 > 0$. Then, for $y > 0$, the double integral

$$I_2 := \int_{0}^{\infty} \int_{0}^{y} (t - y) f_1(t) f_2(y) \, dt \, dy$$

is given by

$$I_2 = \frac{-\lambda_1 \lambda_2^{-i - 2}}{\Gamma(i)} \left( 1 + \frac{\lambda_1}{\lambda_2} \right)^{-i},$$

(43)

when $\lambda_2 > \lambda_1$.

**Proof:** From (10) and (11),

$$I_2 := \int_{0}^{\infty} \int_{0}^{\infty} (t - y) f_1(t) f_2(y) \, dt \, dy$$

$$= \frac{\lambda_1 \lambda_2^{-i}}{\Gamma(i) \Gamma(k)} \int_{0}^{\infty} \int_{0}^{\infty} (t - y) t^{i-1} e^{-\lambda_1 y} y^{k-1} e^{-\lambda_2 y} \, dt \, dy.$$ (44)

By performing the substitution $t = uy$, we update (44) and get

$$I_2 = \frac{-\lambda_1 \lambda_2^{-i}}{\Gamma(i) \Gamma(k)} \int_{0}^{\infty} y^{i+1} e^{-\lambda_2 y} I_{21} dy,$$ (45)

where $I_{21} := \int_{0}^{\infty} u^{i-1} (1 - u) e^{-(\lambda_1 y) u} \, du$.

The integral representation of the confluent hypergeometric function (see [5] p. 185)

$$I_{21} = \frac{\Gamma(i)}{\Gamma(i + 2)} F_1(d; e; x) = \frac{\Gamma(c)}{\Gamma(d) \Gamma(c - d)} \int_{0}^{1} e^{xt} t^{d-1} (1 - t)^{c-d-1} dt,$$ (46)

valid for $c \in C, d \in C, R(c) > R(d) > 0$, and $x \in C$, allows us to rewrite $I_{21}$ as

$$I_{21} = \frac{\Gamma(i)}{\Gamma(i + 2)} F_1(i; i + 2; -\lambda_1 y),$$ (47)

where $d = i, c = i + 2$ and $x = -\lambda_1 y$. Replacing (47) in (45).
we have that
\[ I_2 = \frac{-\lambda_q^2 \lambda_i^k}{\Gamma(k)\Gamma(i+2)} \int_0^\infty y^{i+1}e^{-\lambda_q y} F_1(i; i+2; -\lambda_q y) dy. \] (48)

When \( \lambda_q > \lambda_i > 0 \), we use the identity (see [5] p. 187)
\[ \int_0^\infty t^{\mu-1}e^{-\lambda t} F_1(d; \mu; qt) dt = \Gamma(\mu)x^{-\mu}(1 - \frac{d}{x})^{-d} \]
with \( x = \lambda_q, \ q = -\lambda_i, \ d = i, \) and \( \mu = i+2, \) for rewrite the integral in (48). Then
\[ I_{22} = \int_0^\infty y^{i+1}e^{-\lambda_q y} F_1(i; i+2; -\lambda_q y) dy \]
\[ = \Gamma(i+2)\lambda_q^{-(i+2)} \left( 1 + \frac{\lambda_i}{\lambda_q} \right) \] (50)

Finally, the result (43) is obtained by replacing (50) in (48).

Proof: Theorem 2 With the notation of Lemma 1
\[ E[|X_k - Y_k|] = -2I_{2}|_{k=1} \] So, the expected transport cost is given by
\[ C_{opt} = \sum_{k=1}^n E[|X_k - Y_k|] \]
\[ = \frac{2\lambda_q}{\lambda_q - \lambda_i} \left\{ \frac{1}{\Gamma(k)} \left( \frac{\lambda_q \lambda_i}{\lambda_q + \lambda_i} \right)^k \right\}. \] (51)

By applying the identity (7) in (51), with \( x = \lambda_q \lambda_i/(\lambda_q + \lambda_i), \) we conclude that
\[ C_{opt} = \frac{2\lambda_q \lambda_i}{\lambda_q (\lambda_q + \lambda_i) \Gamma(n)} \left( \frac{n}{\lambda_i + \lambda_q} \right). \] (52)

where
\[ l_n = \frac{n(n+1)}{2} \frac{1}{\lambda_i - \lambda_q} + \frac{2}{\lambda_q} S(n, \lambda_i, \lambda_q), \]
\[ s_n = \frac{n(n+1)}{2} \left( \frac{1}{\lambda_q} + \frac{1}{\lambda_i} \right) \times S(n, \lambda_i, \lambda_q) \] (53)
and
\[ S(n, \lambda_i, \lambda_q) = \sum_{k=1}^n \frac{(pq)^k}{B(k+1,k)}. \] (55)

Proof: For \( \nu = 0 \), from (27) we have:
\[ E[|X_k - Y_k|] = k \frac{\lambda_q}{\lambda_q^2} \left[ \frac{I_p(k, k+1)}{\lambda_q} - \frac{I_p(k+1, k)}{\lambda_i} \right]. \] (56)

By applying identity
\[ I_p(a, b) = I_p(a - 1, b + 1) - \frac{p^{a-1}q^b}{bB(b,a)} \] in (56), for \( a = k + 1 \) and \( b = k \), this results in:
\[ E[|X_k - Y_k|] \geq k \frac{\lambda_q}{\lambda_i^2} \frac{1}{\lambda_q^2} + \frac{2}{\lambda_i B(k+1,k)}. \]

So, the lower bound of the sum is:
\[ \sum_{k=1}^n S_{k,0} \geq \frac{n(n+1)}{2} \frac{1}{\lambda_i} + \frac{2}{\lambda_i} \sum_{k=1}^n \frac{(pq)^k}{B(k+1,k)}, \] (57)
where \( S_{k,0} := E[|X_k - Y_k|]. \)

To obtain the upper limit of the sum, we replace the identities
\[ I_p(a, b) = I_p(a, b + 1) - \frac{\lambda_i}{\nu B(a,b)} \]
and
\[ I_p(a, b) = I_p(a + 1, b) + \frac{\lambda_i}{\nu B(a,b)} \] in (56). Then, for \( \nu = \lambda_q = \lambda_i \), we get:
\[ \frac{I_p(k, k+1)}{\lambda_q} - \frac{I_p(k+1, k)}{\lambda_q} = \left( \frac{1}{\lambda_q} - \frac{1}{\lambda_i} \right) I_p(k, k) \]
\[ + \left( \frac{1}{\lambda_q} + \frac{1}{\lambda_i} \right) \left( \frac{pq}{\nu B(k+1,k)} \right). \] (58)

The upper limit of the sum is obtained from (56), (58) and the fact that
\[ I_x(k, k) \leq 0, \quad \forall k \in \mathbb{Z}^+. \] That is:
\[ E[|X_k - Y_k|] \leq k \frac{\lambda_q}{\lambda_i^2} \frac{1}{\lambda_i^2} + 2 \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^2} \frac{(pq)^k}{B(k+1,k)} \] (59)

and
\[ \sum_{k=1}^n S_{k,0} \leq \frac{n(n+1)}{2} \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^2} \sum_{k=1}^n \frac{(pq)^k}{B(k+1,k)}. \] (60)

Finally, the proof finishes by combining inequalities (57) and (60).

Corollary 3: Consider two independent Poisson processes with arrival times \( X_1, X_2, \cdots \) and \( Y_1, Y_2, \cdots \) and arrival rates \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), respectively; \( \lambda_1 \geq \lambda_2 \). Then
\[ C_{opt}(\lambda, \lambda, \lambda) = \sum_{k=1}^n \frac{(pq)^k}{B(k+1,k)}. \] (52)

Proof: From (52), in Theorem 2 we have:
\[ \frac{2\lambda}{\lambda} \times S(n, \lambda) \leq \sum_{k=1}^n E[|X_k - Y_k|] \leq \frac{2\lambda}{\lambda} \times S(n, \lambda). \]
That is
\[ \sum_{k=1}^n E[|X_k - Y_k|] = \frac{2\lambda}{\lambda} \times S(n, \lambda). \] (62)

The proof of (61) follows directly from (62) and (56), because
\[ p = q = 1/2 \] and
\[ S(n, \lambda) = \sum_{k=1}^n k2^{-2k}(\frac{k}{k}). \]

Equation (61) is one of the main results of Kranakis [13].
A. Graphic illustrations of $C_{opt}$

In order to illustrate our results regarding $C_{opt}$, here we show some graphs of the interval of $C_{opt}$. These graphs were generated by considering some fixed values of the parameters $\lambda_1$ and $\lambda_2$ through simulation. First, Figure 1 contains the graph of the Poisson process $\{N_i(t)\}_{t \geq 1}$, $\{N_i(t)\}_{t \geq 1}$ with rates $\lambda_1 = 0.8$, $\lambda_2 = 0.6$, respectively. For these processes, the respective arrival times $X_i$ and $Y_i$ with distributions $\text{Gamma}(i, \lambda_1)$ and $\text{Gamma}(j, \lambda_2)$ are illustrated in Figure 2. Next, the interval $[l_n, s_n]$ defined in (52) is illustrated in Figure 3.

![Arrival Times](image)

Fig. 1. Poisson processes $\{N_1(t)\}_{t \geq 1}$ and $\{N_2(t)\}_{t \geq 1}$ with $\lambda_1 = 0.8$ and $\lambda_2 = 0.6$.

![Arrival Time in Poisson process](image)

Fig. 2. Arrival time of the Poisson process with rates $\lambda_1 = 0.8$ and $\lambda_2 = 0.6$.

![Fig. 3. Minimum expected cost intervals of transport: $C_{opt}$ (Exact Cost), $s_n$ (upper bound) and $l_n$ (lower bound) as in (53) and (43).](image)

IV. STATISTICAL INFERENCE OF $C_{opt}$

A. Estimation

Let $X_i = (X_{i1}, X_{i2}, \ldots, X_{im})$ and $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{im})$; $i \in \{1, 2, \ldots, n\}$ be random samples from the Poisson processes with arrival rates $\lambda_1$ and $\lambda_2$ and respective arrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$. Then:

$$X_{ij} \sim \text{Gamma}(i, \lambda_1) \text{ and } Y_{ij} \sim \text{Gamma}(i, \lambda_2)$$

for all $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$.

Consider the sample minimum cost

$$\hat{C}_{opt}(n, m) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} |X_{ij} - Y_{ij}|. \quad (63)$$

Here, we prove that (63) is a good estimator of $C_{opt}(\lambda_1, \lambda_2, n)$ obtained in (27). In addition, we prove the asymptotic normality of (63) and then define a confidence interval of $C_{opt}(\lambda_1, \lambda_2, n)$.

Since $\{ \sum_{i=1}^{n} |X_{ij} - Y_{ij}| \}_{j \geq 1}$ is an infinite sequence of independent and identically distributed (i.i.d.) terms with expected value

$$E[\hat{C}_{opt}(n, m)] = C_{opt}(\lambda_1, \lambda_2, n), \quad (64)$$

by the strong law of large numbers, (see(2)) we have that $\hat{C}_{opt}(n, m)$ converges almost surely to the expected value $C_{opt}(\lambda_1, \lambda_2, n)$, that is:

$$\hat{C}_{opt}(n, m) \xrightarrow{a.s.} C_{opt}(\lambda_1, \lambda_2, n), \quad m \to \infty \quad (65)$$

or

$$P\left( \lim_{m \to \infty} \hat{C}_{opt}(n, m) = C_{opt}(\lambda_1, \lambda_2, n) \right) = 1.$$

Therefore, from (64) and (65), $\hat{C}_{opt}(n, m)$ is an unbiased estimator of $C_{opt}(\lambda_1, \lambda_2, n)$. On the other hand, since the
From (65) and (67), we define the confidence interval for the theoretical cost variance of the sample minimum cost \( \hat{C}_{opt}(n, m) \). That is:
\[
\hat{C}_{opt}(n, m) \sim N\left( C_{opt}, \frac{n(n+1)}{2m} \left( \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right) \right), \quad m \to \infty.
\] (67)

From (65) and (67), we define the confidence interval for the theoretical cost variance of the sample minimum cost \( C_{opt}(\lambda_1, \lambda_2, n) \) with confidence level of \( 1 - \alpha \), by:
\[
I_{100(1-\alpha)%}(C_{opt}) = [ \mathcal{L}_n, \mathcal{U}_n ],
\] (68)

where
\[
\mathcal{L}_n = \hat{C}_{opt} - z_{\alpha/2} \sqrt{\text{Var}(\hat{C}_{opt})}
\] and
\[
\mathcal{U}_n = \hat{C}_{opt} + z_{\alpha/2} \sqrt{\text{Var}(\hat{C}_{opt})}
\]

### B. Numerical Illustrations
The performance of statistic (63) was tested by Monte Carlo simulation with 10 combinations of \( \lambda_1 \) and \( \lambda_2 \), as defined in Table 1. We use Algorithm 1 implemented in the computational software [16].

#### Algorithm 1: Monte Carlo Simulation for Sample Minimum Cost

**Input:** Rates: \( \lambda_1 \) and \( \lambda_2 \)
Number of Replications: \( m \)
Sizes of Sample (vector): \( n \)

**Output:** Sample Minimum Cost (\( \hat{C}_{opt}(n, m) \))

```plaintext
function generate_sample_cost
    \( \hat{C}_{opt} = [ ] \)
    for \( j \leftarrow 1 \) to length(\( n \))
        for \( i \leftarrow 1 \) to \( n \)
            \( P_1: \) Generate Random Sample (size=m) of Gamma(\( i, \lambda_1 \));
            \( P_2: \) Generate Random Sample (size=m) of Gamma(\( i, \lambda_2 \));
            Dif.Abs := Determine the absolute values of \( (P_2 - P_1) \);
            Mean.Dif := Calculate the means of Dif.Abs;
            Sum.Mean := Add the values of Mean.Dif;
            \( \hat{C}_{opt} := \) Sum.Mean;
        return \( \hat{C}_{opt} \)
```

Tables [IV] report the results of the mean estimates of the sample minimum cost \( \hat{C}_{opt}(n, m) \) as (63), the values of the theoretical cost \( C_{opt}(\lambda_1, \lambda_2, n) \) as (64), the bias, and the mean square error (MSE) of \( C_{opt}(n, m) \). The bias and MSE of \( \hat{C}_{opt}(n, m) \) decrease as \( m \) grows. Table [I] reports the results for \( m = 500 \), Table [II] for \( m = 1000 \) and Table [IV] for \( m = 1e6 \). In this last case the bias and the MSE are small, so \( \hat{C}_{opt}(n, m) \) is a consistent estimator of \( C_{opt}(\lambda_1, \lambda_2, n) \).

#### Table I

<table>
<thead>
<tr>
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<th>( C_{opt}(n, m) )</th>
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<th>bias</th>
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The results of the corresponding confidence interval
\[ I_{95\%}(C_{opt}) = [L_n, U_n], \]
obtained in (68), are shown in Table VI. The results are satisfactory.

The adjustment of the theoretical cost by the sample cost.

The asymptotic normality of
\[ I \]
is well estimated by
\[ \lambda \]
is illustrated well in Figures 4 and 5.

Graphically, the convergence of \( \hat{C}_{opt}(n, m) \) to \( C_{opt}(\lambda_1, \lambda_2, n) \) is illustrated well in Figures 4 and 5. The confidence interval
\[ I_{95\%}(C_{opt}) = [L_n, U_n], \]
shown in Figure 6 indicates that with 95% confidence, the cost
\[ C_{opt}(\lambda_1, \lambda_2, n) \] is well estimated by \( \hat{C}_{opt}(n, m) \). Finally, the asymptotic normality of \( C_{opt}(n, m) \), proved in (67), is illustrated in Figure 7. The four graphs show the good adjustment of the theoretical cost by the sample cost.

---

TABLE IV
\[ C_{opt}(\lambda_1, \lambda_2, n) \], MEAN ESTIMATES, MSE AND BIAS OF \( \hat{C}_{opt}(n, m) \) WITH \( n = 50 \) AND \( m = 1 \times 5 \).

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TABLE V
\[ C_{opt}(\lambda_1, \lambda_2, n) \], MEAN ESTIMATES, MSE AND BIAS OF \( \hat{C}_{opt}(n, m) \) WITH \( n = 50 \) AND \( m = 1 \times 6 \).

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TABLE VI
CONFIDENCE INTERVAL FOR \( C_{opt}(\lambda_1, \lambda_2, n) \), WITH 95% OF CONFIDENCE.

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<th>( \hat{C}_{opt}(n, m) )</th>
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Empirical and theoretical dens.

Q–Q plot

Empirical and theoretical CDFs

P–P plot

Fig. 7. Asymptotic normality of $\hat{C}_{opt}(n, m)$ as $n \to \infty$: histogram of $\hat{C}_{opt}(n, m)$ versus normal density (top left), Q–Q plot (top right), empirical versus theoretical cumulative distributions (bottom left), and P–P plots (bottom right).

V. CONCLUSION

In this article, we derived an exact expression and an interval for the sum of the expected absolute difference between two Poisson processes that can have different rates. Our results generalize those of [13], and to apply our results we calculated the minimum transport cost of a random two-color combination when two sensors are initially placed according to two Poisson process with different or equal laws. We performed a complete statistical inference study, proved asymptotic normality of the cost estimator, and performed a simulation study to show the consistency of the cost estimator.

REFERENCES