An Interval for the Sum of the Expected Absolute Difference between Distinct Poisson Processes

Adolfo M. D. Silva and Cira E. G. Otiniano

***Abstract*—In this work, a closed analytical formula for the expected absolute difference between two independent Poisson processes with arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$ and respective arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ is determined. When considering that a pair of two-color sensors $\{X_k, Y_j\}$ are initially placed according to the processes described, the transport cost that minimizes the energy consumption is given by the sum of the expected absolute differences between the two processes. Here, an exact interval for the transport cost is obtained. In addition, we show that the sample cost is a strongly consistent and unbiased estimator of the theoretical transport cost. The consistency of the sample cost is illustrated with Monte Carlo simulation experiments and with some graphic illustrations.**

***Index Terms*—Transport cost, Expected distance, Poisson process, Sensors.**

I. Introduction

MOBILE sensors are used for data monitoring and communication for various purposes, such as oceanographic research [15], tropical air analysis [4], robotics [17], and security monitoring [14], among others.

One of the main research topics in this area is the determination of an optimal allocation of the sensors in order to generate good coverage at a minimum cost [3].

Through mobile sensor technology, good coverage can be achieved by placing the sensors in the desired positions. However, mobile sensors are generally equipped with a battery and the energy expenditure is much greater during the movement of the sensor than during its detection function. Therefore, it is important to minimize the movements of the sensor to increase its useful life and maintain the reliability of the network to which it belongs.

There are two approaches to studying the minimum expected cost of transport: the sum or maximum of the movements of the sensors from their initial positions to the destination. With respect to the sum, Huesmann and Sturm [10] given a mathematical approach to the optimal transport from Lebesgue to Poisson process. An empirical approach to the cost of optimal incomplete transportation can be found in [6]. For an unified approach of a series of papers about behaviour asymptotic of a binomial and a Poisson sum which arose as (average) displacement costs when moving randomly placed sensors to anchor positions, see [8].

Ajtai and Komlós [1] considered 2n sensors, n blue $X_1, X_2, \cdots X_n$ and n red $Y_1, Y_2, \cdots Y_n$, distributed independently and uniformly in a unit square, and proved that the expected minimum cost of transportation, denoted by T_n and defined by $T_n := \min_{\pi} \sum_{i=1}^{n} d(X_{\pi(i)}, Y_i)$, belong $\Theta\big(\sqrt{n \log n}\big)$. Kranakis [13], when assuming that the sensors move randomly on a line according to two independent and identically distributed Poisson processes with arrival rate λ and respective arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2 \cdots$ determined an interval for the expected minimum cost of transport, defined by $C_T = \sum_{k=1}^{n} E\big[|X_k - Y_k|\big]$. Kapelko [12] generalized the result of Kranakis [13]. He considered the same hypotheses as [13] and determined an asymptotic expression for the expected minimum cost at power $a > 0$, $C_T^a = \sum_{k=1}^{n} E\big[|X_k - Y_k|^a\big]$. Recently, Kapelko [11], when considering two identical and independent general random processes, determined asymptotic expressions for the expected minimum transport cost at power $b > 0$, C_T^b. A more general transportation cost problem than that addressed in the articles cited above occurs when it is assumed that the sensors move according to two independent stochastic processes, that do not necessarily have the same distribution. In this paper, we study this more general problem. Our results generalize Kranakis's results [13].

We obtain the transport cost $C_{opt} = C_T$ for a particular case and an exact interval for C_{opt}, by considering a network of two sensors $\{X_i, Y_j\}$, where $X_1, X_2, \cdots$ are blue and $Y_1, Y_2, \cdots$ are red, which are initially randomly allocated according to a Poisson process with arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Note that λ_1 can be different from λ_2, so the sensors $\{X_i\}$ and $\{Y_i\}$ follow a different law. In addition to obtaining an interval for the expected transport cost, here we carry out a statistical inference study and verify that the sample transport cost is a consistent estimator of the theoretical transport cost found.

Kranakis [13], Kapelko [12] and Kapelko [11] based their results on combinatorial theory, but for the proof of our results we also use results of the following special functions: gamma function, upper and lower incomplete gamma functions, beta function, and incomplete beta function. These functions are defined, respectively, by:

$$\Gamma(a) := \int_0^{\infty} t^{a-1} e^{-t} dt \;, \tag{1}$$

$$\Gamma(a, x) := \int_x^{\infty} t^{a-1} e^{-t} dt, \tag{2}$$

$$\gamma(a, x) := \int_0^{x} t^{a-1} e^{-t} dt \;, \tag{3}$$

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Adolfo M. D. Silva is an alumnus of the Department of Statistics, University of Brasilia, Brazil e-mail: adolfomanoel@hotmail.com

Cira E. G. Otiniano is Associate Professor at the Department of Statistics, University of Brasilia, Brazil e-mail: ciragotiniano@gmail.com

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}dt, \text{and} \tag{4}$$

and

$$B_x(a,b) := \int_0^x t^{a-1}(1-t)^{b-1}dt. \tag{5}$$

The following identities (see [9]) are also used:

$$\Gamma(a) = \gamma(a,x) + \Gamma(a,x), \tag{6}$$

$$\Gamma(n+1,x) = n!\, e^{-x} \sum_{r=0}^{n} \frac{x^r}{r!}, \tag{7}$$

$$\gamma(n+1,x) = n!\big(1 - e^{-x} \sum_{r=0}^{n} \frac{x^r}{r!}\big), \tag{8}$$

and

$$\begin{aligned} \frac{\Gamma(x+h)}{\Gamma(x)} &= x^{(h)}: \quad \text{Pochhammer polynomial} \\ &= x(x+1)(x+2)\cdots(x+h-1), \text{if } h \geq 1. \end{aligned} \tag{9}$$

The rest of the paper is organized as follows: Section 2 describes our main results; Section 3, presents the statistical inference results about transport cost and illustrations of the results generated from Monte Carlo simulation experiments; and Section 4 concludes.

II. Expected Distance

In this section we present Theorem 1 in which we determine a closed analytic expression for $E\big[|X_k - Y_k|\big]$. Let X_i and Y_k be random variables that represent the i-th and k-th arrival times of two independent Poisson processes with rates λ_1 and λ_2. Then, X_i and Y_k have gamma distribution. With the notation

$$X_i \sim Gama(i,\lambda_1) \quad \text{e} \quad Y_k \sim Gama(k,\lambda_2),$$

the random variables X_i and Y_k have probability density functions (pdf‘s)

$$f_{X_i}(x) := f_1(x) = \frac{\lambda_1^i}{\Gamma(i)} x^{i-1} e^{-\lambda_1 x} \qquad x > 0 \tag{10}$$

and

$$f_{Y_k}(y) := f_2(y) = \frac{\lambda_2^k}{\Gamma(k)} y^{k-1} e^{-\lambda_2 y} \qquad y > 0, \tag{11}$$

respectively. The shape parameters are i and k and the scale parameters are $\lambda_1 > 0$ and $\lambda_2 > 0$. The particular cases of our results are in Corollaries 1 and 2. These results correspond to the main results of [13].

Theorem 1: Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively; $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$. Then

$$E|X_{k+r} - Y_k| = \frac{k+r}{\lambda_1} - \frac{k}{\lambda_2} + 2k(k+r)\binom{2k+r}{k}\Delta B_p, \tag{12}$$

for non-negative integers $r \geq 0$ and $k \geq 1$, where

$$\Delta B_p = \frac{B_p(k+r,k+1)}{\lambda_2} - \frac{B_p(k+r+1,k)}{\lambda_1}.$$

Proof: By using the conditional expectation property, we have:

$$\begin{aligned} E|X_i - Y_k| &= E\big[E\big(|X_i - Y_k| \;\big|\; Y_k\big)\big] \\ &= \int_0^\infty E|X_i - y|\; f_2(y)\; dy. \end{aligned} \tag{13}$$

To find the the expected value of (13), consider:

$$E|X_i - y| = I_1 + I_2, \quad \text{with} \tag{14}$$

$$I_1 = \int_0^y -(x-y) f_1(x)\, dx \quad \text{and} \quad I_2 = \int_y^\infty (x-y) f_1(x)\, dx.$$

By combining (10), I_1 and I_2, we deduce that:

$$I_1 = \frac{\lambda_1^i}{\Gamma(i)} \int_y^\infty x^i e^{-\lambda_1 x}\, dx - \frac{i}{\lambda_1} + \frac{y\lambda_1^i}{\Gamma(i)} \int_0^y x^{i-1} e^{-\lambda_1 x}\, dx. \tag{15}$$

and

$$I_2 = \frac{\lambda_1^i}{\Gamma(i)} \int_y^\infty x^i e^{-\lambda_1 x}\, dx - \frac{y\lambda_1^i}{\Gamma(i)} \int_y^\infty x^{i-1} e^{-\lambda_1 x}\, dx. \tag{16}$$

Now (15) and (16) are replaced in equation (14). The expected value result in terms of the incomplete gamma functions (3) and (2) is:

$$\begin{aligned} E|X_i - y| = &\frac{2}{\lambda_1 \Gamma(i)} \Gamma(i+1, \lambda_1 y) - \frac{i}{\lambda_1} + \\ &+ \frac{y}{\Gamma(i)} \Big(\gamma(i,\lambda_1 y) - \Gamma(i,\lambda_1 y)\Big). \end{aligned} \tag{17}$$

Now by substituting (17) in (13) we obtain an expression composed of the following three new integrals:

$$E|X_i - Y_k| = -\frac{i}{\lambda_1} + \frac{2}{\lambda_1 \Gamma(i)} J_1 + \frac{1}{\Gamma(i)} J_2 - \frac{1}{\Gamma(i)} J_3, \tag{18}$$

where

$$J_1 := \int_0^\infty \Gamma(i+1,\lambda_1 y) f_2(y)\, dy,$$

$$J_2 := \int_0^\infty y\; \gamma(i,\lambda_1 y) f_2(y)\; dy$$

and

$$J_3 := \int_0^\infty y\; \Gamma(i,\lambda_1 y) f_2(y)\, dy.$$

These integrals are calculated using the series representation of the incomplete gamma functions (7), (8) and the density (11). After algebraic manipulations, we deduce that:

$$J_1 = \frac{\Gamma(i+1)}{\Gamma(k)} q^k \sum_{s=0}^{i} \left[p^s \frac{\Gamma(s+k)}{s!}\right], \tag{19}$$

$$J_2 = \frac{\Gamma(i)k}{\lambda_2} - \frac{\Gamma(i)}{\lambda_2\Gamma(k)}q^{k+1}\sum_{s=0}^{i-1}\left[p^s\frac{\Gamma(s+k+1)}{s!}\right] \quad (20)$$

and

$$J_3 = \frac{\Gamma(i)}{\lambda_2\Gamma(k)}q^{k+1}\sum_{s=0}^{i-1}\left[p^s\frac{\Gamma(s+k+1)}{s!}\right]. \quad (21)$$

with $p = \lambda_1/(\lambda_1+\lambda_2)$ and $q = 1-p$.

By combining integrals (19), (20) and (21) in (18), we get:

$$\begin{aligned} E|X_i - Y_k| &= \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i\, q^k}{\lambda_1\Gamma(k)}\sum_{s=0}^{i}p^s\frac{\Gamma(s+k)}{s!} \\ &\quad - \frac{q^{k+1}}{\lambda_2\,\Gamma(k)}\sum_{s=0}^{i-1}p^s\frac{\Gamma(s+k+1)}{s!} \\ &= \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i\, q^k}{\lambda_1}\sum_{s=0}^{i}\binom{s+k-1}{s}p^s \\ &\quad - \frac{2k\, q^{k+1}}{\lambda_2}\sum_{s=0}^{i-1}\binom{s+k}{s}p^s, \end{aligned} \quad (22)$$

where $p = \lambda_1/(\lambda_1+\lambda_2)$ and $q = 1-p$.

Finally, we update equation (22), to obtain:

$$\begin{aligned} &E|X_i - Y_k| = \\ &= \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i\, q^k}{\lambda_1}\left[\frac{1-(i+1)\binom{i+k}{k-1}B_p(i+1,k)}{(1-x)^k}\right] \\ &- \frac{2k\, q^{k+1}}{\lambda_2}\left[\frac{1-i\binom{i+k}{k}B_p(i,k+1)}{q^{k+1}}\right] \\ &= \frac{i}{\lambda_1} - \frac{k}{\lambda_2} + 2ik\binom{i+k}{k}\left(\frac{B_p(i,k+1)}{\lambda_2} - \frac{B_p(i+1,k)}{\lambda_1}\right), \end{aligned} \quad (23)$$

by using the identity (see [7]),

$$\sum_{s=0}^{L}\binom{n+s}{s}p^s = \frac{1-(L+1)\binom{L+n+1}{n}B_p(L+1,n+1)}{(1-p)^{n+1}}. \quad (24)$$

Replacing i by $k+r$ in (23) finishes the proof. ■

An expression equivalent to (12), in terms of the regularized incomplete beta function, is provided below. To obtain this result, just replace the expressions

$$B(k+r,k+1) = \frac{1}{(k+r)\binom{2k+r}{k}} \quad (25)$$

and

$$B(k+r+1,k) = \frac{1}{k\binom{2k+r}{k}} \quad (26)$$

in the regularized incomplete beta function

$$I_p(a,b) := \frac{B_p(a,b)}{B(a,b)}.$$

Then

$$\begin{aligned} E|X_{k+r} - Y_k| = \frac{k+r}{\lambda_1} - \frac{k}{\lambda_2} + \frac{2k\, I_p(k+r,k+1)}{\lambda_2} \\ - \frac{2(k+r)\, I_p(k+r+1,k)}{\lambda_1} \end{aligned} \quad (27)$$

is equivalent to (12).

Corollary 1: Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. If $r = 0$, $k \in \mathbb{Z}_{\geq 1}$ and $\lambda_1 = \lambda_2 = \lambda > 0$, then

$$E|X_k - Y_k| = \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k} \quad (28)$$

Proof: From Theorem 1, for $r = 0$ and $p = \lambda_1/(\lambda_1+\lambda_2) = 1/2$, we have

$$E|X_{k+r} - Y_k| = \frac{2k^2}{\lambda}\binom{2k}{k}\left[B_{\frac{1}{2}}(k,k+1) - B_{\frac{1}{2}}(k+1,k)\right]. \quad (29)$$

The identity

$$B_x(a;n+1-a) = B(a;n+1-a)\sum_{j=a}^{n}\binom{n}{j}x^j(1-x)^{n-j}, \quad (30)$$

(see [7]), allows rewriting the difference in equation (29) as:

$$\begin{aligned} B_{\frac{1}{2}}(k,k+1) - B_{\frac{1}{2}}(k+1,k) &= \frac{B(k,k+1)}{2^{2k}}\binom{2k}{k} \\ &= \frac{1}{k2^{2k}}, \end{aligned} \quad (31)$$

by using the identity

$$B(k,k+1) = \frac{1}{2k}\binom{2k-1}{k-1}^{-1} = \frac{1}{k}\binom{2k}{k}^{-1}. \quad (32)$$

The result (28) is obtained by replacing (31) by (29). ■

Corollary 2: Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. If $r > 0$, $k \in \mathbb{Z}_{\geq 1}$ and $\lambda_1 = \lambda_2 = \lambda > 0$, then

$$S_{k,r} = \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k}\left(1 + \sum_{s=0}^{r-1}\frac{r-s}{(2k+s)\,2^s}\frac{(2k+1)^{(s)}}{(k+1)^{(s)}}\right), \quad (33)$$

where $S_{k,r} = E|X_{k+r} - Y_k|$.

Proof: For $\lambda_1 = \lambda_2 = \lambda$ and $r > 0$, from Theorem 1, we have:

$$S_{k,r} = \frac{r}{\lambda} + \frac{2k(k+r)}{\lambda}\binom{2k+r}{k}Q(k,r). \quad (34)$$

with $Q(k,r) = \left[B_{\frac{1}{2}}(k+r,k+1) - B_{\frac{1}{2}}(k+r+1,k)\right]$.

Equation (34) is updated by rewriting $B_{\frac{1}{2}}(k+r,k+1)$

and $B_{\frac{1}{2}}(k+r+1,k)$ with identity (30) as

$$\begin{aligned} S_{k,r} &= \frac{-r}{\lambda} + \frac{2(k+r)}{\lambda 2^k}\sum_{s=0}^{k+r}\binom{s+k-1}{s}\frac{1}{2^s} \\ &\quad - \frac{2k}{\lambda 2^{k+1}}\sum_{s=0}^{k+r-1}\binom{s+k}{s}\frac{1}{2^s} \\ &= -\frac{r}{\lambda} + \frac{2(k+r)}{2^k\lambda\Gamma(k)} \times \sum_{j=0}^{1} H_j, \end{aligned} \tag{35}$$

where

$$\begin{aligned} H_0 &= \sum_{s=0}^{k+r}\frac{(k-1)!}{2^s}\binom{s+k-1}{s} \\ &\overset{(s-k:=t)}{=} \Gamma(k)\left[2^{k-1} + 2^{-k}\sum_{t=0}^{r}\binom{t+2k-1}{k-1}2^{-t}\right] \end{aligned} \tag{36}$$

and

$$\begin{aligned} H_1 &= \sum_{s=0}^{k+r-1}\left(-\frac{1}{k+r}\right)\frac{k!}{2^{s+1}}\binom{s+k}{s} \\ &= -\frac{k!}{2(k+r)}\left[2^k - \binom{2k}{k}2^{-k} + \right. \\ &\quad \left. + 2^{-k}\sum_{t=0}^{r-1}\frac{2k+t}{k}\binom{2k+t-1}{k-1}2^{-t}\right]. \end{aligned} \tag{37}$$

By replacing H_0 and H_1 in (35), we get:

$$\begin{aligned} S_{k,r} =& \frac{k2^{-2k}}{\lambda}\binom{2k}{k} + \frac{2^{-2k+1}}{\lambda}\sum_{t=0}^{r-1}(r-t)\binom{2k+t-1}{k-1}2^{-t} \\ &+ \frac{2^{-2k}}{\lambda}\left[2^{-r}(2k+r)C_{2k+r-1}^{k-1} + \sum_{t=0}^{r} t\binom{2k+t-1}{k-1}2^{-t}\right]. \end{aligned} \tag{38}$$

Now, the identity

$$\sum_{t=0}^{r} t\binom{2k+t-1}{k-1}2^{-t} = 2kC_{2k-1}^{k-1} - 2^{-r}(2k+r)C_{2k+r-1}^{k-1},$$

valid for $k \geq 1$, is applied in (38), so:

$$\begin{aligned} S_{k,r} =& \frac{k2^{-2k}}{\lambda}\binom{2k}{k} + \frac{2^{-2k+1}}{\lambda}\sum_{t=0}^{r-1}(r-t)\binom{2k+t-1}{k-1}2^{-t} \\ &+ \frac{k2^{-2k+1}}{\lambda}\binom{2k-1}{k-1} \\ =& \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k} + \frac{2^{-2k+1}}{\lambda}\sum_{t=0}^{r-1}(r-t)\binom{2k+t-1}{k-1}2^{-t}. \end{aligned} \tag{39}$$

Finally, by replacing the binomial identity

$$\binom{2k+s-1}{k-1} = k\binom{2k}{k}\frac{1}{2k+s}\frac{(2k+1)^{(s)}}{(k+1)^{(s)}} \tag{40}$$

in (39), the result (33) is obtained. ∎

III. Minimum expected transport cost

In this section, we present an interval for the expected transport cost of a pair $\{X_i, Y_k\}$ of sensors placed randomly in the interval $[0,\infty)$. The position of the i-th sensor (blue) and the k-th (red) are determined by the arrival times X_i and Y_k, according to two Poisson process with arrival rates λ_1 and λ_2, respectively. This expected transport cost corresponds to:

$$C_{opt}(\lambda_1,\lambda_2,n) = \sum_{k=1}^{n} E\big[|X_k - Y_k|\big].$$

In Theorem 2 we obtain a closed formula in terms of incomplete gamma function for $C_{opt}(\lambda_1,\lambda_2,n)$ and in Theorem 3 we obtain an exact interval for $C_{opt}(\lambda_1,\lambda_2,n)$.

Theorem 2: Consider two independent Poisson processes with arrival rate $\lambda_1 > 0$ and $\lambda_2 > 0$ and respectives arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$. If $k \geq 1$ is integer and $\lambda_2 > \lambda_1$, then:

$$C_{opt}(\lambda_1,\lambda_2,n) = \frac{2\,\lambda_1\, e^{\lambda_1\lambda_2/(\lambda_1+\lambda_2)}}{\lambda_2\,(\lambda_1+\lambda_2)\,\Gamma(n)}\,\Gamma\!\left(n\,,\,\frac{\lambda_1\,\lambda_2}{\lambda_1+\lambda_2}\right). \tag{41}$$

Lemma 1: Let $f_1(\cdot)$ and $f_2(\cdot)$ be densities of the random variables $X_i \sim Gamma(i,\lambda_1)$ and $Y_k \sim Gamma(k,\lambda_2)$, respectively, with $\lambda_1 > 0$ and $\lambda_2 > 0$. Then, for $y > 0$, the double integral

$$I_2 := \int_0^\infty\int_0^y (t-y)\, f_1(t) f_2(y)\, dt\, dy \tag{42}$$

is given by

$$I_2 = \frac{-\lambda_1^i\lambda_2^{k-i-2}}{\Gamma(k)}\left(1+\frac{\lambda_1}{\lambda_2}\right)^{-i}, \tag{43}$$

when $\lambda_2 > \lambda_1$.

Proof: From (10) and (11),

$$\begin{aligned} I_2 &:= \int_0^\infty\int_0^y (t-y)\, f_1(t) f_2(y)\, dt\, dy \\ &= \frac{\lambda_1^i\lambda_2^k}{\Gamma(i)\Gamma(k)}\int_0^\infty\int_0^y (t-y)\, t^{i-1}e^{-\lambda_1 t}y^{k-1}e^{-\lambda_2 y}\, dt\, dy. \end{aligned} \tag{44}$$

By performing the substitution $t = uy$, we update (44) and get

$$I_2 = \frac{-\lambda_1^i\lambda_2^k}{\Gamma(i)\Gamma(k)}\int_0^\infty y^{i+1}e^{-\lambda_2 y} I_{21} dy, \tag{45}$$

where $I_{21} := \int_0^1 u^{i-1}(1-u)\, e^{-(\lambda_1 y)u}\, du$.

The integral representation of the confluent hypergeometric function (see [5] p. 185)

$${}_1F_1(d;c;x) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)}\int_0^1 e^{xt}\, t^{d-1}(1-t)^{c-d-1}\, dt, \tag{46}$$

valid for $c \in \mathbb{C}$, $d \in \mathbb{C}$, $\mathcal{R}(c) > \mathcal{R}(d) > 0$, and $x \in \mathbb{C}$, allows us to rewrite I_{21} as

$$I_{21} = \frac{\Gamma(i)}{\Gamma(i+2)}\,{}_1F_1(i;i+2;-\lambda_1 y), \tag{47}$$

where $d = i$, $c = i+2$ e $x = -\lambda_1 y$. Replacing (47) in (45), we have that

$$I_2 = \frac{-\lambda_1^i \lambda_2^k}{\Gamma(k)\Gamma(i+2)} \int_0^\infty y^{i+1} e^{-\lambda_2 y} {}_1F_1(i; i+2; -\lambda_1 y)\, dy. \tag{48}$$

When $\lambda_2 > \lambda_1 > 0$, we use the identity (see [5] p. 187)

$$\int_0^\infty t^{\mu-1} e^{-xt} {}_1F_1(d;\, \mu\,; qt)\, dt = \Gamma(\mu) x^{-\mu} \left(1 - \frac{q}{x}\right)^{-d}, \tag{49}$$

with $x = \lambda_2,\ q = -\lambda_1,\ d = i,$ and $\ \mu = i+2$, for rewrite the integral in (48). Then

$$\begin{aligned} I_{22} &= \int_0^\infty y^{i+1} e^{-\lambda_2 y} {}_1F_1(i; i+2; -\lambda_1 y)\, dy \\ &= \Gamma(i+2) \lambda_2^{-(i+2)} \left(1 + \frac{\lambda_1}{\lambda_2}\right)^{-i}. \end{aligned} \tag{50}$$

Finally, the result (43) is obtained by replacing (50) in (48). ■

Proof: (***Theorem 2***) With the notation of Lemma 1, $E\big[|X_k - Y_k|\big] = -2I_2\big|_{i=k}$. So, the expected transport cost is given by

$$\begin{aligned} C_{opt} &= \sum_{k=1}^n E\big[|X_k - Y_k|\big] \\ &= \frac{2}{\lambda_2^2} \sum_{k=1}^n \left\{ \frac{1}{\Gamma(k)} \left(\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \right)^k \right\}. \end{aligned} \tag{51}$$

By applying the identity (7) in (51), with $x = \lambda_1\lambda_2/(\lambda_1 + \lambda_2)$, we conclude that

$$C_{opt} = \frac{2\,\lambda_1\, e^{\lambda_1\lambda_2/(\lambda_1+\lambda_2)}}{\lambda_2\,(\lambda_1+\lambda_2)\,\Gamma(n)}\, \Gamma\!\left(n\,,\, \frac{\lambda_1\,\lambda_2}{\lambda_1+\lambda_2}\right).$$

■

Theorem 3: Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively; $\lambda_1 \geq \lambda_2$. Then

$$C_{opt}(\lambda_1, \lambda_2, n) \in \big[\, l_n\ ,\ s_n\, \big], \tag{52}$$

where

$$l_n = \frac{n(n+1)}{2} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{2}{\lambda_2} \times S(n, \lambda_1, \lambda_2), \tag{53}$$

$$s_n = \frac{n(n+1)}{2} \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \times S(n, \lambda_1, \lambda_2) \tag{54}$$

and

$$S(n, \lambda_1, \lambda_2) = \sum_{k=1}^n \frac{(pq)^k}{B(k+1, k)}. \tag{55}$$

Proof: For $r = 0$, from (27) we have:

$$E|X_k - Y_k| = \frac{k}{\lambda_1} - \frac{k}{\lambda_2} + 2k\left[\frac{I_p(k, k+1)}{\lambda_2} - \frac{I_p(k+1, k)}{\lambda_1}\right]. \tag{56}$$

By applying identity

$$I_p(a, b) = I_p(a-1, b+1) - \frac{p^{a-1} q^b}{bB(a, b)},$$

in (56), for $a = k+1$ and $b = k$, this results in:

$$E|X_k - Y_k| \geq k\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{2}{\lambda_2} \frac{(pq)^k}{B(k+1, k)}.$$

So, the lower bound of the sum is:

$$\sum_{k=1}^n S_{k,0} \geq \frac{n(n+1)}{2}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{2}{\lambda_2} \sum_{k=1}^n \frac{(pq)^k}{B(k+1, k)}, \tag{57}$$

where $S_{k,0} := E|X_k - Y_k|$.

To obtain the upper limit of the sum, we replace the identities

$$I_p(a, b) = I_p(a, b+1) - \frac{p^a q^b}{bB(a, b)}$$

and

$$I_p(a, b) = I_p(a+1, b) + \frac{p^a q^b}{aB(a, b)}$$

in (56). Then, for $a = b = k$, we get:

$$\begin{aligned} \frac{I_p(k, k+1)}{\lambda_2} - \frac{I_p(k+1, k)}{\lambda_1} &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) I_p(k, k) \\ &\quad + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \frac{(pq)^k}{kB(k, k)}. \end{aligned} \tag{58}$$

The upper limit of the sum is obtained from (56), (58) and the fact that
$I_x(k, k) \leq 1,\ \ \forall\, k \in \mathbb{Z}^+$. That is:

$$E|X_k - Y_k| \leq k\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + 2\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \frac{(pq)^k}{B(k, k)} \tag{59}$$

and

$$\sum_{k=1}^n S_{k,0} \leq \frac{n(n+1)}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \sum_{k=1}^n \frac{(pq)^k}{B(k+1, k)}. \tag{60}$$

Finally, the proof finishes by combining inequalities (57) and (60). ■

Corollary 3: Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 = \lambda_2 = \lambda$. Then

$$C_{opt}(\lambda, \lambda, n) = \frac{2n}{3\lambda} \binom{n + \frac{1}{2}}{n}. \tag{61}$$

Proof: From (52), in Theorem 2, we have

$$\frac{2}{\lambda} \times S(n, \lambda)\ \leq\ \sum_{k=1}^n E\big[|X_k - Y_k|\big]\ \leq\ \frac{2}{\lambda} \times S(n, \lambda)\,.$$

That is

$$\sum_{k=1}^n E\big[|X_k - Y_k|\big] = \frac{2}{\lambda} \times S(n, \lambda). \tag{62}$$

The proof of (61) follows directly from(62) and (26), because $p = q = 1/2$ and $S(n, \lambda) = \sum_{k=1}^n k 2^{-2k} \binom{2k}{k}$. ■

Equation (61) is one of the main results of Kranakis [13].

A. Graphic illustrations of C_{opt}

In order to illustrate our results regarding C_{opt}, here we show some graphs of the interval of C_{opt}. These graphs were generated by considering some fixed values of the parameters λ_1 and λ_2 through simulation. First, Figure 1 contains the graph of the Poisson process $\{N_1(t)\}_{t\geq 1}$, $\{N_2(t)\}_{t\geq 1}$ with rates $\lambda_1 = 0.8$, $\lambda_2 = 0.6$, respectively. For these processes, the respective arrival times X_i and Y_j with distributions $\text{Gamma}(\text{i}, \lambda_1)$ and $\text{Gamma}(\text{j}, \lambda_2)$ are illustrated in Figure 2. Next, the interval $\left[\, l_n \, , \, s_n \,\right]$ defined in (52) is illustrated in Figure 3.

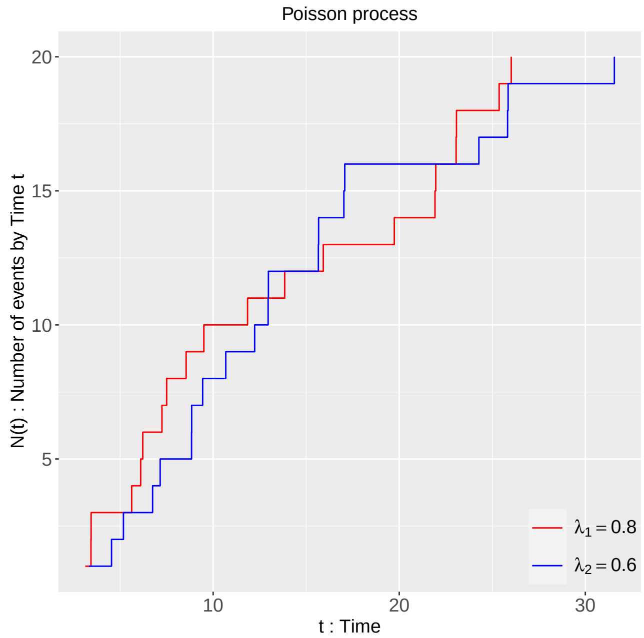

Fig. 1. Poisson processes $\{N_1(t)\}_{t\geq 1}$ and $\{N_2(t)\}_{t\geq 1}$ with $\lambda_1 = 0.8$ and $\lambda_2 = 0.6$.

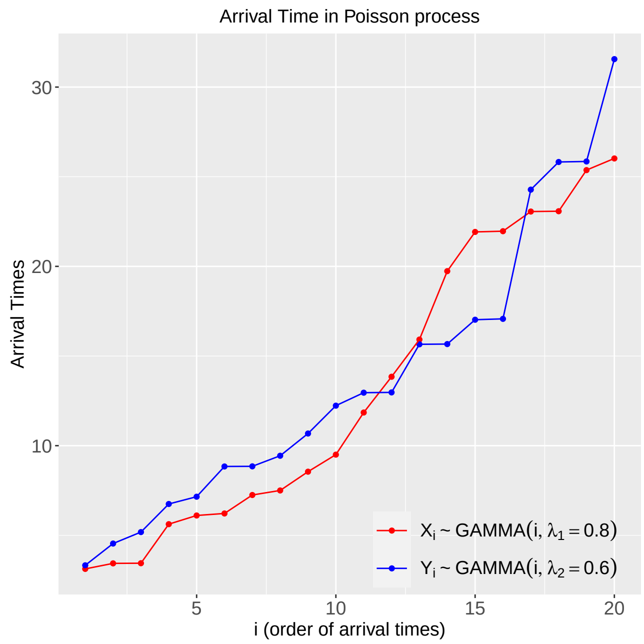

Fig. 2. Arrival time of the Poisson process with rates $\lambda_1 = 0.8$ and $\lambda_2 = 0.6$.

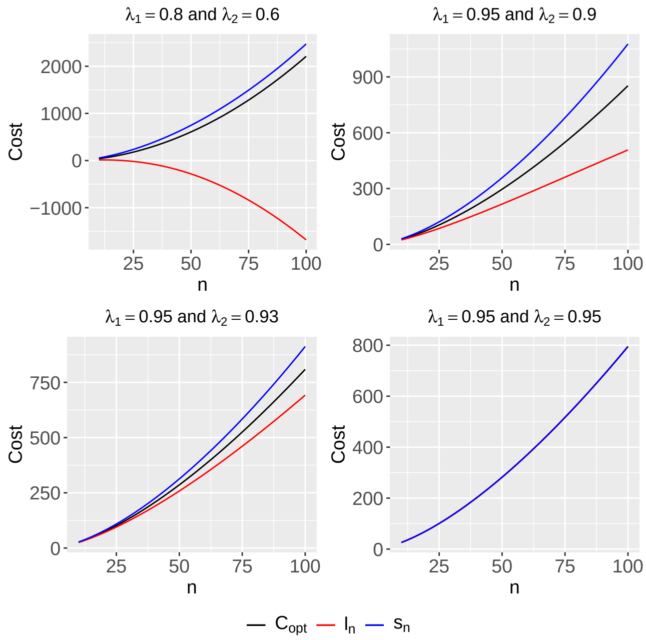

Fig. 3. Minimum expected cost intervals of transport: C_{opt} (Exact Cost), s_n (upper bound) and l_n (lower bound) as in (53) and (43).

IV. Statistical Inference of C_{opt}

A. Estimation

Let $X_i = (X_{i1}, X_{i2}, \ldots, X_{im})$ and $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{im})$; $i \in \{1, 2, \cdots, n\}$ be random samples from the Poisson processes with arrival rates λ_1 and λ_2 and respective arrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$. Then:

$$X_{ij} \sim Gamma(i, \lambda_1) \text{ and } Y_{ij} \sim Gamma(i, \lambda_2)$$

for all $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$.

Consider the sample minimum cost

$$\hat{C}_{opt}(n, m) = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{m} \left| X_{ij} - Y_{ij} \right|. \tag{63}$$

Here, we prove that (63) is a good estimator of $C_{opt}(\lambda_1, \lambda_2, n)$ obtained in (27). In addition, we prove the asymptotic normality of (63) and then define a confidence interval of $C_{opt}(\lambda_1, \lambda_2, n)$.

Since $\{ \sum_{i=1}^{n} \left| X_{ij} - Y_{ij} \right| \}_{j\geq 1}$ is an infinite sequence of independent and identically distributed (i.i.d.) terms with expected value

$$E\left[\hat{C}_{opt}(n, m)\right] = C_{opt}(\lambda_1, \lambda_2, n), \tag{64}$$

by the strong law of large numbers, (see[2]) we have that $\hat{C}_{opt}(n, m)$ converges almost surely to the expected value $C_{opt}(\lambda_1, \lambda_2, n)$, that is:

$$\hat{C}_{opt}(n, m) \xrightarrow{a.s.} C_{opt}(\lambda_1, \lambda_2, n), \quad m \to \infty \tag{65}$$

or

$$P\left(\lim_{m\to\infty} \hat{C}_{opt}(n, m) = C_{opt}(\lambda_1, \lambda_2, n) \right) = 1.$$

Therefore, from (64) and (65), $\hat{C}_{opt}(n, m)$ is an **unbiased estimator** of $C_{opt}(\lambda_1, \lambda_2, n)$. On the other hand, since the variance

$$Var(\hat{C}_{opt}) = \frac{n(n+1)}{2m}\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}\right) < \infty \tag{66}$$

then as m approaches infinity, the sample minimum cost $\hat{C}_{opt}(n,m)$ converges, in distribution, to $N\bigg(E(\hat{C}_{opt}), Var(\hat{C}_{opt})\bigg)$. That is:

$$\hat{C}_{opt}(n,m) \overset{d}{\sim} N\left(C_{opt}, \frac{n(n+1)}{2m}\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}\right)\right),\ m \to \infty. \tag{67}$$

From (65) and (67), we define the confidence interval for $C_{opt}(\lambda_1, \lambda_2, n)$, with confidence level of $1-\alpha$, by:

$$I_{100(1-\alpha)\%}(C_{opt}) = [\ \mathcal{L}_n,\ \mathcal{U}_n\ ], \tag{68}$$

where

$$\mathcal{L}_n = \hat{C}_{opt} - z_{\alpha/2}\sqrt{Var(\hat{C}_{opt})} \quad \text{and}$$
$$\mathcal{U}_n = \hat{C}_{opt} + z_{\alpha/2}\sqrt{Var(\hat{C}_{opt})}$$

B. Numerical illustrations

The performance of statistic (63) was tested by Monte Carlo simulation with 10 combinations of λ_1 and λ_2, as defined in Table 1. We use Algorithm 1 implemented in the computational software [16].

Algorithm 1: Monte Carlo Simulation for Sample Minimum Cost

Input: Rates: λ_1 and λ_2
Number of Replications: m
Sizes of Sample (vector): n
Output: Sample Minimum Cost ($\hat{C}_{opt}(n,m)$)
Function generate.Sample.Cost
  $\hat{C}_{opt} = [\,]$
  for $j \leftarrow 1$ *to length(n)* **do**
    for $i \leftarrow 1$ *to* n_j **do**
      P_1: Generate Random Sample (size=m) of Gamma(i, λ_1);
      P_2: Generate Random Sample (size=m) of Gamma(i, λ_2).
    Dif.Abs:= Determine the absolute values of $(P_2 - P_1)$;
    Mean.Dif:= Calculate the means of *Dif.Abs*;
    Sum.Mean:= Add the values of *Mean.Dif*.
  $\hat{C}_{opt}$:= *Sum.Mean*.
  return $\hat{C}_{opt}$

Tables I-V report the results of the mean estimates of the sample minimum cost $\hat{C}_{opt}(n,m)$ as (63), the values of the theoretical cost $C_{opt}(\lambda_1, \lambda_2, n)$ as (52), the bias, and the mean square error (MSE) of $\hat{C}_{opt}(n,m)$. The bias and MSE of $\hat{C}_{opt}(n,m)$ decrease as m grows. Table I reports the results for $m = 500$, Table II for $m = 1000$ and Table V for $m = 1e6$. In this last case the bias and the MSE are small, so $\hat{C}_{opt}(n,m)$ is a consistent estimator of $C_{opt}(\lambda_1, \lambda_2, n)$.

TABLE I
$C_{opt}(\lambda_1, \lambda_2, n)$, MEAN ESTIMATES, MSE AND BIAS OF $\hat{C}_{opt}(n,m)$ WITH $n = 50$ AND $m = 500$.

λ_1	λ_2	$C_{opt}(\lambda_1,\lambda_2,n)$	$\hat{C}_{opt}(n,m)$	MSE	bias
0.55	0.92	951.65	950.02	14.10	-1.63
0.87	0.42	1578.21	1581.17	26.58	2.96
0.59	0.93	816.06	818.44	15.98	2.39
0.70	0.66	404.71	408.61	26.34	3.91
0.86	0.70	451.87	451.46	8.82	-0.41
0.97	0.47	1405.62	1406.26	14.66	0.64
0.43	0.93	1600.54	1598.93	19.33	-1.61
0.87	0.56	841.31	839.06	16.59	-2.26
0.74	0.79	361.95	359.99	12.58	-1.96
0.53	0.90	1007.33	1005.11	17.19	-2.23

TABLE II
$C_{opt}(\lambda_1, \lambda_2, n)$, MEAN ESTIMATES , MSE AND BIAS OF $\hat{C}_{opt}(n,m)$ WITH $n = 50$ AND $m = 1000$.

λ_1	λ_2	$C_{opt}(\lambda_1,\lambda_2,n)$	$\hat{C}_{opt}(n,m)$	MSE	bias
0.55	0.92	951.65	954.80	15.63	3.15
0.87	0.42	1578.21	1579.04	9.60	0.83
0.59	0.93	816.06	819.84	19.49	3.79
0.70	0.66	404.71	408.16	17.46	3.45
0.86	0.70	451.87	449.20	11.45	-2.67
0.97	0.47	1405.62	1406.27	7.56	0.66
0.43	0.93	1600.54	1596.28	26.56	-4.26
0.87	0.56	841.31	837.15	23.09	-4.16
0.74	0.79	361.95	360.87	5.54	-1.08
0.53	0.90	1007.33	1007.90	6.44	0.57

TABLE III
$C_{opt}(\lambda_1, \lambda_2, n)$, MEAN ESTIMATES , MSE AND BIAS OF $\hat{C}_{opt}(n,m)$ WITH $n = 50$ AND $m = 1E4$.

λ_1	λ_2	$C_{opt}(\lambda_1,\lambda_2,n)$	$\hat{C}_{opt}(n,m)$	MSE	bias
0.55	0.92	951.65	951.03	0.95	-0.62
0.87	0.42	1578.21	1577.51	1.38	-0.70
0.59	0.93	816.06	814.87	1.92	-1.19
0.70	0.66	404.71	404.87	0.58	0.16
0.86	0.70	451.87	451.19	0.90	-0.69
0.97	0.47	1405.62	1406.41	1.34	0.79
0.43	0.93	1600.54	1601.29	1.40	0.75
0.87	0.56	841.31	841.84	0.85	0.52
0.74	0.79	361.95	361.81	0.46	-0.13
0.53	0.90	1007.33	1007.02	0.71	-0.31

TABLE IV
$C_{opt}(\lambda_1, \lambda_2, n)$, MEAN ESTIMATES , MSE AND BIAS OF $\hat{C}_{opt}(n, m)$ WITH $n = 50$ AND $m = 1E5$.

λ_1	λ_2	$C_{opt}(\lambda_1, \lambda_2, n)$	$\hat{C}_{opt}(n, m)$	MSE	bias
0.55	0.92	951.65	951.47	0.09	-0.18
0.87	0.42	1578.21	1578.34	0.11	0.14
0.59	0.93	816.06	816.00	0.05	-0.06
0.70	0.66	404.71	404.65	0.06	-0.06
0.86	0.70	451.87	451.57	0.14	-0.30
0.97	0.47	1405.62	1405.99	0.21	0.37
0.43	0.93	1600.54	1601.10	0.40	0.56
0.87	0.56	841.31	841.78	0.27	0.46
0.74	0.79	361.95	361.93	0.04	-0.02
0.53	0.90	1007.33	1007.48	0.08	0.15

TABLE V
$C_{opt}(\lambda_1, \lambda_2, n)$, MEAN ESTIMATES , MSE AND BIAS OF $\hat{C}_{opt}(n, m)$ WITH $n = 50$ AND $m = 1E6$.

λ_1	λ_2	$C_{opt}(\lambda_1, \lambda_2, n)$	$\hat{C}_{opt}(n, m)$	MSE	bias
0.55	0.92	951.65	951.62	0.01	-0.02
0.87	0.42	1578.21	1578.16	0.01	-0.05
0.59	0.93	816.06	816.10	0.01	0.04
0.70	0.66	404.71	404.71	0.01	0.00
0.86	0.70	451.87	451.85	0.00	-0.02
0.97	0.47	1405.62	1405.64	0.01	0.02
0.43	0.93	1600.54	1600.55	0.01	0.01
0.87	0.56	841.31	841.14	0.04	-0.17
0.74	0.79	361.95	361.91	0.01	-0.04
0.53	0.90	1007.33	1007.34	0.01	0.01

The results of the corresponding confidence interval $I_{95\%}(C_{opt}) = [\ \mathcal{L}_n,\ \mathcal{U}_n\ ]$, obtained in (68), are shown in Table VI. The results are satisfactory.

TABLE VI
CONFIDENCE INTERVAL FOR $C_{opt}(\lambda_1, \lambda_2, n)$, WITH 95% OF CONFIDENCE .

λ_1	λ_2	$\hat{C}_{opt}(n, m)$	$C_{opt}(\lambda_1, \lambda_2, n)$	$[L_{95\%}(C_{opt}),$	$U_{95\%}(C_{opt})]$
0.26	0.25	3065.70	3065.53	[3062.16,	3077.46]
0.55	0.28	8801.69	8802.98	[8797.76,	8808.89]
0.91	0.30	11188.47	11188.42	[11184.60,	11194.31]
0.22	0.04	93329.40	93341.57	[93297.73,	93363.09]
0.46	0.90	5389.82	5391.14	[5384.39,	5391.20]
0.78	0.99	1479.48	1478.87	[1477.50,	1482.05]
0.61	0.60	1243.93	1244.34	[1241.60,	1248.07]
0.20	0.56	15906.65	15905.23	[15895.58,	15910.23]
0.20	0.11	20375.64	20375.64	[20366.78,	20395.68]
0.41	0.07	57903.33	57913.29	[57896.82,	57936.20]

Graphically, the convergence of $\hat{C}_{opt}(n, m)$ to $C_{opt}(\lambda_1, \lambda_2, n)$ is illustrated well in Figures 4 and 5. The confidence interval $I_{95\%}(C_{opt}) = [\ \mathcal{L}_n,\ \mathcal{U}_n\ ]$ shown in Figure 6 indicates that with 95% confidence, the cost $C_{opt}(\lambda_1, \lambda_2, n)$ is well estimated by $\hat{C}_{opt}(n, m)$. Finally, the asymptotic normality of $\hat{C}_{opt}(n, m)$, proved in (67), is illustrated in Figure 7. The four graphs show the good adjustment of the theoretical cost by the sample cost.

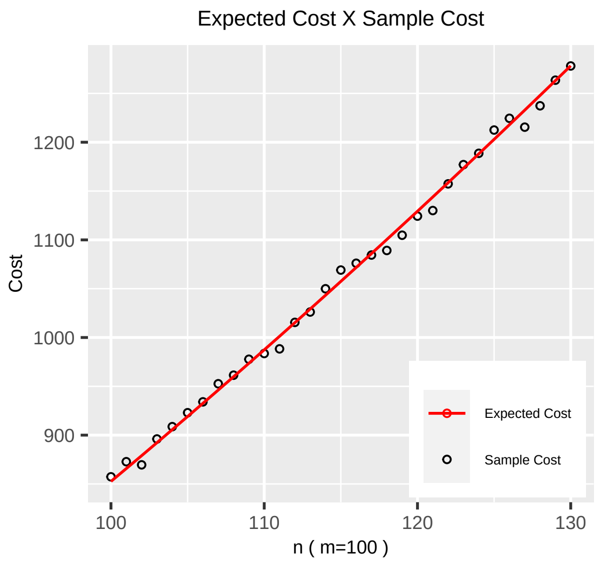

Fig. 4. Graph of $\hat{C}_{opt}(n, m)$ versus $C_{opt}(\lambda_1, \lambda_2, n)$, for $\lambda_1 = 0.95$ and $\lambda_2 = 0.90$.

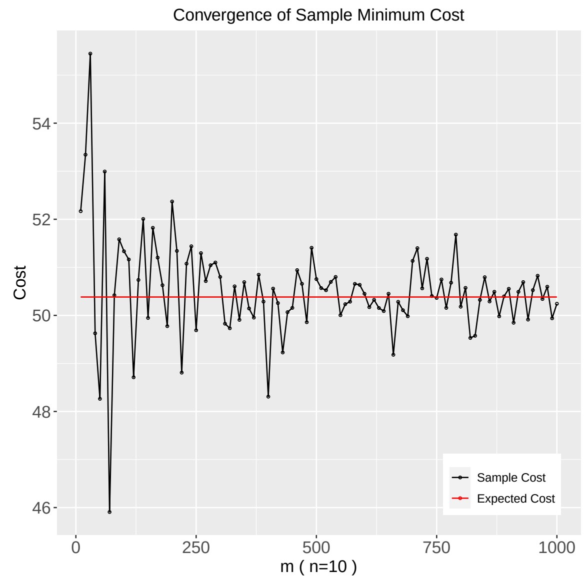

Fig. 5. Graph of $\hat{C}_{opt}(n, m)$ varying m (black) and $C_{opt}(\lambda_1, \lambda_2, n)$ (red), for $\lambda_1 = 0.55$ and $\lambda_2 = 0.95$.

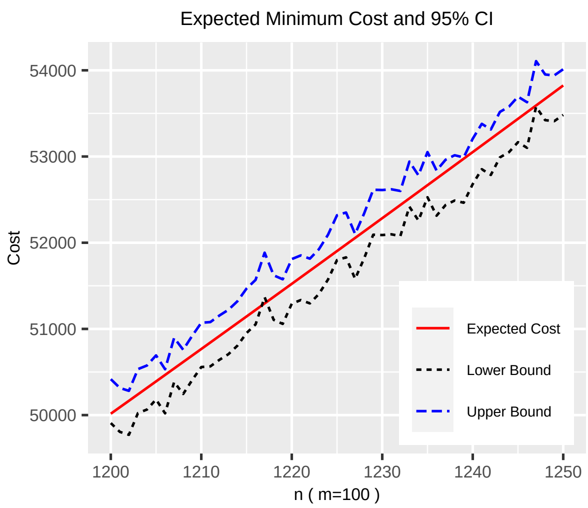

Fig. 6. Illustration of the confidence interval for $C_{opt}(\lambda_1, \lambda_2, n)$, with 95% confidence, for $\lambda_1 = 0.95$ and $\lambda_2 = 0.90$.

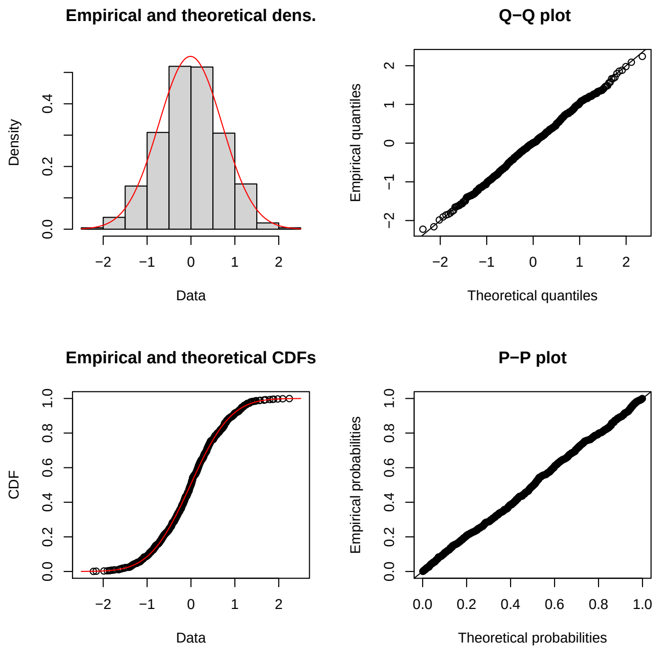

Fig. 7. Asymptotic normality of $\hat{C}_{opt}(n,m)$ as (67): histogram of $\hat{C}_{opt}(n,m)$ versus normal density (top left), Q-Q plot (top right), empirical versus theoretical cumulative distributions (bottom left), and P-P plots (bottom right).

V. Conclusion

In this article, we derived an exact expression and an interval for the sum of the expected absolute difference between two Poisson processes that can have different rates. Our results generalize those of [13], and to apply our results we calculated the minimum transport cost of a random two-color combination when two sensors are initially placed according to two Poisson process with different or equal laws. We performed a complete statistical inference study, proved asymptotic normality of the cost estimator, and performed a simulation study to show the consistency of the cost estimator.

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