Quantized Adaptive Bounded-H-infinity Tracking Control for a Class of Stochastic Nonaffine Nonlinear Systems

Xiaoyu Qu, Xiaohua Li, Xiaojie Cao

***Abstract*—Aiming at a class of nonaffine stochastic nonlinear systems with external disturbances and quantized input, this paper studies the quantized adaptive robust H∞ tracking control problem. A novel quantized adaptive robust bounded-H∞ tracking controller is proposed under the condition that the system is bounded stable. This control strategy ensures all the signals are bounded in probability. Meanwhile, the system has a good tracking performance and an H∞ disturbance attenuation performance. To demonstrate the effectiveness and superiority of the designed controller, the simulation results for a Brusselator model are given.**

***Keywords*—adaptive control, backstepping technique, bounded-H∞ tracking control, disturbance attenuation, stochastic nonaffine systems, quantized input**

I. Introduction

In our practical life, there exist a great deal of dynamic systems with nonaffine structure [1]–[2], such as some multi-agent systems, networked manipulator systems, etc. In order to save precious communication resources, a quantized control is introduced. Simultaneously, compared with strict-feedback nonlinear systems, nonaffine pure-feedback systems are evidently more universal and representative in real life. Because the mathematical models of nonaffine pure-feedback systems does not have affine appearance of variables, it is hard to directly obtain the virtual control and actual control inputs. Because nonaffine pure-feedback systems have such problems, it leads to more difficulty in control design process and stability analysis. At present, there are many researches considering the control problem for this kind of systems [3]–[4]. To transform the original system into affine form, an auxiliary subsystem was introduced in the reference [3]. But the disadvantage of this method is that the system order and the computational burden were increased. Consequently, in order to avoid the disadvantage, reference [4] focused on the control problem of the pure-feedback systems. To solve the problem about affine appearance of variables, this paper firstly uses the implicit function theorem. Moreover, this method has become the primary method to solve the controller design problem of nonaffine systems in the control field.

Because there are various uncertainties in practical systems, the robust control problem for nonaffine nonlinear systems is studied. Recently, there have been many research results in this field [5]–[7]. However, because of the constant terms appearing in control design process using the backstepping technique, the H∞ performance criterion cannot be satisfied. Therefore, this technique has been rarely used in this situation.

In recent years, with the development of stochastic nonlinear system theory, its control problem has been studied widely, a lot of control schemes have been presented [8]–[10]. However, in the research papers using backstepping technique, H∞ control problem is rarely considered because of the above reason. Up to now, there has been no literature studying H∞ robust control problem for nonaffine stochastic nonlinear systems by means of backstepping technique.

At the meanwhile, with the wide application of network control, quantitative control has achieved a great deal of research results in the nonlinear systems [11]–[13]. However, as far as the author knows, there is no paper on quantized control for stochastic nonaffine systems.

According to the above discussion, the H∞ robust adaptive tracking control problem is researched for a class of nonaffine stochastic nonlinear systems with external disturbances and quantized input in this paper. The control strategy can ensure that the signals are bounded in probability. Meanwhile, the proposed controller has a good tracking performance and an H∞ disturbance attenuation performance. In addition, a concept on bounded-H∞ is introduced to solve the H∞ control problem for the stochastic systems which can only be designed to be bounded in probability. Here, the principal advantages are listed.

(1) This paper, for the first time, focuses on the quantized adaptive bounded-H∞ tracking control problem for a class of stochastic nonaffine nonlinear systems with external disturbances and quantized input. Thereby, a novel control strategy is proposed, and a quantized adaptive bounded-H∞ tracking controller is obtained. (2) Due to the introduction of bounded-H∞ control, the influences of external disturbances are able to be attenuated by the designed controller. (3) The control strategy designed only needs to estimate an adaptive

Manuscript received June 01, 2021; revised September 24, 2021. This work was supported by the Natural Science Foundation of Liaoning Province (Grant No.20180550319), the Foundation of Liaoning Province Educational Administration (Grant No.2019 LNJC09) and Science Technology Innovation for Graduate Students of University of Science and Technology Liaoning (Grant No. LKDYC202008).

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parameter online. A great deal of calculations is avoided. Thus, the computational burden is greatly reduced.

II. Problem formulation and preliminaries

A. Problem Formulation

Consider the following system.

$$\begin{cases} dx_i = \left(f_i(\bar{x}_i, x_{i+1}) + \phi_i(\bar{x}_i)\varpi_i(t) \right) dt + \psi_i^{\mathrm{T}}(\bar{x}_i) d\omega, \\ dx_n = \left(f_n(\bar{x}_n, Q(u)) + \phi_n(\bar{x}_n)\varpi_n(t) \right) dt + \psi_n^{\mathrm{T}}(\bar{x}_n) d\omega, \\ y = x_1, \end{cases} \quad (1)$$

where $x = [x_1, x_2, \cdots, x_n]^{\mathrm{T}} \in \mathrm{R}^n$ is the state variable, $\bar{x}_i = [x_1, x_2, \cdots, x_i]^{\mathrm{T}} \in \mathrm{R}^i$ $(i = 1, 2, \cdots, n)$, $y \in \mathrm{R}$ denotes the output; ω refers to an $r-$dimensional standard Brownian motion which is defined on the complete probability space (Ω, F, P). Ω, F and P represent a sample space, a $\sigma-$ field, and a probability measure, respectively. At the same time, $f_i(\cdot): \mathrm{R}^i \to \mathrm{R}$ is unknown smooth and nonaffine functions, $\phi_i(\cdot): \mathrm{R}^i \to \mathrm{R}$, $\psi_i(\cdot): \mathrm{R}^i \to \mathrm{R}^r$ denote unknown smooth nonlinear functions and satisfy $f_i(0) = \phi_i(0) = \psi_i(0) = 0$. At the meanwhile, $\varpi_i(t)$ denotes unknown external disturbances with $\|\varpi(t)\| \in L_F^2([0,T], \mathrm{R})$. The quantized input $Q(u)$ can be able to be defined like [12].

$$Q(u) = \begin{cases} u_k \operatorname{sgn}(u), & \frac{u_k}{1+\sigma} < |u| \le u_k, \dot{u} < 0 \quad or \\ & u_k < |u| \le \frac{u_k}{1-\sigma}, \dot{u} > 0 \\ u_k(1+\sigma)\operatorname{sgn}(u), & u_k < |u| \le \frac{u_k}{1-\sigma}, \dot{u} < 0 \quad or \\ & \frac{u_k}{1-\sigma} < |u| \le \frac{u_k(1+\sigma)}{1-\sigma}, \dot{u} > 0 \\ 0, & 0 \le |u| < \frac{u_{\min}}{1+\sigma}, \dot{u} < 0 \quad or \\ & \frac{u_{\min}}{1+\sigma} \le |u| \le u_{\min}, \dot{u} > 0 \\ Q(u(t^-)), & \text{othercases} \end{cases} \quad (2)$$

where $u_k = \rho^{1-i} u_{\min}$ $(k = 1, 2, \cdots)$ and $\sigma = (1-\rho)/(1+\rho)$, $0 < \rho < 1$ can be viewed as the measure of the quantization density, the range of the dead-zone of $Q(u)$ can be represented as $u_{\min} > 0$. At the same time, $Q(u(t^-))$ is able to be regarded as the status prior of $Q(u)$. Then, in the hysteresis quantizer (2), $Q(u) \in U = \{0, \pm u_i, \pm u_i(1+\sigma), i = 1, 2, \cdots\}$. Furthermore, the quantized input $Q(u)$ can be decomposed as follow.

$$Q(u) = J(u)u(t) + L(t),$$

where $1 - \sigma \le J(u) \le 1 + \sigma$, $|L(t)| \le u_{\min}$.

Our control target is to design a controller so that the system (1) has an H_∞ disturbance attenuation performance and a reference signal y_d can be followed by the output y. Concurrently, the signals remain bounded in probability.

For the convenience of the following design, we define

$$g_i(\bar{x}_i, x_{i+1}) = \frac{\partial f_i(\bar{x}_i, x_{i+1})}{\partial x_{i+1}}, \quad (i = 1, 2, \cdots, n), \quad (3)$$

with $x_{n+1} = Q(u)$.

B. Preliminaries

For the sake of the main results, we give the following assumptions, definitions, and lemmas.

Assumption 1 The signs of function $g_i(\bar{x}_i, x_{i+1})$ are known, and there are two constants b_M and b_m such that

$$0 < b_m \le g_i(\bar{x}_i, x_{i+1}) \le b_M < \infty, \ \forall (\bar{x}_i, x_{i+1}) \in \mathrm{R}^i \times \mathrm{R}. \quad (4)$$

Remark 1 Those two constants are only used for stability analysis, and the controller design does not need them, so there's no need to know their true value.

Assumption 2 The desired reference signal y_d and $y_d^{(i)}$ are continuous and bounded. Here, $y_d^{(i)}$ are time derivatives, $(i = 1, 2, \cdots, n)$.

Firstly, we consider the following system

$$dx = f(x)dt + h(x)d\omega, \quad (5)$$

where $f(x): \mathrm{R}^n \to \mathrm{R}^n$ and $h(x): \mathrm{R}^n \to \mathrm{R}^{n \times r}$ are locally Lipschitz functions satisfying $f(0) = 0$ and $h(0) = 0$.

Corollary 1 [16] For the system (5), if there exists a positive definite, radially unbounded, twice continuously differentiable function $\bar{V} = V + m$, $m > 0$, and the constants $a_1 > 0$, $b_1 > 0$ such that

$$L\bar{V}(x) \le -a_1 \bar{V} + b_1,$$

subsequently, 1) the system has a unique solution almost surely; 2) the system is bounded in probability.

Definition 1[17] For the system (1), if there exists a finite T and a continuous differentiable function $\bar{V}(x)$ satisfying $LV(x) \le -a_0 V(x) + b_0$ in reference [15] with $\bar{V}(x(0)) > 0$, we can get

$$E\left[\int_0^T \|z_1(s)\|^2 ds\right] \le \gamma^2 E\left[\int_0^T \|\varpi(s)\|^2 ds\right] + E\left[V(x(0))\right],$$

where $z_1 = x_1 - y_d$ stands for the tracking error, $\varpi(t) \in \mathrm{L}_2[0,T]$ is the bounded external disturbance. Meanwhile, γ is a disturbance attenuation coefficient, then the stochastic system (1) has H_∞ performance against external disturbance for all $T \ge 0$.

An RBF neural network (RBFNN), in this paper, is introduced to model an unknown continuous function $f(Z)$.

$$f(Z) = W^{*\mathrm{T}} S(Z) + \delta(Z), \forall Z \in \Omega_Z \subset \mathrm{R}^q \quad (6)$$

where W^* represents an optimal weight value vector, and $S(Z) = [s_1(Z), s_2(Z), \cdots, s_l(Z),]^{\mathrm{T}}$ can be represented as the basis function vector. Here, $s_i(Z)$ is selected as

$$s_i(Z) = \exp[-\frac{(Z - u_i)^{\mathrm{T}}(Z - u_i)}{r_i^2}], \quad (1 \le i \le l), \quad (7)$$

where $u_i = [u_{i1}, u_{i2}, \cdots, u_{iq}]^{\mathrm{T}}$ and r_i are the center and the width of $s_i(Z)$. It should be noted that $\delta(Z)$ is the approximation error, and it meets such conditions that $|\delta| \le \delta_{\max}$. Here, $\delta_{\max}$ is a bounded positive number.

Lemma 1 [20] Consider the equations (7), let $\rho=\frac{1}{2}\min_{i\neq j}\|u_i-u_j\|$, afterwards, an upper bound of $\|S(Z)\|$ is defined as

$$\|S(Z)\|\le\sum_{k=0}^{\infty}3q(k+2)^{q-1}e^{-2\rho^2k^2/r^2}:\ =s \tag{8}$$

with s being a limited value. And it should be noted that s is irrelevant to the input variable and the node numbers of the RBFNN.

III. Main results

In this section, based on the above lemmas, corollaries, etc. the quantized adaptive bounded-H_∞ tracking controller will be designed for the stochastic system (1). Moreover, for convenience of description, x_i and t are omitted in the following functions. In addition, let $S_i(Z_i)=S_i$. The design scheme bases on the following coordinate transformation, it can be defined as

$$z_i=x_i-\alpha_{i-1}(\overline{x}_{i-1},\overline{y}_d^{(i-1)},\hat{\theta}),\quad i=1,2,\cdots,n \tag{9}$$

with $\alpha_0=y_d$, $\overline{y}_d^{(i-1)}=\left[y_d,\dot{y}_d,\cdots,y_d^{(i-1)}\right]^{\mathrm{T}}$. Here, $\hat{\theta}=\theta-\tilde{\theta}$ is the estimation of θ. The estimation error of θ is expressed as $\tilde{\theta}$. θ can be specified as

$$\theta=\max\left\{b_M^2\left\|W_i^*\right\|^2,i=1,2,\cdots,n\right\}. \tag{10}$$

Next, the quantized adaptive bounded-H_∞ neural tracking controller and adaptive laws will be established as follows.

$$\alpha_i=-(k_i+h^2)z_i-\frac{1}{2a_i}z_i^3\hat{\theta}S_i^{\mathrm{T}}(Z_i)S_i(Z_i), \tag{11}$$

$$u=-\frac{(k_n+h^2)}{(1-\sigma)}z_n-\frac{1}{2a_n(1-\sigma)}z_n^3\hat{\theta}S_n^{\mathrm{T}}(Z_n)S_n(Z_n), \tag{12}$$

$$\dot{\hat{\theta}}=\sum_{i=1}^{n}\frac{\lambda}{2a_i}z_i^6S_i^{\mathrm{T}}(Z_i)S_i(Z_i)-m_0\hat{\theta}, \tag{13}$$

where $1\le i\le n-1$, a_i, k_i, m_0 and λ are positive design parameters, $h=n/\gamma^2+1$, γ denotes a positive disturbance attenuation parameter, and $S_i(Z_i)$ is the basis function vector of NNs with $Z_1=[x_1,y_d,\dot{y}_d]^{\mathrm{T}}\in\Omega_{Z_1}\subset\mathrm{R}^3$, $Z_i=[\overline{x}_i^T,\hat{\theta},\overline{y}_d^{(i)\mathrm{T}}]^{\mathrm{T}}\in\Omega_{Zi}\subset\mathrm{R}^{2i+1}$ $(2\le i\le n)$. Particularly, when $i=n$, α_n is the actual control input $u(t)$.

The design procedure is as follows.

Step 1: Consider the stochastic system (1), from the coordinate transformation with $i=1$, the following equation can be obtained.

$$dz_1=(f_1(x_1,x_2)+\phi_1\varpi_1-\dot{y}_d)dt+\psi_1^{\mathrm{T}}d\omega. \tag{14}$$

Select a Lyapunov function candidate as

$$V_1=\frac{1}{4}hb_mz_1^4+\frac{hb_m^2}{2\lambda}\tilde{\theta}^2. \tag{15}$$

According to the definition of infinity differential operator in [14], the infinity differential operator of (15) can be written as

$$\begin{aligned}LV_1&=hb_mz_1^3(f_1(x_1,x_2)+\phi_1\varpi_1-\dot{y}_d)\\&\quad+\frac{3}{2}hb_mz_1^2\|\psi_1\|^2-\lambda^{-1}hb_m^2\tilde{\theta}\dot{\hat{\theta}}.\end{aligned} \tag{16}$$

By utilizing Young's Inequality in [18], we can get (17) and (18)

$$hb_mz_1^3\phi_1\varpi_1\le\frac{1}{4}hb_m^2z_1^6\phi_1^2+h\varpi_1^2, \tag{17}$$

$$\begin{aligned}\frac{3}{2}hb_mz_1^2\|\psi_1\|^2&\le\frac{9}{16}l_1+\frac{1}{l_1}h^2b_m^2z_1^4\|\psi_1\|^4\\&\le\frac{9}{16}l_1+h^3b_m^2z_1^4+\frac{1}{4l_1^2}hb_m^2z_1^4\|\psi_1\|^8.\end{aligned} \tag{18}$$

Remark 2 In (18), $l_1>0$ is chosen as an arbitrarily small constant. The actual value of l_1 does not need to be known because it is not used to construct the controller.

Substituting (17) and (18) into (16) results in

$$\begin{aligned}LV_1&\le hb_mz_1^3(f_1(x_1,x_2)+\frac{1}{4}b_mz_1^3\phi_1^2-\dot{y}_d+\frac{1}{4l_1^2}b_mz_1\|\psi_1\|^8)\\&\quad+h^3b_m^2z_1^4+\frac{9}{16}l_1+h\varpi_1^2-\lambda^{-1}hb_m^2\tilde{\theta}\dot{\hat{\theta}}.\end{aligned} \tag{19}$$

To prove the system (1) to satisfy bounded-H_∞ performance index, define an auxiliary function as

$$H_1=LV_1+h^2(\|z_1\|^2-\gamma^2\|\varpi_1\|^2)=LV_1+h^2(z_1^2-\gamma^2\varpi_1^2). \tag{20}$$

By employing Young's Inequality, it follows that

$$h^2z_1^2\le\frac{1}{4l_1}hz_1^4+l_1h^3. \tag{21}$$

Substituting (19) and (21) into (20) produces

$$\begin{aligned}H_1&\le h^3b_m^2z_1^4+hb_mz_1^3(f_1(x_1,x_2)+\frac{1}{4}b_mz_1^3\phi_1^2+\frac{1}{4l_1^2}b_mz_1\|\psi_1\|^8\\&\quad-\dot{y}_d+\frac{1}{4l_1b_m}z_1)+\frac{9}{16}l_1-\lambda^{-1}hb_m^2\tilde{\theta}\dot{\hat{\theta}}+l_1h^3+h(1-h\gamma^2)\varpi_1^2.\end{aligned} \tag{22}$$

Define a new function

$$\eta_1=\frac{b_m}{2}z_1^3-\dot{y}_d+\frac{3}{4}z_1+\frac{1}{4l_1^2}b_mz_1\|\psi_1\|^8+\frac{1}{4}b_mz_1^3\phi_1^2+\frac{1}{4l_1b_m}z_1.$$

Then (22) can be converted into

$$\begin{aligned}H_1&\le hb_mz_1^3\left(f_1(x_1,x_2)+\eta_1\right)-\frac{3}{4}hb_mz_1^4-\frac{hb_m^2}{2}z_1^6+h^3b_m^2z_1^4\\&\quad+\frac{9}{16}l_1-\lambda^{-1}hb_m^2\tilde{\theta}\dot{\hat{\theta}}+l_1h^3+h(1-h\gamma^2)\varpi_1^2.\end{aligned} \tag{23}$$

Due to $\partial\eta_1/\partial x_2=0$, according to Assumption 1, we can get

$$g_1(x_1,x_2)=\frac{\partial\left[f_1(x_1,x_2)+\eta_1\right]}{\partial x_2}\ge b_m>0. \tag{24}$$

According to [21], on the basis of the implicit function theorem, for every value of x_1 and η_1, there exists a smooth ideal control input $x_2=\alpha_1^*(x_1,\eta_1)$, it can make (25) holds

$$f_1(x_1,\alpha_1^*)+\eta_1=0. \tag{25}$$

Next, making use of the mean-value theorem, like [21], there exists $\mu_1(0<\mu_1<1)$, it can make the following inequality holds

$$f_1(x_1,x_2)=f_1(x_1,\alpha_1^*)+g_{1\mu_1}(x_2-\alpha_1^*), \tag{26}$$

where $g_{1\mu_1}:=g_1(x_1,x_{2\mu_1})$, $x_{2\mu_1}=\mu_1x_2+(1-\mu_1)\alpha_1^*$. It should be noted that Assumption 1 is also valid for $g_{1\mu_1}$.

Therefore, according to the formula (25), substituting (26) into (23) leads to

$$\begin{aligned}H_1 \le\, & hb_m z_1^3 g_{1\mu_1}(x_2-\alpha_1^*)+h^3b_m^2 z_1^4+\frac{9}{16}l_1-\frac{3}{4}hb_m z_1^4\\ & -\frac{hb_m^2}{2}z_1^6-\lambda^{-1}hb_m^2\tilde{\theta}\dot{\hat{\theta}}+l_1h^3+h(1-h\gamma^2)\varpi_1^2.\end{aligned}\tag{27}$$

Since α_1^* contains the unknown function η_1, we make use of an RBFNN $W_1^{*\mathrm{T}}S_1(Z_1)$ to approximate α_1^*. Utilizing Young's Inequality and (10), it can make the following inequality holds

$$\begin{aligned}-hb_m z_1^3 g_{1\mu_1}\alpha_1^* &= -hb_m z_1^3 g_{1\mu_1}(W_1^{*\mathrm{T}}S_1+\delta_1)\\ &\le \frac{hb_m^2}{2a_1}z_1^6\theta S_1^{\mathrm{T}}S_1+\frac{1}{2}a_1h+\frac{hb_m^2}{2}z_1^6+\frac{hb_M^2}{2}\delta_{1\max}^2 \quad (28)\\ &= \frac{hb_m^2}{2a_1}z_1^6\theta S_1^{\mathrm{T}}S_1+\frac{hb_m^2}{2}z_1^6+\frac{1}{2}h(a_1+b_M^2\delta_{1\max}^2),\end{aligned}$$

Substituting α_i in (11) with $i=1$, if $\hat{\theta}(0)\ge 0$, according to Assumption 1, we can get

$$hb_m z_1^3 g_{1\mu_1}\alpha_1 \le -(k_1+h^2)hb_m^2 z_1^4-\frac{hb_m^2}{2a_1}z_1^6\hat{\theta}S_1^{\mathrm{T}}S_1. \tag{29}$$

Remark 3 If $\hat{\theta}(0)\ge 0$ for all $t\ge 0$, We can easily obtain the conclusion that $\hat{\theta}(t)\ge 0$. Due to $\hat{\theta}(t)\ge 0$, the formula (29) can hold. Moreover, the condition of $\hat{\theta}(0)\ge 0$ will be adopted in each design steps.

Utilizing Young's Inequality, here, we choose $p=4/3,\ q=4$, it yields

$$hb_m z_1^3 z_2 g_{1\mu_1} \le \frac{3}{4}hb_m z_1^4+\frac{1}{4}hb_m b_M^4 z_2^4. \tag{30}$$

Subsequently, combining (27) with (28) ~ (30) yields

$$\begin{aligned}H_1 \le\, & -c_1 z_1^4+\frac{1}{4}hb_m b_M^4 z_2^4+\frac{hb_m^2\tilde{\theta}}{\lambda}\Big(\frac{\lambda}{2a_1}z_1^6 S_1^{\mathrm{T}}S_1-\dot{\hat{\theta}}\Big)+\frac{9}{16}l_1\\ & +\frac{1}{2}h(a_1+b_M^2\delta_{1\max}^2)+l_1h^3+h(1-h\gamma^2)\varpi_1^2,\end{aligned}\tag{31}$$

with $c_1=k_1hb_m^2>0$.

Step i $(2\le i\le n-1)$**:** The derivative of $z_i=x_i-\alpha_{i-1}$ is

$$\begin{aligned}dz_i =\, & \big(f_i(\bar{x}_i,x_{i+1})+\phi_i\varpi_i-L\alpha_{i-1}\big)dt\\ & +(\psi_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j)^{\mathrm{T}}d\omega,\end{aligned}\tag{32}$$

where

$$L\alpha_{i-1}=\tau_{i-1}+\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\varpi_j,$$

$$\begin{aligned}\tau_{i-1}=\, & \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}f_j(\bar{x}_j,x_{j+1})+\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)}\\ & +\frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\psi_p^{\mathrm{T}}\psi_q.\end{aligned}\tag{33}$$

A stochastic Lyapunov functional candidate is chosen as

$$V_i=\frac{1}{4}hb_m z_i^4. \tag{34}$$

According to the definition of infinity differential operator in [14] and (34), one has

$$\begin{aligned}LV_i =\, & hb_m z_i^3\Big(f_i(\bar{x}_i,x_{i+1})+\phi_i\varpi_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\varpi_j-\tau_{i-1}-\varsigma_i(Z_i)\Big)\\ & +\frac{3}{2}hb_m z_i^2\left\|\psi_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j\right\|^2+hb_m z_i^3\Big(\varsigma_i(Z_i)-\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\Big).\end{aligned}\tag{35}$$

Remark 4 Here, $\varsigma_i(Z_i)$ is a smooth function, which is adopted for the following proof. It will appear in the next each step. The functions $\varsigma_i(Z_i)$ $(i=2,\cdots,n)$ will be deduced later.

By making use of Young's Inequality, it can make (36) ~ (38) hold

$$\begin{aligned}\frac{3}{2}hb_m z_i^2\left\|\psi_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j\right\|^2 \le\, & \frac{9}{16}l_1+h^3b_m^2 z_i^4\\ & +\frac{1}{4l_1^2}hb_m^2 z_i^4\left\|\psi_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j\right\|^8,\end{aligned}\tag{36}$$

$$-hb_m z_i^3\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\varpi_j \le \frac{1}{4}hb_m^2 z_i^6\sum_{j=1}^{i-1}\Big(\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\Big)^2+h\sum_{j=1}^{i-1}\varpi_j^2, \tag{37}$$

$$hb_m z_i^3\phi_i\varpi_i \le \frac{1}{4}hb_m^2 z_i^6\phi_i^2+h\varpi_i^2. \tag{38}$$

Substituting (36) ~ (38) into (35) yields

$$\begin{aligned}LV_i \le\, & hb_m z_i^3\Big(f_i(\bar{x}_i,x_{i+1})+\frac{1}{4}b_m\phi_i^2 z_i^3+\frac{1}{4}b_m z_i^3\sum_{j=1}^{i-1}\Big(\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\Big)^2\Big)\\ & +hb_m z_i^3\Big(\frac{1}{4l_1^2}b_m z_i\left\|\psi_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j\right\|^8-\tau_{i-1}-\varsigma_i(Z_i)\Big)\\ & +\frac{9}{16}l_1+h^3b_m^2 z_i^4+hb_m z_i^3\Big(\varsigma_i(Z_i)-\frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\Big)+h\sum_{j=1}^{i}\varpi_j^2.\end{aligned}\tag{39}$$

Define an auxiliary function

$$\begin{aligned}H_i &= LV_1+\sum_{j=2}^{i}LV_j+h^2\Big(\|z_1\|^2-\gamma^2\sum_{j=1}^{i}\|\varpi_j\|^2\Big)\\ &= LV_1+\sum_{j=2}^{i}LV_j+h^2\Big(z_1^2-\gamma^2\sum_{j=1}^{i}\varpi_i^2\Big)\\ &= H_{i-1}+LV_i-h^2\gamma^2\varpi_i^2.\end{aligned}\tag{40}$$

According to (39), (40) can be rewritten as

$$\begin{aligned}H_i \le\, & -\sum_{j=1}^{i-1}c_j z_j^4-\frac{3}{4}hb_m z_i^4-\frac{hb_m^2}{2}z_i^6+\frac{1}{2}h\sum_{j=1}^{i-1}(a_j+b_M^2\delta_{1\max}^2)\\ & +\frac{hb_m^2\tilde{\theta}}{\lambda}\Big(\sum_{j=1}^{i-1}\frac{\lambda}{2a_{j1}}z_j^6 S_j^{\mathrm{T}}S_j-\dot{\hat{\theta}}\Big)+l_1h^3+\frac{9i}{16}l_1\\ & +hb_m z_i^3\big(f_i(\bar{x}_i,x_{i+1})+\eta_i\big)+h^3b_m^2 z_i^4\\ & +hb_m\sum_{j=2}^{i}z_j^3\Big(\varsigma_j(Z_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\Big)\\ & +h\sum_{j=1}^{i}(i+1-j-h\gamma^2)\varpi_j^2,\end{aligned}\tag{41}$$

where

$$\begin{aligned}\eta_i =\, & \frac{b_m}{2}z_i^3+\frac{3}{4}z_i+\frac{1}{4}b_M^4 z_i+\frac{1}{4}b_m\phi_i^2 z_i^3+\frac{1}{4}b_m z_i^3\sum_{j=1}^{i-1}\Big(\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\Big)^2\\ & +\frac{1}{4l_1^2}b_m z_i\left\|\psi_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j\right\|^8-\tau_{i-1}-\varsigma_i(Z_i).\end{aligned}\tag{42}$$

Subsequently, repeating (24) ~ (26), (41) can be rewritten as.

$$
\begin{aligned}
H_i \le & -\sum_{j=1}^{i-1} c_j z_j^4 - \frac{3}{4} h b_m z_i^4 - \frac{h b_m^2}{2} z_i^6 + \frac{1}{2} h \sum_{j=1}^{i-1} (a_j + b_M^2 \delta_{j\max}^2) \\
& + l_1 h^3 + \frac{h b_m^2 \tilde{\theta}}{\lambda} \Big(\sum_{j=1}^{i-1} \frac{\lambda}{2a_j} z_j^6 S_j^{\mathrm{T}} S_j - \dot{\hat{\theta}} \Big) + \frac{9i}{16} l_1 + h^3 b_m^2 z_i^4 \\
& + h b_m z_i^3 g_{i\mu_i} (x_{i+1} - \alpha_i^*) + h \sum_{j=1}^{i} (i+1-j-h\gamma^2) \varpi_j^2 \\
& + h b_m \sum_{j=2}^{i} z_j^3 \Big(\varsigma_j(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big).
\end{aligned} \tag{43}
$$

Here, an RBFNN can be utilized to approximate α_i^*, and making use of Young's Inequality, it can make (44) holds

$$
\begin{aligned}
-z_i^3 h b_m g_{i\mu_i} \alpha_i^* &= -z_i^3 h b_m g_{i\mu_i} (W_i^{*\mathrm{T}} S_i + \delta_i) \\
&\le \frac{h b_m^2}{2a_i} z_i^6 \theta S_i^{\mathrm{T}} S_i + \frac{1}{2} a_i h + \frac{h b_m^2}{2} z_i^6 + \frac{h b_M^2}{2} \delta_{i\max}^2 \\
&= \frac{h b_m^2}{2a_i} z_i^6 \theta S_i^{\mathrm{T}} S_i + \frac{h b_m^2}{2} z_i^6 + \frac{1}{2} h (a_i + b_M^2 \delta_{i\max}^2).
\end{aligned} \tag{44}
$$

By applying α_i in (11), we can obtain the inequality (46).

$$
h b_m z_i^3 g_{i\mu_i} \alpha_i \le -(k_i + h^2) h b_m^2 z_i^4 - \frac{h b_m^2}{2a_i} z_i^6 \hat{\theta} S_i^{\mathrm{T}} S_i. \tag{45}
$$

In addition, similarly to (30), the following inequality holds.

$$
h b_m z_i^3 z_{i+1} g_{i\mu_i} \le \frac{3}{4} h b_m z_i^4 + \frac{1}{4} h b_m b_M^4 z_{i+1}^4. \tag{46}
$$

Substituting (44), (45) and (46) into (43) finally leads to

$$
\begin{aligned}
H_i \le & -\sum_{j=1}^{i} c_j z_j^4 + \frac{1}{4} h b_m b_M^4 z_{i+1}^4 + \frac{1}{2} h \sum_{j=1}^{i} (a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 \\
& + \frac{h b_m^2 \tilde{\theta}}{\lambda} \Big(\sum_{j=1}^{i} \frac{\lambda}{2a_j} z_j^6 S_j^{\mathrm{T}} S_j - \dot{\hat{\theta}} \Big) + h \sum_{j=1}^{i} (i+1-j-h\gamma^2) \varpi_j^2 \\
& + \frac{9i}{16} l_1 + h b_m \sum_{j=2}^{i} z_j^3 \Big(\varsigma_j(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big),
\end{aligned} \tag{47}
$$

with $c_j = k_j h b_m^2 > 0$.

Step n**:** Because of $z_n = x_n - \alpha_{n-1}$, one has

$$
\begin{aligned}
dz_n &= \big(f_n(\bar{x}_n, Q(u)) + \phi_n \varpi_n - L\alpha_{n-1} \big) dt \\
&\quad + \Big(\psi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_j \Big)^{\mathrm{T}} d\omega,
\end{aligned} \tag{48}
$$

where

$$
\begin{aligned}
L\alpha_{n-1} &= \tau_{n-1} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j \varpi_j, \\
\tau_{n-1} &= \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} f_j(\bar{x}_j, x_{j+1}) + \sum_{j=0}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \\
&\quad + \frac{1}{2} \sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \psi_p^{\mathrm{T}} \psi_q.
\end{aligned} \tag{49}
$$

Choose the quartic auxiliary Lyapunov function candidate as

$$
V_n = \frac{1}{4} h b_m z_n^4 + k_0, \tag{50}
$$

with $k_0 > 0$ being an arbitrarily small constant.

Remark 5: Here, the added k_0 will not influence the analysis of system stability. It serves as an auxiliary constant for bounded-H$_\infty$ performance proof, and k_0 is not used in the design of the controller.

Next, it follows from (50) that:

$$
\begin{aligned}
LV_n = & h b_m z_n^3 \Big(f_n(\bar{x}_n, Q(u)) + \phi_n \varpi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j \varpi_j - \tau_{n-1} \\
& - \varsigma_n(Z_n) \Big) + \frac{3}{2} h b_m z_n^2 \Big\| \psi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_j \Big\|^2 \\
& + h b_m z_n^3 \Big(\varsigma_n(Z_n) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big).
\end{aligned} \tag{51}
$$

Utilizing Young's Inequality, it produces

$$
\begin{aligned}
\frac{3}{2} h b_m z_n^2 \Big\| \psi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_j \Big\|^2 \le & \frac{9}{16} l_1 + h^3 b_m^2 z_n^4 \\
& + \frac{1}{4 l_1^2} h b_m^2 z_n^4 \Big\| \psi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_j \Big\|^8,
\end{aligned} \tag{52}
$$

$$
-h b_m z_n^3 \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j \varpi_j \le \frac{1}{4} h b_m^2 z_n^6 \sum_{j=1}^{n-1} \Big(\frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j \Big)^2 + h \sum_{j=1}^{n-1} \varpi_j^2. \tag{53}
$$

$$
h b_m z_n^3 \phi_n \varpi_n \le \frac{1}{4} h b_m^2 z_n^6 \phi_n^2 + h \varpi_n^2. \tag{54}
$$

From (52) ~ (54), (51) can be expressed as

$$
\begin{aligned}
LV_n \le & h b_m z_n^3 \Big(\big(f_n(\bar{x}_n, Q(u)) + \frac{1}{4} b_m \phi_n^2 z_n^3 - \tau_{n-1} - \varsigma_n(Z_n) \big) \\
& + h b_m z_n^3 \Big(\frac{1}{4} b_m z_n^3 \sum_{j=1}^{n-1} \Big(\frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j \Big)^2 \\
& + \frac{1}{4 l_1^2} b_m z_n \Big\| \psi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_j \Big\|^8 \Big) + \frac{9}{16} l_1 + h^3 b_m^2 z_n^4 \\
& + h b_m z_n^3 \Big(\varsigma_n(Z_n) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big) + h \sum_{j=1}^{n} \varpi_j^2.
\end{aligned} \tag{55}
$$

Define an auxiliary function

$$
\begin{aligned}
H_n &= LV_1 + \sum_{j=2}^{n} LV_j + h^2 \Big(\|z_1\|^2 - \gamma^2 \sum_{j=1}^{n} \|\varpi_j\|^2 \Big) \\
&= H_{n-1} + LV_n - h^2 \gamma^2 \varpi_n^2.
\end{aligned} \tag{56}
$$

Combining (56) with (55) results in

$$
\begin{aligned}
H_n \le & -\sum_{j=1}^{n-1} c_j z_j^4 - \frac{h b_m}{2} z_n^6 - \frac{h^2 b_m^2 b_M^2}{2} z_n^6 + \frac{1}{2} h \sum_{j=1}^{n-1} (a_j + b_M^2 \delta_{j\max}^2) \\
& + l_1 h^3 + \frac{h b_m^2 \tilde{\theta}}{\lambda} \Big(\sum_{j=1}^{n-1} \frac{\lambda}{2a_j} z_j^6 S_j^{\mathrm{T}} S_j - \dot{\hat{\theta}} \Big) + \frac{9n}{16} l_1 + h^3 b_m^2 z_n^4 \\
& + h b_m z_n^3 \big(f_n(\bar{x}_n, Q(u)) + \eta_n \big) + h b_m \sum_{j=2}^{n} z_j^3 \Big(\varsigma_j(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big) \\
& + h \sum_{j=1}^{n} (n+1-j-h\gamma^2) \varpi_j^2,
\end{aligned} \tag{57}
$$

with

$$
\begin{aligned}
\eta_n = & \frac{h b_m b_M^2}{2} z_n^3 + \frac{b_m}{2} z_n^3 + \frac{1}{4} b_M^4 z_n + \frac{1}{4} b_m \phi_n^2 z_n^3 \\
& + \frac{1}{4} b_m z_n^3 \sum_{j=1}^{n-1} \Big(\frac{\partial \alpha_{n-1}}{\partial x_j} \phi_j \Big)^2 - \tau_{n-1} - \varsigma_n(Z_n) \\
& + \frac{1}{4 l_1} b_m z_n \Big\| \psi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \psi_j \Big\|^8
\end{aligned} \tag{58}
$$

Note that $\partial \eta_n / \partial u = 0$, we can get

$$g_n(\bar{x}_n, Q(u)) = \frac{\partial\left[f_n(\bar{x}_n, Q(u)) + \eta_n\right]}{\partial Q(u)} \ge b_m > 0. \tag{59}$$

On the basis of the implicit function theorem, for every value of $\bar{x}_n$ and η_n, there exists a smooth ideal control input $Q(u) = \alpha_n^*(\bar{x}_n, \eta_n)$, it yields

$$f_n(\bar{x}_n, \alpha_n^*) + \eta_n = 0. \tag{60}$$

Adopting the mean-value theorem, same as the previous step, there exists $\mu_n (0 < \mu_n < 1)$, it can make the following inequality holds

$$f_n(\bar{x}_n, Q(u)) = f_n(\bar{x}_n, \alpha_n^*) + g_{n\mu_n}(Q(u) - \alpha_n^*), \tag{61}$$

where $g_{n\mu_n} := g_n(\bar{x}_n, x_{(n+1)\mu_n}),\ x_{(n+1)\mu_n} = \mu_n Q(u) + (1-\mu_n)\alpha_n^*$.

Note that Assumption 1 is still valid for $g_{n\mu_n}$.

Further, according to (59) ~ (61) and $Q(u) = J(u)u(t) + L(t)$ with $|L(t)| \le u_{\min}$, the following inequality can be derived.

$$\begin{aligned} H_n \le& -\sum_{j=1}^{n-1} c_j z_j^4 - \frac{h^2 b_m^2 b_M^2}{2} z_n^6 - \frac{h b_m}{2} z_n^6 + \frac{1}{2} h \sum_{j=1}^{n-1} (a_j + b_M^2 \delta_{j\max}^2) \\ &+ l_1 h^3 + \frac{h b_m^2 \tilde{\theta}}{\lambda} \Big(\sum_{j=1}^{n-1} \frac{\lambda}{2a_j} z_j^6 S_j^{\mathrm{T}} S_j - \dot{\hat{\theta}}\Big) + \frac{9n}{16} l_1 + h^3 b_m^2 z_n^4 \\ &+ h b_m \sum_{j=2}^{n} z_j^3 \Big(\varsigma_j(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}\Big) + h \sum_{j=1}^{n} (n+1-j-h\gamma^2) \varpi_j^2 \\ &+ h b_m z_n^3 g_{n\mu_n} (J(u)u(t) + L(t) - \alpha_n^*). \end{aligned} \tag{62}$$

By utilizing Young's Inequality, it produces

$$h b_m z_n^3 g_{n\mu_n} L(t) \le \frac{h^2 b_m^2 b_M^2}{2} z_n^6 + \frac{1}{2} u_{\min}^2, \tag{63}$$

An RBFNN is employed to approximate α_n^*. Adopting the processing method as (28), one has

$$\begin{aligned} -z_n^3 h b_m g_{n\mu_n} \alpha_n^* \le& \frac{h b_m^2}{2a_n} z_n^6 \theta S_n^{\mathrm{T}} S_n + \frac{h b_m}{2} z_n^6 \\ &+ \frac{1}{2} h (a_n + b_M^2 \delta_{n\max}^2). \end{aligned} \tag{64}$$

Now, substituting the actual control input (12) into the item $h b_m z_n^3 g_{n\mu_n} J(u) u$ in (62), we can get

$$h b_m z_n^3 g_{n\mu_n} J(u) u \le -(k_n + h^2) h b_m^2 z_n^4 - \frac{h b_m^2}{2a_n} z_n^6 \hat{\theta} S_n^{\mathrm{T}} S_n. \tag{65}$$

Then, substituting (63) ~ (65) into (62) produces

$$\begin{aligned} H_n \le& -\sum_{j=1}^{n} c_j z_j^4 + \frac{1}{2} h \sum_{j=1}^{n} (a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 \\ &+ \frac{h b_m^2 \tilde{\theta}}{\lambda} \Big(\sum_{j=1}^{n} \frac{\lambda}{2a_j} z_j^6 S_j^{\mathrm{T}} S_j - \dot{\hat{\theta}}\Big) + \frac{9n}{16} l_1 + \frac{1}{2} u_{\min}^2 \\ &+ h \sum_{j=1}^{n} (n+1-j-h\gamma^2) \varpi_j^2 \\ &+ h b_m \sum_{j=2}^{n} z_j^3 \Big(\varsigma_j(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}\Big), \end{aligned} \tag{66}$$

where $c_j = k_j h b_m^2 > 0$.

Next, by using $\dot{\hat{\theta}}$ in (13), we can get

$$\begin{aligned} H_n \le& -\sum_{j=1}^{n} c_j z_j^4 + \frac{1}{2} h \sum_{j=1}^{n} (a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 + \frac{9n}{16} l_1 \\ &+ h \sum_{j=1}^{n} (n+1-j-h\gamma^2) \varpi_j^2 + \frac{h m_0 b_m^2}{\lambda} \tilde{\theta} \hat{\theta} \\ &+ \frac{1}{2} u_{\min}^2 + h b_m \sum_{j=2}^{n} z_j^3 \Big(\varsigma_j(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}\Big). \end{aligned} \tag{67}$$

Next, in order to ensure that $-h b_m \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}$ is negative, we apply Lemma 1 and the adaptive law $\dot{\hat{\theta}}$ in (13), it follows that

$$\begin{aligned} -h b_m \sum_{j=2}^{n} z_j^3 \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} =& -h b_m \sum_{j=2}^{n} z_j^3 \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Big(\sum_{k=1}^{n} \frac{\lambda}{2a_k} z_k^6 S_k^{\mathrm{T}} S_k - m_0 \hat{\theta}\Big) \\ \le& -h b_m \sum_{j=2}^{n} z_j^3 \Big(-m_0 \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \hat{\theta} - \frac{\lambda s^2}{2a_j} z_j^3 \sum_{k=2}^{j} \left| z_k^3 \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right| \Big) \\ &+ h b_m \sum_{j=2}^{n} z_j^3 \Big(\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \sum_{k=1}^{j-1} \frac{\lambda}{2a_k} z_k^6 S_k^{\mathrm{T}} S_k\Big). \end{aligned} \tag{68}$$

The auxiliary function $\varsigma_j(Z_j)$ can be obtained.

$$\begin{aligned} \varsigma_j(Z_j) =& -m_0 \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \hat{\theta} + \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \sum_{k=1}^{j-1} \frac{\lambda}{2a_k} z_k^6 S_k^{\mathrm{T}} S_k \\ &- \frac{\lambda s^2}{2a_j} z_j^3 \sum_{k=2}^{j} \left| z_k^3 \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right|, \ (2 \le j \le n) \end{aligned}.$$

So $h b_m \sum_{j=2}^{n} z_j^3 \Big(\varsigma_j(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \dot{\hat{\theta}}\Big) \le 0$.

It follows from $\tilde{\theta} = \theta - \hat{\theta}$ and the complete square inequality $\tilde{\theta}\hat{\theta} \le -\frac{\tilde{\theta}^2}{2} + \frac{\theta^2}{2}$ that

$$\frac{h m_0 b_m^2}{\lambda} \tilde{\theta} \hat{\theta} \le -\frac{h m_0 b_m^2}{2\lambda} \tilde{\theta}^2 + \frac{h m_0 b_m^2}{2\lambda} \theta^2, \tag{69}$$

Then, we can rewrite the function H_n

$$\begin{aligned} H_n \le& -\sum_{j=1}^{n} c_j z_j^4 - \frac{h m_0 b_m^2}{2\lambda} \tilde{\theta}^2 + \frac{1}{2} h \sum_{j=1}^{n} (a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 \\ &+ \frac{9n}{16} l_1 + h \sum_{j=1}^{n} (n+1-j-h\gamma^2) \varpi_j^2 \\ &+ \frac{h b_m^2 m_0}{2\lambda} \theta^2 + \frac{1}{2} u_{\min}^2. \end{aligned} \tag{70}$$

According to the formula (70), choose the design parameter $h = n/\gamma^2 + 1$, one has

$$\begin{aligned} H_n \le& -\sum_{j=1}^{n} c_j z_j^4 - \frac{h m_0 b_m^2}{2\lambda} \tilde{\theta}^2 + \frac{1}{2} h \sum_{j=1}^{n} (a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 \\ &+ \frac{9n}{16} l_1 + \frac{h b_m^2 m_0}{2\lambda} \theta^2 + \frac{1}{2} u_{\min}^2. \end{aligned} \tag{71}$$

Considering (15), (34) and (50), we can choose the whole Lyapunov function as

$$V(Z) = \frac{1}{4} h b_m \sum_{j=1}^{n} z_j^4 + \frac{1}{2\lambda} h b_m^2 \tilde{\theta}^2 + k_0, \tag{72}$$

with $Z = [z_1, \cdots, z_n, \tilde{\theta}]$. Due to $k_0 > 0$, so

$$V(Z) > 0. \tag{73}$$

According to $LV(Z) = \sum_{j=1}^{n} LV_j$, we have

$$LV(Z) \le -\sum_{j=1}^{n} c_j z_j^4 - \frac{hm_0 b_m^2}{2\lambda}\tilde{\theta}^2 + \frac{1}{2}h\sum_{j=1}^{n}(a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 + \frac{9n}{16}l_1 + \frac{hb_m^2 m_0}{2\lambda}\theta^2 + \frac{1}{2}u_{\min}^2 + h^2\gamma^2\|\varpi\|^2 . \quad (74)$$

Next, the proof of stability and bounded-H∞ performance for the system (1) will be given according to (74). It should be noted that the formula (74) is obtained under the condition that control laws (11) ~ (13) hold.

Based on the above deduction, Theorem 1 is able to be gained.

Theorem 1 For the system (1) satisfying Assumption 1-2. If the control laws and adaptive law are selected as (11) ~ (13) with the given positive design parameters λ , m_0 , a_i , k_i , h and the quantized input parameters σ , $u_{\min}$, afterwards all the signals are uniformly ultimately bounded in probability. At the meanwhile, the system has a good tracking performance and an H∞ performance against disturbance.

Proof 1) For the stability

Consider the stability analysis of the system (1). Let $\varpi_i(t) = 0,\ 1 \le i \le n$, then $\|\varpi\| = 0$. It can be gotten from (74) that

$$LV(Z) \le -\sum_{j=1}^{n} c_j z_j^4 - \frac{hm_0 b_m^2}{2\lambda}\tilde{\theta}^2 + \frac{1}{2}h\sum_{j=1}^{n}(a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 + \frac{9n}{16}l_1 + \frac{hb_m^2 m_0}{2\lambda}\theta^2 + \frac{1}{2}u_{\min}^2 , \quad (75)$$

let

$$a_0 = \min\{4c_j, m_0, j = 1, 2, \cdots, n\}$$

$$b_0 = \frac{1}{2}h\sum_{j=1}^{n}(a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 + \frac{9n}{16}l_1 + \frac{hb_m^2 m_0}{2\lambda}\theta^2 + \frac{1}{2}u_{\min}^2 + a_0 k_0 .$$

Then (75) can be rewritten as

$$LV \le -a_0 V + b_0,\ \ t \ge 0. \quad (76)$$

Therefore, the stochastic nonlinear system (1) is uniformly ultimately bounded in probability.

Further, the inequality (77) can be proved from (76).

$$E\left[V(Z(t))\right] \le e^{-a_0 t}(E\left[V(Z(0))\right] - \frac{b_0}{a_0}) + \frac{b_0}{a_0}, \quad (77)$$

which implies that

$$0 \le E\left[\bar{V}(Z(t))\right] \le E\left[\bar{V}(Z(0))\right]e^{-a_0 t} + \frac{\bar{b}_0}{a_0}, \quad (78)$$

where

$$\bar{V}(Z(t)) = \frac{1}{4}hb_m\sum_{j=1}^{n} z_j^4 + \frac{1}{2\lambda}hb_m^2\tilde{\theta}^2,$$

$$\bar{b}_0 = \frac{1}{2}h\sum_{j=1}^{n}(a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 + \frac{9n}{16}l_1 + \frac{hb_m^2 m_0}{2\lambda_1}\theta^2 + \frac{1}{2}u_{\min}^2 .$$

When $t \to \infty$, it produces

$$E\left[\bar{V}(Z(t))\right] \le \frac{\bar{b}_0}{a_0}. \quad (79)$$

Thus, from (79), we can easily learn that all the signals are semi-globally uniformly ultimately bounded by our proposed control strategy. In addition, the convergence bound of the error variables is independent of the constant k_0 .

2) For the H∞ performance

From (56), one has

$$H_n = LV + h^2(\|z_1\|^2 - \gamma^2\|\varpi\|^2) . \quad (80)$$

According to (71), we know that there must be a positive constant $\tilde{q}$ such that

$$H_n \le \frac{1}{2}h\sum_{j=1}^{n}(a_j + b_M^2 \delta_{j\max}^2) + l_1 h^3 + \frac{9n}{16}l_1 + \frac{hb_m^2 m_0}{2\lambda}\theta^2 + \frac{1}{2}u_{\min}^2 \le \tilde{q}V(Z) . \quad (81)$$

Combining (80) with (81), (82) can be obtained.

$$LV(Z) \le \tilde{q}V(Z) + h^2(\gamma^2\|\varpi\|^2 - \|z_1\|^2) . \quad (82)$$

For (82), we take integrations and expectations, it yields

$$E\left[V(Z(t))\right] - E\left[V(Z(0))\right] \le E\left[\int_0^t \tilde{q}V(Z(s))ds\right] + E\left[\int_0^t h^2(\gamma^2\|\varpi(s)\|^2 - \|z_1(s)\|^2)ds\right]. \quad (83)$$

Then

$$\begin{aligned} E\left[V(Z(t))\right] &\le E\left[\int_0^t qV(Z(s))ds\right] + E\left[V(Z(0))\right] \\ &\quad + E\left[\int_0^t h^2(\gamma^2\|\varpi(s)\|^2 - \|z_1(s)\|^2)ds\right] \\ &= E\left[\int_0^t \chi V(Z(s))ds\right] + E\left[\varphi(t)\right], \end{aligned} \quad (84)$$

with $\varphi(t) = \int_0^t h^2(\gamma^2\|\varpi(s)\|^2 - \|z_1(s)\|^2)ds + V(Z(0)),\ \chi = \tilde{q}$.

Applying Gronwall Inequality in [19] to (84) gives

$$E\left[V(Z(t))\right] \le \chi\int_0^t E\left[\varphi(s)\right]e^{E\left[\int_s^t \chi(\mu)d\mu\right]}ds + E\left[\varphi(t)\right]. \quad (85)$$

Next, in order to prove $E\left[\varphi(t)\right] > 0$, we utilize the method of reduction to absurdity. At the first time, we assume

$$E\left[\varphi(t)\right] \le 0, \quad (86)$$

then, it yields

$$E\left[V(Z(t))\right] \le \int_0^t \chi E\left[\varphi(s)\right]e^{E\left[\int_s^t \chi(\mu)d\mu\right]}ds \le 0. \quad (87)$$

Combining (86) with (87) results in

$$E\left[V(Z(t))\right] \le 0. \quad (88)$$

It is noticeable that (88) and (73) contradict each other. So, we can know $E\left[\varphi(t)\right] > 0$. Afterwards, we can get

$$E\left[h^2\int_0^t \|z_1(s)\|^2 ds\right] < E\left[h^2\int_0^t \gamma^2\|\varpi(s)\|^2 ds\right] + E\left[V(Z(0))\right]. \quad (89)$$

because of $h > 1$, (90) can be obtained.

$$E\left[\int_0^t \|z_1(s)\|^2 ds\right] < E\left[\int_0^t \gamma^2\|\varpi(s)\|^2 ds\right] + E\left[V(Z(0))\right]. \quad (90)$$

So far, the designed controller has satisfied bounded-H∞ performance index. Hence, we know that the system (1) has the H∞ performance.

IV. Simulation Example

A practical model in this section is given to demonstrate the performance and advantage of the proposed controller. Consider the following Brusselator model in [22],

$$\begin{cases} dx_1 = ((C-(D+1)x_1)+x_1^2x_2+x_1e^{-x_1}\varpi_1+\beta_1(x_1,x_2))dt \\ \qquad +\psi_1(x_1)d\omega, \\ dx_2 = (Dx_1+(2+\cos(x_1))Q(u)-x_1^2x_2)dt \\ \qquad (\sin(x_1+x_2^2)e^{-x_1}\varpi_2+\beta_1(x_1,x_2))dt \\ \qquad +\psi_2(x_1,x_2)d\omega, \\ y = x_1, \end{cases} \tag{89}$$

where $Q(u)$ denotes the quantized input defined in (2), $\beta_1(x_1,x_2)=0.3x_2^3$, $\beta_1(x_1,x_2)=0.4x_1^3x_2^5$, $\psi_1(x_1)=0.2\sin(x_1^3)$, $\psi_2(\bar{x}_2)=0.6x_1\cos(x_2)$, the reference signal is chosen as $y_d=0.5(\sin t+\sin 0.5t)$, the external disturbance functions are $\varpi_1=\sin(t)e^{-0.02t}$ and $\varpi_2=\cos(2t)e^{-0.05t}$. According to Theorem 1, the design parameters and the quantized input parameters are chosen as $C=1, D=0.5$, $k_1=20$, $k_2=20$, $a_1=a_2=1$, $\lambda=1$, $m_0=0.6$, $\sigma=0.1$, $u_{\min}=0.3$. To avoid the excessive controller gain, we choose $h=2$ when $t<0.05$ and $h=8$ when $t\geq 0.05$. Then control laws $\alpha_1(Z_1)$, $u(Z_2)$ and adaptive law $\dot{\hat{\theta}}$ can be obtained. The initial values of the simulation are $[x_1(0),\ x_2(0)]^{\mathrm{T}}=$ $[0.3,\ 0.2]^{\mathrm{T}}$ and $\hat{\theta}(0)=0$. RBFNNs are chosen as follows: $W_1^{\mathrm{T}}S_1(Z_1)$ contains 7 nodes with centers $u_l(l=1,\cdots,7)$ evenly distributed in interval $[-3,3]$, and the widths $r_l=2$, $W_2^{\mathrm{T}}S_2(Z_2)$ contains 7^3 nodes with centers $u_l(l=1,\cdots,343)$ evenly distributed in interval $[-3,3]\times[-3,3]\times[-3,3]$, and the widths are $r_l=2$. The simulation results are shown in Fig.1~Fig.6.

In order to verify the effect of disturbance attenuation, the simulation results with external disturbances or without external disturbances are given at the same time in Fig.1~Fig.6. Among them, Fig.1~Fig.4 denote the system output y, the tracking error z_1, the control signal u and the quantized signal $Q(u)$, respectively. Fig.5 and Fig. 6 show the curves of the state variable x_2 and the adaptive parameter $\hat{\theta}$. For a comparison, in these figures, the simulation results for the proposed scheme in this paper and the method proposed in [23] are given simultaneously under the same external disturbances. Here, the simulation parameters of the method in [23] are chosen as same as the paper [23].

As can be seen from Fig.1-Fig.4, the quantized adaptive bounded-H∞ tracking controller designed in this paper has a better tracking effect and the disturbance attenuation performance than the control method in [23]. In addition, in the simulation results, it is demonstrated that the tracking error can converge to a small neighborhood of the origin, all the signals of the system are uniformly ultimately bounded in probability.

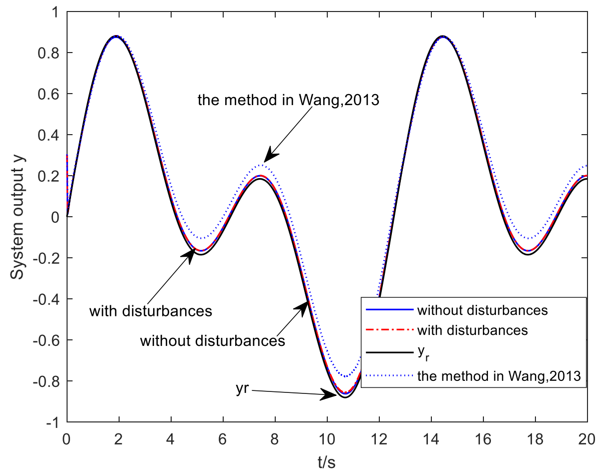

Fig. 1. Comparison of system tracking effect

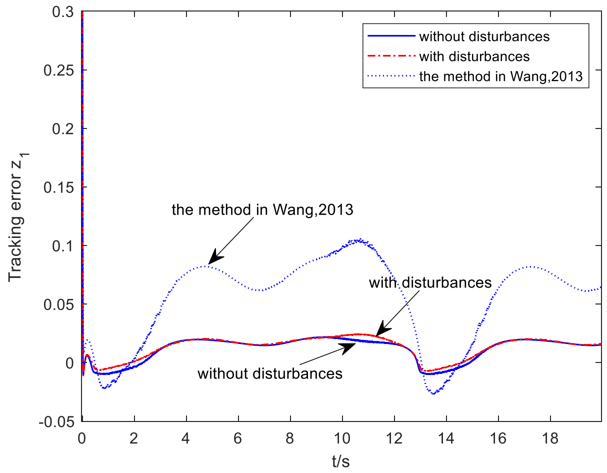

Fig. 2. Comparison of tracking error z_1 curves

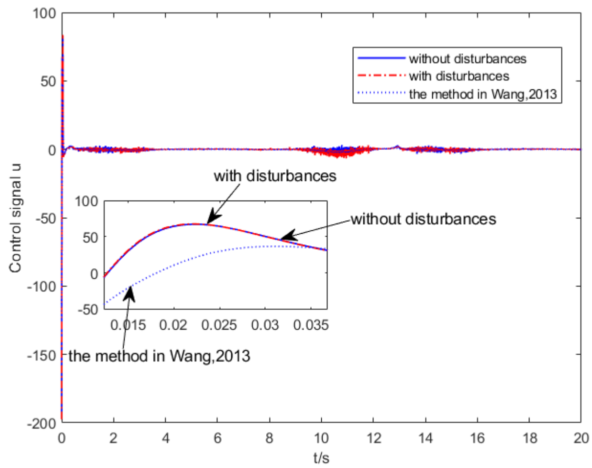

Fig. 3. Comparison of control input u curves

V. Conclusion

Aiming at a class of nonaffine pure-feedback stochastic nonlinear systems with external disturbances and quantized input, a novel quantized adaptive bounded-H∞ tracking control method in this paper is designed for the first time by using the backstepping technique and H∞ control theory. A developed quantized adaptive bounded-H∞ tracking control strategy ensures that all the signals remain bounded in probability. In addition, the system is guaranteed to have an H∞ disturbance attenuation performance. This is a meaningful work for nonlinear system control theory. Eventually, the results illustrate the effectiveness and advantage of the designed controller.

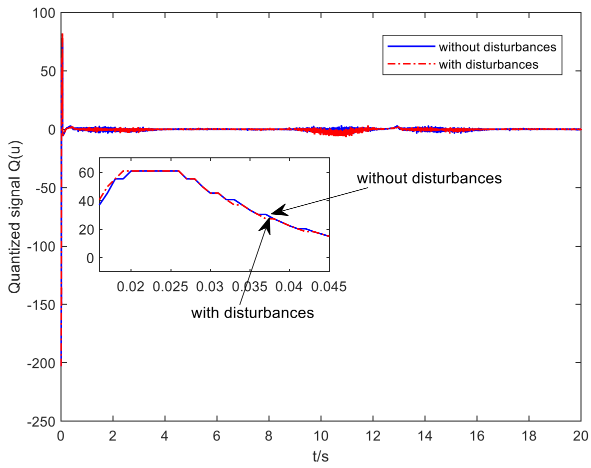

Fig.4. Comparison of quantized signal $Q(u)$ curves

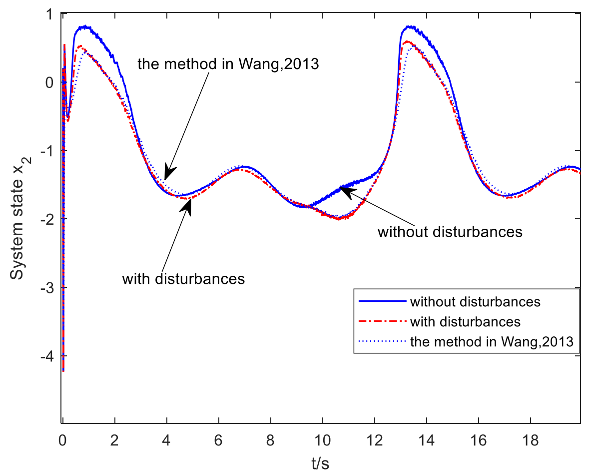

Fig. 5. Comparison of system state x_2 curves

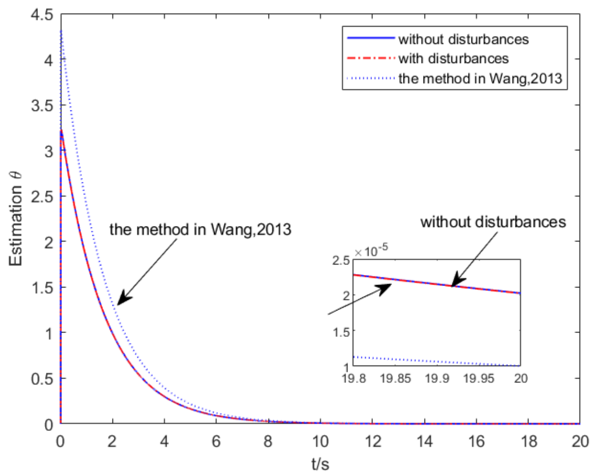

Fig. 6. Comparison of estimation $\hat{\theta}$ curves

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