Seidel Energy of Partial Complementary Graph

Amrithalakshmi, Swati Nayak*, Sabitha D'Souza and Pradeep G. Bhat

***Abstract*—The partial complement of a graph G with respect to a set S denoted by $G \oplus S$ is the graph obtained by removing the edges of $\langle S \rangle$ and adding edges which are not in $\langle S \rangle$ in G. In this paper we introduce the concept of Seidel energy of partial complement of a graph. Some bounds are obtained for Seidel energy of partial complementary graph. We compute Seidel energy and Seidel spectrum for partial complement of several classes of graph.**

***Index Terms*—Partial complements, Seidel matrix, Seidel energy, Seidel eigenvalues.**

I. Introduction

Let $G = (V, E)$ be a graph and $S \subseteq V$. The partial complement of a graph G with respect to S, denoted by $G \oplus S$, is a graph (V, E_S), where for any two vertices $u, v \in V$, $uv \in E_S$ if and only if one of the following conditions hold good:

1) $u \notin S$ or $v \notin S$ and $uv \in E$.
2) $u, v \in S$ and $uv \notin E$.

Alternatively, we can also define partial complement of graph G with respect to a set S as graph obtained from G by removing edges of $\langle S \rangle$ and adding the edges which are not in $\langle S \rangle$.
Let $G \oplus S$ be partial complement of a graph G with respect to S. Partial complement adjacency matrix [5] of $G \oplus S$ is $n \times n$ matrix defined by $A_p(G \oplus S) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i \neq j \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

We refer to [2] and [7] for all notations and terminologies.

J. Liu and B. Liu defined the Seidel energy of a graph in generalization for Laplacian energy and analyzed the Seidel energy bounds using the rank of the Seidel matrix and extended the concept of energy to Hermite matrix. In Seidel switching and graph energy, Willem H. Haemers investigates how Seidel switching changes the spectrum but not the energy and presents an infinite family of examples with maximal energy. We refer to [1] and [6] for more information on the energy of graphs.

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Definition 2. *[3] The Seidel matrix of a simple graph G with n vertices and m edges, denoted by $S(G) = (s_{ij})$ is a real square symmetric matrix of order n defined as*

$$s_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ 0, & i = j. \end{cases}$$

Definition 3. *[3] The Seidel energy of the graph G with n vertices and m edges is defined as*

$$SE(G) = \sum_{i=1}^{n} |s_i|,$$

where $s_1, s_2, \ldots, s_n$ are the eigenvalues of the Seidel matrix $S(G)$.

Definition 4. *The Seidel matrix of partial complementary graph $G \oplus S$ with n vertices and m_S edges, denoted by $S_p(G \oplus S)$ is defined as*

$$s_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ 1, & \text{if } v_i \in S \\ 0, & \text{if } i = j. \end{cases}$$

Definition 5. *The Seidel energy of the partial complementary graph $G \oplus S$ with n vertices and m_S edges is defined as*

$$SE_p(G \oplus S) = \sum_{i=1}^{n} |s_i|,$$

where $s_1, s_2, \ldots, s_n$ are the eigenvalues of the partial complement Seidel matrix $S_p(G \oplus S)$.

Theorem 6. *The Seidel eigenvalues $s_1, s_2, \ldots, s_n$ of the Seidel matrix of partial complementary graph $G \oplus S$ satisfies the following relations:*

1) $\sum_{i=1}^{n} s_i = |S|$.
2) $\sum_{i=1}^{n} s_i^2 = |S|^2 + n^2 - n$.

Proof:

1) Sum of principal diagonal elements of $S_p(G \oplus S) = |S|$. Also sum of eigenvalues of $S_p(G \oplus S)$=trace of $S_p(G \oplus S) = |S|$.
2) We know that sum of squares of eigenvalues of $S_p(G \oplus$

S) is trace of $S_p^2(G \oplus S)$.

$$\begin{aligned}\sum_{i=1}^{n} s_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}s_{ji}\\ &= \sum_{i=1}^{n} s_{ii}^2 + \sum_{i\neq j} s_{ij}s_{ji}\\ &= \sum_{i=1}^{n} s_{ii}^2 + 2\sum_{i<j} s_{ij}^2\\ &= |S|^2+2\left[m_S(-1)^2 + \left(\frac{n^2-n}{2} - m_S\right)1^2\right]\\ &= |S|^2+n^2-n.\end{aligned}$$

■

II. Bounds for Seidel energy of partial complementary graph

Theorem 7. *If $G \oplus S$ is partial complementary graph on n vertices with induced subgraph $\langle S\rangle$, then* $\sqrt{|S|+n^2-n+n(n-1)[\det S_p(G\oplus S)]^{2/n}}$ $\leq SE_p(G\oplus S) \leq \sqrt{n(|S|+n^2-n)}$.

Proof: By taking $a_i = 1$ and $b_i = |s_i|$ in Cauchy-Schwarz inequality, we get

$$\left(\sum_{i=1}^{n}|s_i|\right)^2 \leq n\sum_{i=1}^{n}|s_i|^2$$

From Theorem 6,

$$\left(\sum_{i=1}^{n}|s_i|\right)^2 \leq n(|S|^2+n^2-n)$$

$$SE_p(G\oplus S) \leq \sqrt{n(|S|^2+n^2-n)}.$$

By Arithmetic mean and Geometric mean inequality,

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|s_i||s_j| &\geq \left[\prod_{i\neq j}|s_i||s_j|\right]^{\frac{1}{n(n-1)}}\\ &\geq [\det S_p(G\oplus S)]^{2/n}\\ \sum_{i\neq j}|s_i||s_j| &\geq n(n-1)\,[\det S_p(G\oplus S)]^{2/n}.\end{aligned}$$

Consider,

$$\begin{aligned}[SE_p(G\oplus S)]^2 &= \left(\sum_{i=1}^{n}|s_i|\right)^2\\ &= \sum_{i=1}^{n}|s_i|^2+\sum_{i\neq j}|s_i||s_j|\end{aligned}$$

$$SE_p(G\oplus S) \geq \sqrt{|S|+n^2-n+n(n-1)[\det S_p(G\oplus S)]^{2/n}}.$$

■

Lemma 8. *[4] Let $a, a_1, a_2, \ldots a_n, A$ and $b, b_1, b_2, \ldots, b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$, $\forall i = 1, 2, \ldots, n$. Then the following inequality is valid.*

$$\left|n\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_i\sum_{i=1}^{n} b_i\right| \leq \alpha(n)(A-a)(B-b)$$

and equality holds if and only if $a_1 = a_2 = \ldots = a_n$ and $b_1 = b_2 = \ldots = b_n$.

Theorem 9. *Let $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$ be non-increasing order of eigenvalues of $S_p(G\oplus S)$. Then $SE_p(G\oplus S) \geq \sqrt{n(n^2-n+|S|)-\alpha(n)(|\lambda_1|-|\lambda_n|)^2}$, where $\alpha(n) = n[\frac{n}{2}](1-\frac{1}{n}[\frac{n}{2}])$.*

Proof: Taking $a_i = |\lambda_i|$, $b_i = |\lambda_i|$, $a = b = |\lambda_n|$ and $A = B = |\lambda_1|$ in Lemma 8, we obtain

$$\left|n\sum_{i=1}^{n}|\lambda_i|^2-\left(\sum_{i=1}^{n}\lambda_i\right)^2\right| \leq \alpha(n)(|\lambda_1|-|\lambda_n|)^2 \qquad (10)$$

but,

$$\sum_{i=1}^{n}|\lambda_i|^2= n^2-n+|S|.$$

Inequality (10) becomes $n(n^2-n+|S|)-[SE_p(G\oplus S)]^2 \leq \alpha(n)(|\lambda_1|-|\lambda_n|)^2$.

$SE_p(G\oplus S) \geq \sqrt{n(n^2-n+|S|)-\alpha(n)(|\lambda_1|-|\lambda_n|)^2}$, where $\alpha(n) = n[\frac{n}{2}](1-\frac{1}{n}[\frac{n}{2}])$, $[\frac{n}{2}]$ denotes the integral part of a real number. ■

Theorem 11. *Let $|\lambda_1|\geq |\lambda_2|\geq \ldots \geq |\lambda_n|> 0$ be a non-increasing order of eigenvalues of $G\oplus S$. Then*

$$SE_p(G\oplus S) \geq \frac{n^2-n+|S|+n|\lambda_1||\lambda_n|}{|\lambda_1|+|\lambda_n|}.$$

Proof: Let $a_i \neq 0, b_i, r$ and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds.

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \leq (r+R)\sum_{i=1}^{n} a_ib_i$$

By putting $b_i = |\lambda_i|$, $a_i = 1, r = |\lambda_n|$ and $R = |\lambda_1|$, we obtain

$$\sum_{i=1}^{n}|\lambda_i|^2+|\lambda_1||\lambda_n|\sum_{i=1}^{n} 1 \leq (|\lambda_1|+|\lambda_n|)\sum_{i=1}^{n}|\lambda_i|$$

$$n^2-n+|S|+|\lambda_1||\lambda_n|n \leq (|\lambda_1|+|\lambda_n|)SE_p(G\oplus S)$$

$$SE_p(G\oplus S) \geq \frac{n^2-n+|S|+n|\lambda_1||\lambda_n|}{|\lambda_1|+|\lambda_n|}.$$

■

Theorem 12. *Let $\rho(G\oplus S)$ be the spectral radius of $S_p(G\oplus S)$ of order n and size m_S. Then*

$$\sqrt{\frac{n^2-n+|S|}{n}} \leq \rho(G\oplus S) \leq \sqrt{n^2-n+|S|}.$$

Proof: Consider,

$$\begin{aligned}\rho^2(G\oplus S) &= \max_{1\leq i\leq n}\{|\lambda_i|^2\}\\ &\leq \sum_{i=1}^{n}\lambda_i^2 = n^2-n+|S|.\\ \rho(G\oplus S) &\leq \sqrt{n^2-n+|S|}.\end{aligned}$$

Next consider,

$$\begin{aligned}n\rho^2(G\oplus S) &\geq \max_{1\leq i\leq n}\{|\lambda_i|^2\}\\ &\geq n^2-n+|S|.\end{aligned}$$

Thus,

$$\rho(G\oplus S) \geq \sqrt{\frac{n^2-n+|S|}{n}}.$$

Hence, $\sqrt{\dfrac{n^2-n+|S|}{n}} \leq \rho(G\oplus S) \leq \sqrt{n^2-n+|S|}$. ■

Theorem 13. *If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of $S_p(G\oplus S)$ on n vertices and m_S edges, then $SE_p(G\oplus S) \leq \lambda_1 + \sqrt{(n-1)(n^2-n+|S|-\lambda_1^2)}$.*

Proof: Applying Cauchy Schwarz inequality for $(n-1)$ terms,

$$\left(\sum_{i=2}^{n}\lambda_i\right)^2 \leq \left(\sum_{i=2}^{n}1\right)\left(\sum_{i=2}^{n}\lambda_i^2\right)$$

$$[SE_p(G\oplus S)-\lambda_1]^2 \leq (n-1)(n^2-n+|S|-\lambda_1^2)$$

$$SE_p(G\oplus S) \leq \lambda_1 + \sqrt{(n-1)(n^2-n+|S|-\lambda_1^2)}.$$

■

Theorem 14. *For $G\oplus S$ on n vertices, m_S edges and $2m_S \geq n$,*

$$E_p(G\oplus S) \leq \frac{n^2-n+|S|}{n} + \sqrt{(n-1)\left[n^2-n+|S|-\left(\frac{n^2-n+|S|}{n}\right)^2\right]}.$$

Proof: From Theorem 13, we have,

$$SE_p(G\oplus S) \leq \lambda_1 + \sqrt{(n-1)(n^2-n+|S|-\lambda_1^2)}.$$

Let

$$f(x) = x + \sqrt{(n-1)(n^2-n+|S|-x^2)}.$$

For decreasing function,

$$f'(x) \leq 0 \Rightarrow 1 - \frac{2x(n-1)}{2\sqrt{(n-1)(n^2-n+|S|-x^2)}} \leq 0$$

$$\Rightarrow x \geq \sqrt{\frac{n^2-n+|S|}{n}}.$$

Since $n^2-n+|S| \geq n$,

we have $\sqrt{\dfrac{n^2-n+|S|}{n}} \leq \dfrac{n^2-n+|S|}{n} \leq \lambda_1$

Thus,

$$E_p(G\oplus S) \leq \frac{n^2-n+|S|}{n} + \sqrt{(n-1)\left[n^2-n+|S|-\left(\frac{n^2-n+|S|}{n}\right)^2\right]}.$$

■

III. Seidel energy of partial complements of some families of Graphs

For various classes of graphs, we now compute Seidel energy and the spectrum of partial complements. We adopt approach of eigenvector to prove Theorems 15, 19. 22 and 28. In this approach, the result is proved by showing $S_pW = \lambda W$ for certain vector W and by making use of fact that geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $S_p(G\oplus S)$ is real and symmetric.

Theorem 15. *Partial complement Seidel energy of complete graph K_n with $|S|= k$ vertices is $SE_p(K_n\oplus S) = (n-k-1) + \sqrt{4nk-4k^2+n^2-2n+1}$.*

Proof:

$$S_p(K_n\oplus S) = \begin{bmatrix} J_{k\times k} & -J_{k\times(n-k)} \\ -J_{(n-k)\times k} & (I-J)_{(n-k)\times(n-k)} \end{bmatrix}_{n\times n}$$

is the Seidel matrix of partial complement of $K_n\oplus S$.

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with S_p.

Consider

$$(S_p(K_n\oplus S)-\lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(J-\lambda I)X - JY \\ -JX + [-J+(1-\lambda)I]Y \end{bmatrix} \tag{16}$$

Case 1: Let $X = X_j = e_1 - e_j, j = 2,3,\ldots,k$ and $Y = O_{n-k}$.

From equation (16), $[J-\lambda I]X_j - JO_{n-k} = -\lambda X_j$.

Then, $\lambda = 0$ is an eigenvalue with multiplicity of at least $(k-1)$ since there are $(k-1)$ independent vectors of the form X_j.

Case 2: Let $X = O_{k-1}$ and $Y = Y_j = e_1 - e_j, j = 2,3,\ldots,n-k$.

From equation (16), $[-J+(1-\lambda)I]Y_j = (1-\lambda)Y_j$.

So $\lambda = 1$ is an eigenvalue with multiplicity of at least $(n-k-1)$ since there are $n-k-1$ independent vectors of the form Y_j.

Case 3: Let $Y = \mathbf{1}_{n-k}$ and $X = \left(\dfrac{n-k}{k-\lambda}\right)\mathbf{1}_k$, where λ is any root of the equation

$$\lambda^2 + \lambda(n-2k-1) + 2k^2 - 2nk + k = 0.$$

From equation (16),

$$\begin{aligned} &-J\left(\frac{n-k}{k-\lambda}\right)\mathbf{1}_k + [-J+(1-\lambda)I]\mathbf{1}_{n-k} \\ &= (1-\lambda)\mathbf{1}_k + \left(\frac{k+\lambda-1}{\lambda+1}\right)(n-k)\mathbf{1}_k \\ &= -k\mathbf{1}_{n-k}\frac{n-k}{k-\lambda} + (-n+k+1-\lambda)\mathbf{1}_{n-k} \\ &= \frac{\lambda^2+\lambda(n-2k-1)+2k^2-2nk+k}{k-\lambda}\mathbf{1}_{n-k} \end{aligned}$$

So

$$\lambda = \frac{2k+1-n}{2} + \frac{\sqrt{(2k+1-n)^2+4(2k^2-2nk+k)}}{2}$$

and

$$\lambda = \frac{2k+1-n}{2} - \frac{\sqrt{(2k+1-n)^2+4(2k^2-2nk+k)}}{2}$$

are the eigenvalues with multiplicity of at least one.

Thus partial complement Seidel spectrum of complete graph is

$$\left\{\begin{matrix} 0 & k-1 \\ 1 & n-k-1 \\ \dfrac{2k+1-n}{2} + \dfrac{\sqrt{(2k+1-n)^2+4(2k^2-2nk+k)}}{2} & 1 \\ \dfrac{2k+1-n}{2} - \dfrac{\sqrt{(2k+1-n)^2+4(2k^2-2nk+k)}}{2} & 1 \end{matrix}\right\}.$$

So,

$$SE_p(K_n\oplus S) = (n-k-1) + \sqrt{4nk-4k^2+n^2-2n+1}$$

. ■

Theorem 17. *Let $K_{1,n-1}\oplus S$ be the partial complement of star graph with $|S|= k$ vertices including central vertex. Then $SE_p(K_{1,n-1}\oplus S) = (k+n-3) + \sqrt{(n-2k+1)^2 + 4(k+2kn-2k^2-2n+2)}$.*

Proof: $S_p(K_{1,n-1} \oplus S)$
$$= \begin{bmatrix} \mathbf{1} & J_{1\times(k-1)} & -J_{1\times(n-k)} \\ J_{(k-1)\times 1} & (2I-J)_{k-1} & J_{(k-1)\times(n-k)} \\ -J_{(n-k)\times 1} & J_{(n-k)\times(k-1)} & (J-I)_{n-k} \end{bmatrix}_{n\times n}$$
is Seidel matrix of partial complement of $K_{1,n-1} \oplus S$.

On $|S_p(K_{1,n-1}\oplus S)-\lambda I|$, applying row operation $R_i' \longrightarrow R_i-R_{i+1}, i=2,3,\ldots,k-1,k+1,\ldots,n-k-1$ and column operations $C_i' \longrightarrow C_i+C_{i-1}+\ldots+C_2, i=k,k-1,\ldots,3$, $C_j' \longrightarrow C_j+C_{j-1}+\ldots+C_{k+1}, j=n-k,n-k-1,\ldots,k+2$ gives $(\lambda-2)^{k-2}(\lambda+1)^{n-k-1}\det(A)$, where $\det(A)$ is of order 3.

i.e,
$$\det(A) = \begin{vmatrix} 1-\lambda & k-1 & k-n \\ 1 & 1-\lambda-k+2 & n-k \\ -1 & k-1 & -\lambda+n-k-1 \end{vmatrix}$$
$$= (\lambda-2)[\lambda^2-(n-2k+1)\lambda+2n-k-2kn + 2k^2-2].$$

Therefore Seidel spectrum of $K_{1,n-1} \oplus S$ is
$$\begin{pmatrix} 2 & -1 & P+Q & P-Q \\ k-1 & n-k-1 & 1 & 1 \end{pmatrix},$$
where $P = \dfrac{n-2k+1}{2}$ and
$Q = \dfrac{\sqrt{(n-2k+1)^2+4(k+2kn-2k^2-2n+2)}}{2}$.
Hence $SE_p(K_{1,n-1} \oplus S) = (k+n-3) +\sqrt{(n-2k+1)^2+4(k+2kn-2k^2-2n+2)}$ is the Seidel energy of $K_{1,n-1} \oplus S$. ∎

Theorem 18. *Let $K_{l,m} \oplus S$ be partial complement of complete bipartite graph with partites V_1 and V_2 of l and m vertices respectively and $\langle S\rangle$ be an induced subset of V which consists of p vertices of V_1 and $k-p$ vertices of V_2. Then Seidel energy of $K_{l,m} \oplus S$ is*
$SE_p(K_{l,m}\oplus S) = (n+k-3)+\sqrt{n^2-4k^2+4kn-6n+9}$.

Proof: The Seidel characteristic polynomial of $K_{l,m}\oplus S$ is given by $|S_p(K_{l,m} \oplus S) - \lambda I| = \left|\begin{array}{c|c} P & Q \\ \hline Q & R \end{array}\right|_{n\times n}$

where,
$$P = \left|\begin{array}{c|c} [(2-\lambda)I-J]_p & J_{p\times k-p} \\ \hline J_{k-p\times p} & [(2-\lambda)I-J]_{k-p} \end{array}\right|,$$
$$Q = \left|\begin{array}{c|c} -J_{M\times p} & J_{M\times k-p} \\ \hline J_{l-p\times p} & -J_{l-p\times k-p} \end{array}\right|,$$
and
$$R = \left|\begin{array}{c|c} [J-(\lambda+1)I]_M & -J_{M\times l-p} \\ \hline -J_{l-p\times M} & [J-(\lambda+1)I]_{l-p} \end{array}\right|,$$
where $M = (n-l)-(k-p)$.

Step 1: Applying row operation $R_i' \longrightarrow R_i - R_{i+1}$, for $i = 1,2,\ldots,p-1,p+1,\ldots,k-p-1,k-p+1,\ldots,M-1,M+1,\ldots,l-p-1$ for the above determinant, we get $(\lambda-2)^{k-2}(\lambda+1)^{n-k-2}\det(B)$.

Step 2: In $\det(B)$, performing column operations $C_i' \longrightarrow C_i+C_{i-1}+\ldots+C_1, i=p,p-1,\ldots,2$, $C_j' \longrightarrow C_j + C_{j-1}+\ldots+C_{p+1}, j=k-p,k-p-1,\ldots,p+2$, $C_r' \longrightarrow C_r+C_{r-1}+\ldots+C_{k-p+1}, r=M,M-1,\ldots,k-p+2$ and $C_s' \longrightarrow C_s+C_{s-1}+\ldots+C_{M+1}, s=l-p,l-p-1,\ldots,M+2$, we get $\det(C)$.

On expansion of $\det(C)$, it reduces to order 4. On further simplification, we get polynomial $(\lambda^2-\lambda-2)[\lambda^2+(2k-n-1)\lambda+2n-k-2kn+2k^2-2]$.
Hence Seidel characteristic polynomial of $K_{l,m} \oplus S$ is $(\lambda-2)^{k-1}(\lambda+1)^{n-k-1}[\lambda^2+(2k-n-1)\lambda+2n-k-2kn+2k^2-2]$.

Also Seidel spectrum of $K_{l,m} \oplus S$ is
$$\begin{pmatrix} 2 & -1 & P+Q & P-Q \\ k-1 & n-k-1 & 1 & 1 \end{pmatrix},$$
where $P = \dfrac{n-2k+1}{2}$ and
$Q = \dfrac{\sqrt{n^2-4k^2+4kn-6n+9}}{2}$.
Therefore Seidel energy of $K_{l,m} \oplus S$ is
$SE_p(K_{l,m}\oplus S) = (n+k-3)+\sqrt{n^2-4k^2+4kn-6n+9}$. ∎

Theorem 19. *Let $S_n^0 \oplus S$ be the partial complement of a crown graph with $|S|= k$.*
(i) *$SE_p(S_n^0 \oplus S) = 5(n-1)+\sqrt{8n^2-28n+25}$ for $k=n$.*
(ii) *$SE_p(S_n^0 \oplus S) = 2(3n-4)$ for $k=2n$.*

Proof: **(i)** Let $S_p = \begin{bmatrix} (2I-J)_n & (2I-J)_n \\ (2I-J)_n & (J-I)_n \end{bmatrix}_{2n\times 2n}$ be the Seidel matrix of $S_n^0 \oplus S$.
Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with S_p.
Consider
$$(S_p-\lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(2-\lambda)I-J]X+(2I-J)Y \\ (2I-J)X+[J-(\lambda+1)I]Y \end{bmatrix} \quad (20)$$

Case 1: Let $X = X_j, j=2,3,\ldots,n$ and $Y = \dfrac{\lambda-2}{2}X_j$, where λ is any root of the equation
$$\lambda^2-\lambda-6=0.$$
From equation (20),
$$(2I-J)X_j+[J-(\lambda+1)I]\left(\frac{\lambda-2}{2}\right)X_j$$
$$= \left[2-\frac{(\lambda+1)(\lambda-2)}{2}\right]X_j.$$
Hence $\lambda = 3$ and $\lambda = -2$ are the eigenvalues with multiplicity of at least $n-1$, as there are $n-1$ eigenvectors of the form X_j.

Case 2: Let $Y = \mathbf{1}_n$ and $X = -\dfrac{\lambda-n+1}{n-2}\mathbf{1}_n$, where λ is any root of the equation
$$\lambda^2-\lambda-2n^2+7n-6=0.$$
From equation (20),
$$[(2-\lambda)I-J]\frac{(-\lambda+n-1)}{n-2}\mathbf{1}_n+(2I-J)\mathbf{1}_n$$
$$= \frac{(\lambda-n+1)(\lambda+n-2)}{n-2}-(n-2).$$
Thus $\lambda = \dfrac{1+\sqrt{8n^2-28n+25}}{2}$ and
$\lambda = \dfrac{1-\sqrt{8n^2-28n+25}}{2}$ are the eigenvalues with multiplicity of at least one.
Thus Seidel spectrum of partial complement of crown graph with $|S|= n$ is

$$\begin{pmatrix} 3 & -2 & \frac{1}{2}+\frac{\sqrt{8n^2-28n+25}}{2} & \frac{1}{2}-\frac{\sqrt{8n^2-28n+25}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

and its Seidel energy is

$$SE_p(S_n^0 \oplus S) = 5(n-1) + \sqrt{8n^2-28n+25}.$$

(2) Let $S_p = \begin{bmatrix} (2I-J)_n & -(2I-J)_n \\ -(2I-J)_n & (2I-J)_n \end{bmatrix}_{2n\times 2n}$ be the Seidel matrix of $S_n^0 \oplus S$.

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with S_p.

Consider

$$(S_p - \lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(2-\lambda)I-J]X + (J-2I)Y \\ (J-2I)X + [(2-\lambda)I-J]Y \end{bmatrix} \quad (21)$$

Case 1: Let $X = X_j, j = 2, 3, \ldots, n$ and $Y = -\frac{\lambda-2}{2}X_j$, where λ is any root of the equation

$$\lambda^2 - 4\lambda = 0.$$

From equation (21),

$$(J-2I)X_j + [(2-\lambda)I-J]\left(-\frac{\lambda-2}{2}\right)X_j = \left[-2 + \frac{(\lambda-2)^2}{2}\right]X_j.$$

Hence $\lambda = 0$ and $\lambda = 4$ are the eigenvalues each with multiplicity of at least $n-1$, as there are $n-1$ eigenvectors of the form X_j.

Case 2: Let $X = \mathbf{1}_n$ and $Y = \frac{\lambda+n-1}{n-2}\mathbf{1}_n$, where λ is any root of the equation

$$\lambda^2 + (2n-4)\lambda = 0.$$

From equation (21),

$$(J-2I)\mathbf{1}_n + [(2-\lambda)I-J]\frac{(\lambda+n-2)}{n-2}\mathbf{1}_n = \frac{(4-2n)\lambda - \lambda^2}{n-2}\mathbf{1}_n.$$

Thus $\lambda = 0$ and $\lambda = 4-2n$ are the eigenvalues with multiplicity of at least one.

Thus Seidel spectrum of partial complement of crown graph with $|S| = 2n$ is $\begin{pmatrix} 0 & 4 & 4-2n \\ n & n-1 & 1 \end{pmatrix}$

and its Seidel energy is $SE_p(S_n^0 \oplus S) = 6n-8$. ∎

Theorem 22. *Let $K_{n\times 2}$ be cocktail party graph with $\langle S \rangle = K_n$. Then*

$$SE_p(K_{n\times 2} \oplus S) = \sqrt{17}(n-1) + \sqrt{4n^2-12n+17}.$$

Proof: Let $S_p = \begin{bmatrix} J_n & (2I-J)_n \\ (2I-J)_n & (I-J)_n \end{bmatrix}_{2n\times 2n}$ be the Seidel matrix of $K_{n\times 2} \oplus S$.

Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with S_p.

Consider

$$(S_p - \lambda I)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} (J-\lambda I)X + (2I-J)Y \\ (2I-J)X + [(1-\lambda)I-J]Y \end{bmatrix} \quad (23)$$

Case 1: Let $X = X_j, j = 2, 3, \ldots, n$ and $Y = \frac{\lambda}{2}X_j$, where λ is any root of the equation $\lambda^2 - \lambda - 4 = 0$.

From equation (23),

$$(2I-J)X_j + [(1-\lambda)I-J]\left(\frac{\lambda}{2}\right)X_j = \left[2 + \frac{(1-\lambda)\lambda}{2}\right]X_j.$$

Hence $\lambda = 2.5616$ and $\lambda = -1.5616$ are the eigenvalues each with multiplicity of at least $n-1$, as there are $n-1$ eigenvectors of the form X_j.

Case 2: Let $X = \mathbf{1}_n$ and $Y = \frac{\lambda-n}{2}\mathbf{1}_n$, where λ is any root of the equation

$\lambda^2 - \lambda + 3n - n^2 - 4 = 0.$

From equation (23),

$$(2I-J)\mathbf{1}_n + [(1-\lambda)I-J]\frac{\lambda-n}{2}\mathbf{1}_n = \frac{2(2-n) + (1-\lambda)(\lambda-n) - n(\lambda-n)}{2}\mathbf{1}_n.$$

Thus $\lambda = \frac{1+\sqrt{4n^2-12n+17}}{2}$ and $\lambda = \frac{1-\sqrt{4n^2-12n+17}}{2}$ are the eigenvalues with multiplicity of at least one.

Therefore Seidel spectrum of partial complement of cocktail party graph with $\langle S \rangle = K_n$ is

$$\begin{pmatrix} 2.5616 & -1.5616 & \frac{1+\sqrt{4n^2-12n+17}}{2} & \frac{1-\sqrt{4n^2-12n+17}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

and its Seidel energy is

$$SE_p(K_{n\times 2} \oplus S) = \sqrt{17}(n-1) + \sqrt{4n^2-12n+17}.$$

∎

Theorem 24. *Let $S(l,m)$ be double star graph of order $l+m+2$ with $\langle S \rangle = K_{1,l}$. Then $S_\phi(S(l,m) \oplus S) = (\lambda+1)^{m-1}(\lambda-2)^{l-1}[\lambda^4 + (l-m-2)\lambda^3 + (m-2l-2lm-2)\lambda^2 + (2m-l+2lm+3)\lambda + 2l+2]$.*

Proof: The Seidel characteristic polynomial of $S(l,m) \oplus S$ is given by

$|S_p(S(l,m) \oplus S) - \lambda I| =$

$$\begin{vmatrix} [(2-\lambda)I-J]_l & J_{l\times 1} & J_{l\times 1} & J_{l\times m} \\ J_{1\times l} & [(1-\lambda)I-J]_1 & J_{1\times 1} & -J_{1\times m} \\ J_{1\times l} & J_{1\times 1} & \lambda I_1 & -J_{1\times m} \\ J_{m\times l} & J_{m\times 1} & -J_{m\times 1} & [J-(1+\lambda)I]_m \end{vmatrix}_{l+m+2}$$

Step 1: Applying row operation $R_i' \longrightarrow R_i - R_{i+1}$, for $i = 1, 2, \ldots, l-1, l+3, \ldots, , l+m+1$ for the above determinant, we get $(\lambda-2)^{l-1}(\lambda+1)^{m-1}\det(B)$.

Step 2: In $\det(B)$, performing column operations $C_i' \longrightarrow C_i + C_{i-1} + \ldots + C_1, i = l, l-1, \ldots, 2$, $C_j' \longrightarrow C_j + C_{j-1} + \ldots + C_{l+3}, j = l+m+2, l+m+1, \ldots, l+4$, we get $\det(C)$.

On expansion of $\det(C)$, it reduces to order 4. On further simplification, we get polynomial $\lambda^4 + (l-m-2)\lambda^3 + (m-2l-2lm-2)\lambda^2 + (2m-l+2lm+3)\lambda + 2l+2$.

Hence Seidel characteristic polynomial of $S(l,m) \oplus S$ is $(\lambda+1)^{m-1}(\lambda-2)^{l-1}[\lambda^4 + (l-m-2)\lambda^3 + (m-2l-2lm-2)\lambda^2 + (2m-l+2lm+3)\lambda + 2l+2]$. ∎

Seidel energy of partial complement of amalgamation of m copies of K_n

A graph amalgamation is a relationship between two graphs (one graph is an amalgamation of another). Amalgamations can be used to reduce a graph to a simpler graph while retaining some structure.

Definition 25. *Let $\{G_1, G_2, G_3, \ldots, G_m\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{0i} called a terminal. The amalgamation $Amal(v_{0i}, G_i)$ is formed by taking all the $G_i's$ and identifying their terminals. In particular, if we take $G_i = K_n$ for $i = 1, 2, \ldots, m$ we get amalgamation of m copies of K_n denoted by $Amal(m, K_n)$, $m \geq 2$. For convenience we denote v_0 as the vertex of amalgamation and $v_{j2}, v_{j3}, \ldots, v_{jn}$ are the remaining vertices of the j^{th} copy of K_n, where $1 \leq j \leq m$.*

Theorem 26. *Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ and $S = \{v_0\}$. Then, $SE_P(Amal(m, K_n) \oplus S) = 3mn - 2n - 5m + 3 + \sqrt{(mn - 2n - m + 4)^2 + 4(2n - 3)}$.*

Proof: Let $S_p =$

$$\begin{pmatrix} J_1 & -J_{1\times n-1} & -J_{1\times n-1} & \ldots & -J_{1\times n-1} \\ -J_{n-1\times 1} & -B_{n-1} & J_{n-1} & \ldots & J_{n-1} \\ -J_{n-1\times 1} & J_{n-1} & -B_{n-1} & \ldots & J_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -J_{n-1\times 1} & J_{n-1} & J_{n-1} & \ldots & -B_{n-1} \end{pmatrix}_{m(n-1)+1}$$

be the Seidel matrix of $Amal((m, K_n) \oplus S)$. Here J is matrix of all 1's and B is the adjacency matrix of complete subgraph.
Step 1: Consider $|\lambda I - S_p|$.
Applying row operation $R'_{v_{ij}} \longrightarrow R_{v_{ij}} - R_{v_{ij+1}}, i = 1, 2, \ldots, m, j = 2, 3, \ldots, n-1$ and column operation $C'_{v_{ij}} \longrightarrow C_{v_{ij}} + C_{v_{ij-1}} + C_{v_{ij-2}} + \ldots + C_{v_{i2}}, i = 1, 2, \ldots, m, j = n, n-1, \ldots, 3$ on $|\lambda I - S_p|$, we get $(\lambda - 1)^{m(n-2)} \det(C)$, where $\det(C)$ is of the order $m + 1$.
Step 2: On performing row operation $R'_i \longrightarrow R_i - R_{i+1}, i = 2, 3, \ldots, m$ and column operation $C'_i \longrightarrow C_i + C_{i-1} + C_{i-2} + \ldots + C_2, i = m+1, m, \ldots, 3$ on $\det(C)$, we obtain $(\lambda + 2n - 3)^{m-1} \det(D)$ which is of order 2.
Step 3: Expanding $\det(D)$ leads to the polynomial $\lambda^2 + (m + 2n - mn - 4)\lambda - 2n + 3$.
Hence Seidel spectrum of partial complement of $Amal(m, K_n)$ is

$$\begin{pmatrix} 1 & 3-2n & \dfrac{P+Q}{2} & \dfrac{P-Q}{2} \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix},$$

where $P = mn + 4 - 2n - m$ and
$Q = \sqrt{(mn + 4 - 2n - m)^2 - 4(3 - 2n)}$.
So $SE_P(Amal(m, K_n) \oplus S) = 3mn - 2n - 5m + 3 + \sqrt{(mn - 2n - m + 4)^2 + 4(2n - 3)}$. ∎

Theorem 27. *Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ with $S = \{v_{ij} | i = 1, 2, \ldots, m, j = 2, 3, \ldots, n\}$. Then $SE_P(Amal(m, K_n) \oplus S) = 3nm - 4m - 2n + 2 + \sqrt{(mn - 2n - m + 2)^2 + 4m(n - 1)}$.*

Proof: Let $S_p =$

$$\begin{pmatrix} 0_1 & -J_{1\times n-1} & -J_{1\times n-1} & \ldots & -J_{1\times n-1} \\ -J_{n-1\times 1} & J_{n-1} & -J_{n-1} & \ldots & -J_{n-1} \\ -J_{n-1\times 1} & -J_{n-1} & J_{n-1} & \ldots & -J_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -J_{n-1\times 1} & -J_{n-1} & -J_{n-1} & \ldots & J_{n-1} \end{pmatrix}_{m(n-1)+1}$$

be the Seidel matrix of partial complement of $Amal(m, K_n)$.
Repeating Step 1 to Step 3 of Theorem 26, we get the polynomial

$\lambda^{m(n-2)}(\lambda - 2n + 2)^{m-1}[\lambda^2 + (mn - 2n - m + 2)\lambda - mn + m]$.

Hence the Seidel spectrum of partial complement of $Amal(m, k_n)$ is

$$\begin{pmatrix} 0 & 2n-2 & \dfrac{P+Q}{2} & \dfrac{P-Q}{2} \\ m(n-2) & m-1 & 1 & 1 \end{pmatrix},$$

where $P = -(mn - 2n - m + 2)$ and
$Q = \sqrt{(mn - 2n - m + 2)^2 + 4m(n - 1)}$.
Therefore, $SE_P(Amal(m, K_n) \oplus S) = 3nm - 4m - 2n + 2 + \sqrt{(mn - 2n - m + 2)^2 + 4m(n - 1)}$. ∎

Seidel Energy of Partial complement of Ladder rung graph

The ladder rung graph LR_n is a regular graph of degree one on $2n$ vertices. Let the vertices of LR_n be $v_1, v_2, \ldots, v_{2n}$ and the vertex v_i is adjacent to $v_{i+1}, i = 1, 3, \ldots, 2n - 1$. We obtain $SE_p(LR_n \oplus S)$, when $S = \{v_1, v_3, \ldots, v_{2n-1}\}$ in the following theorem.

Theorem 28. *Let LR_n be the Ladder rung graph with $S = \{v_1, v_3, \ldots, v_{2n-1}\}$. Then, $SE_p(LR_n \oplus S) = 5(n - 1) + \sqrt{8n^2 - 28n + 25}$.*

Proof: Let $S_p = \begin{bmatrix} (2I - J)_n & -(2I - J)_n \\ -(2I - J)_n & -(I - J)_n \end{bmatrix}_{2n\times 2n}$ be the Seidel matrix of $LR_n \oplus S$.
Let $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with S_p.
Consider

$$(\lambda I - S_p)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} ((\lambda - 2)I + J)X + (2I - J)Y \\ (2I - J)X + [(1 + \lambda)I - J]Y \end{bmatrix} \quad (29)$$

Case 1: Let $X = X_j, j = 2, 3, \ldots, n$ and $Y = -\dfrac{\lambda - 2}{2}X_j$, where λ is any root of the equation $\lambda^2 - \lambda - 6 = 0$.

From equation (29),

$$(2I - J)X_j - [(1 + \lambda)I - J]\left(\frac{\lambda - 2}{2}\right)X_j = \left[2 - \frac{(\lambda + 1)(\lambda - 2)}{2}\right]X_j.$$

Hence $\lambda = -2$ and $\lambda = 3$ are the eigenvalues each with multiplicity of at least $n - 1$, as there are $n - 1$ eigenvectors of the form X_j.

Case 2: Let $X = \mathbf{1}_n$ and $Y = \dfrac{\lambda + n - 2}{n - 2}\mathbf{1}_n$, where λ is any root of the equation $\lambda^2 - \lambda - 2n^2 + 7n - 6 = 0$.

From equation (29),

$$(2I - J)\mathbf{1}_n + [(1 + \lambda)I - J]\frac{\lambda + n - 2}{n - 2}\mathbf{1}_n = \frac{(2 - n)^2 + (\lambda + 1 - n)(\lambda + n - 2)}{n - 2}\mathbf{1}_n.$$

Thus $\lambda = \dfrac{1 + \sqrt{8n^2 - 28n + 25}}{2}$ and
$\lambda = \dfrac{1 - \sqrt{8n^2 - 28n + 25}}{2}$ are the eigenvalues with multiplicity of at least one.
Therefore Seidel spectrum of partial complement of ladder rung graph with respect $\langle S \rangle$ is

$$\begin{pmatrix} -2 & 3 & \dfrac{1 + \sqrt{8n^2 - 28n + 25}}{2} & \dfrac{1 - \sqrt{8n^2 - 28n + 25}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

and its Seidel energy is

$$SE_p(LR_n \oplus S) = 5(n - 1) + \sqrt{8n^2 - 28n + 25}.$$

∎

IV. Conclusion

In this paper, we computed the Seidel energy and Seidel spectrum of partial complements of several graph classes. The Seidel energy of a partial complement of a graph is determined by the subgraph chosen from G. We also examined a few properties and established upper and lower bounds for $SE_P(G \oplus S)$.

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