Seidel Energy of Partial Complementary Graph

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Abstract—The partial complement of a graph $G$ with respect to a set $S$ denoted by $G \oplus S$ is the graph obtained by removing the edges of $\langle S \rangle$ and adding edges which are not in $\langle S \rangle$ in $G$. In this paper we introduce the concept of Seidel energy of partial complement of a graph. Some bounds are obtained for Seidel energy of partial complementary graph. We compute Seidel energy and Seidel spectrum for partial complement of several classes of graph.


1. INTRODUCTION

Let $G = (V, E)$ be a graph and $S \subseteq V$. The partial complement of a graph $G$ with respect to $S$, denoted by $G \oplus S$, is a graph $(V, E_S)$, where for any two vertices $u, v \in V$, $uv \in E_S$ if and only if one of the following conditions hold good:

1) $u \notin S$ or $v \notin S$ and $uv \in E$.
2) $u, v \in S$ and $uv \notin E$.

Alternatively, we can also define partial complement of graph $G$ with respect to a set $S$ as graph obtained from $G$ by removing edges of $\langle S \rangle$ and adding the edges which are not in $\langle S \rangle$.

Let $G \oplus S$ be partial complement of a graph $G$ with respect to $S$. Partial complement adjacency matrix $[5]$ of $G \oplus S$ is $n \times n$ matrix defined by $A_p(G \oplus S) = (a_{ij})$, where

$$ a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i \neq j \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise.} \end{cases} $$

We refer to [2] and [7] for all notations and terminologies.

J. Liu and B. Liu defined the Seidel energy of a graph in generalization for Laplacian energy and analyzed the Seidel energy bounds using the rank of the Seidel matrix and extended the concept of energy to Hermite matrix. In Seidel switching and graph energy, Willem H. Haemers investigates how Seidel switching changes the spectrum but not the energy and presents an infinite family of examples with maximal energy. We refer to [1] and [6] for more information on the energy of graphs.

Definition 2. [3] The Seidel matrix of a simple graph $G$ with $n$ vertices and $m$ edges, denoted by $S(G) = (s_{ij})$ is a real square symmetric matrix of order $n$ defined as

$$ s_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ 0, & \text{if } i = j. \end{cases} $$

Definition 3. [3] The Seidel energy of the graph $G$ with $n$ vertices is defined as

$$ SE(G) = \sum_{i=1}^{n} |s_i|, $$

where $s_1, s_2, \ldots, s_n$ are the eigenvalues of the Seidel matrix $S(G)$.

Definition 4. The Seidel matrix of partial complementary graph $G \oplus S$ with $n$ vertices and $m$ edges, denoted by $S_p(G \oplus S)$ is defined as

$$ s_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ 1, & \text{if } v_i \in S \\ 0, & \text{if } i = j. \end{cases} $$

Definition 5. The Seidel energy of the partial complementary graph $G \oplus S$ with $n$ vertices is defined as

$$ SE_p(G \oplus S) = \sum_{i=1}^{n} |s_i|, $$

where $s_1, s_2, \ldots, s_n$ are the eigenvalues of the partial complement Seidel matrix $S_p(G \oplus S)$.

Theorem 6. The Seidel eigenvalues $s_1, s_2, \ldots, s_n$ of the Seidel matrix of partial complementary graph $G \oplus S$ satisfies the following relations:

1) $\sum_{i=1}^{n} s_i = |S|$
2) $\sum_{i=1}^{n} s_i^2 = |S|^2 + n^2 - n.$

Proof:

1) Sum of principal diagonal elements of $S_p(G \oplus S) = |S|$.
Also sum of eigenvalues of $S_p(G \oplus S)$=trace of $S_p(G \oplus S) = |S|$.
2) We know that sum of squares of eigenvalues of $S_p(G \oplus S)$
Consider, 
\[ b \quad \text{and equality holds if and only if} \quad (\{1\}) \cdot (1 - \frac{1}{n^2}) \].

**Theorem 9.** Let \(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\) be non-increasing order of eigenvalues of \(S_p(G \oplus S)\), Then 
\[ SE_p(G \oplus S) \geq \sqrt{n(n^2 - n + |S|)} - \alpha(n)(|\lambda_1| - |\lambda_n|)^2, \text{ where } \alpha(n) = n(\frac{n^2}{2}) - (\frac{n}{2} - (\frac{1}{n^2})]. \]

**Proof:** Taking \(a_i = |\lambda_i|\), \(b_i = |\lambda_i|\), \(a = b = |\lambda_n|\) and 
\(A = B = |\lambda_1|\) in Lemma 8, we obtain
\[ \left| \sum_{i=1}^{n} |\lambda_i|^2 - \left( \sum_{i=1}^{n} \lambda_i \right)^2 \right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2 \]
but,
\[ \sum_{i=1}^{n} |\lambda_i|^2 = n^2 - n + |S| \].

Inequality (10) becomes 
\[ n(n^2 - n + |S|) - |SE_p(G \oplus S)| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2 \].

**Theorem 11.** Let \(|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| > 0\) be a non-increasing order of eigenvalues of \(G \oplus S\). Then
\[ SE_p(G \oplus S) \geq \frac{n^2 - n + |S| + n|\lambda_1||\lambda_n|}{|\lambda_1| + |\lambda_n|}. \]

**Proof:** Let \(a_i \neq 0, b_i, r\) and \(R\) be real numbers satisfying 
\(ra_i \leq b_i\), then the following inequality holds.
\[ \sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i b_i \]
By putting \(b_i = |\lambda_i|, a_i = 1, r = |\lambda_n|\) and \(R = |\lambda_1|\), we obtain
\[ \sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} |\lambda_i| \]
\[ n^2 - n + |S| + n|\lambda_1||\lambda_n| \leq \sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} |\lambda_i| \]
\[ SE_p(G \oplus S) \geq \frac{n^2 - n + |S| + n|\lambda_1||\lambda_n|}{|\lambda_1| + |\lambda_n|}. \]

**Theorem 12.** Let \(\rho(G \oplus S)\) be the spectral radius of \(S_p(G \oplus S)\) of order \(n\) and size \(m_s\). Then
\[ \sqrt{n^2 - n + |S|} \leq \rho(G \oplus S) \leq \sqrt{n^2 - n + |S|}. \]

**Proof:** Consider,
\[ \rho^2(G \oplus S) = \max_{1 \leq i \leq n} \{|\lambda_i|^2\} \]
\[ \leq \sum_{i=1}^{n} \lambda_i^2 = n^2 - n + |S|. \]

Next consider,
\[ n\rho^2(G \oplus S) \geq \max_{1 \leq i \leq n} \{|\lambda_i|^2\} \]
\[ \geq n^2 - n + |S|. \]

Thus,
\[ \rho(G \oplus S) \geq \sqrt{n^2 - n + |S|}. \]
Hence, \( \sqrt{\frac{n^2 - n + |S|}{n}} \leq \rho(G \oplus S) \leq \sqrt{n^2 - n + |S|} \).

**Theorem 13.** If \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) are the eigenvalues of \( S_p(G \oplus S) \) on \( n \) vertices and \( m_S \) edges, then \( SE_p(G \oplus S) \leq \lambda_1 + \sqrt{(n-1)(n^2-n+|S|)} \).

**Proof:** Applying Cauchy Schwarz inequality for \((n-1)\) terms,
\[
\left( \sum_{i=1}^{n} \lambda_i \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} \lambda_i^2 \right)
\]
\[
[SE_p(G \oplus S) - \lambda_1]^2 \leq (n-1)(n^2-n+|S|-\lambda_1^2)
\]
\[
SE_p(G \oplus S) \leq \lambda_1 + \sqrt{(n-1)(n^2-n+|S|)}.
\]

**Theorem 14.** For \( G \oplus S \) on \( n \) vertices, \( m_S \) edges and \( 2m_S \geq n \),
\[
E_p(G \oplus S) \leq \frac{n^2 - n + |S|}{n}
\]
\[
+ \sqrt{(n-1) \left[ n^2 - n + |S| - \left( \frac{n^2 - n + |S|}{n} \right)^2 \right]}.
\]

**Proof:** From Theorem 13, we have,
\[
SE_p(G \oplus S) \leq \lambda_1 + \sqrt{(n-1)(n^2-n+|S|-\lambda_1^2)}.
\]

Let
\[
f(x) = x + \sqrt{(n-1)(n^2-n+|S|)} - x^2.
\]

For decreasing function,
\[
f'(x) \leq 0 \Rightarrow 1 - \frac{2x(n-1)}{2 \sqrt{(n-1)(n^2-n+|S|)} - x^2} \leq 0
\]
\[
\Rightarrow x \geq \sqrt{\frac{n^2 - n + |S|}{n}}.
\]

Since \( n^2 - n + |S| \geq n \),
we have
\[
\frac{n^2 - n + |S|}{n} \leq \frac{n^2 - n + |S|}{n} \leq \lambda_1
\]
Thus,
\[
E_p(G \oplus S) \leq \frac{n^2 - n + |S|}{n}
\]
\[
+ \sqrt{(n-1) \left[ n^2 - n + |S| - \left( \frac{n^2 - n + |S|}{n} \right)^2 \right]}.
\]

**III. SEIDEL ENERGY OF PARTIAL COMPLEMENTS OF SOME FAMILIES OF GRAPHS**

For various classes of graphs, we now compute Seidel energy and the spectrum of partial complements. We adopt approach of eigenvector to prove Theorems 15, 19, 22 and 28. In this approach, the result is proved by showing \( S_pW = \lambda W \) for certain vector \( W \) and by making use of fact that geometric multiplicity and algebraic multiplicity of each eigenvalue \( \lambda \) is same, as \( S_p(G \oplus S) \) is real and symmetric.

**Theorem 15.** Partial complement Seidel energy of complete graph \( K_n \) with \( |S| = k \) vertices is \( SE_p(K_n \oplus S) = (n-k-1) + \sqrt{4nk - 4k^2 + n^2 - 2n + 1} \).

**Proof:**
\[
S_p(K_n \oplus S) = \begin{bmatrix} J_{k \times k} & -J_{k \times (n-k)} \\ -J_{(n-k) \times k} & (I-J)_{(n-k) \times (n-k)} \end{bmatrix}_{n \times n}
\]
is the Seidel matrix of partial complement of \( K_n \oplus S \).

Let \( W = \begin{bmatrix} X \\ Y \end{bmatrix} \) be an eigenvector of order \( 2n \) partitioned conformally with \( S_p \).

Consider
\[
(S_p(K_n \oplus S) - \lambda I) \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (J - \lambda I)X - JY \\ -JX + [-J + (1 - \lambda)I]Y \end{bmatrix}
\]

Case 1: Let \( X = x_j - e_j, j = 2, 3, \ldots, k \) and \( Y = O_{n-k} \).

From equation (16), \( [J - \lambda I]X_j - JO_{n-k} = -\lambda X_j \).

Then, \( \lambda = 0 \) is an eigenvalue with multiplicity of at least \((k-1)\) since there are \((k-1)\) independent vectors of the form \( X_j \).

Case 2: Let \( X = O_{k-1} \) and \( Y = Y_j = e_1 - e_j, j = 2, 3, \ldots, n-k \).

From equation (16), \( [-J + (1 - \lambda)I]Y_j = (1 - \lambda)Y_j \),

So \( \lambda = 1 \) is an eigenvalue with multiplicity of at least \((n-k-1)\) since there are \(n-k-1\) independent vectors of the form \( Y_j \).

Case 3: Let \( Y = 1_{n-k} \) and \( X = \begin{bmatrix} \frac{n-k}{k-\lambda} \end{bmatrix} 1_k \), where \( \lambda \) is any root of the equation
\[
\lambda^2 + \lambda(n - 2k + 1) + 2k^2 - 2nk + k = 0.
\]

From equation (16),
\[
-J \begin{bmatrix} \frac{n-k}{k-\lambda} \end{bmatrix} 1_k + [-J + (1 - \lambda)I]1_{n-k}
\]
\[
= (1 - \lambda)1_k + \begin{bmatrix} k + \lambda - 1 \\ \lambda + 1 \end{bmatrix} \begin{bmatrix} n-k \\ 1 \end{bmatrix}
\]
\[
= -k1_{n-k} \begin{bmatrix} \frac{n-k}{k-\lambda} + (-n + k - 1) \lambda \end{bmatrix} \begin{bmatrix} 1_k \\ 1 \end{bmatrix}
\]
\[
= \lambda^2 + \lambda(n - 2k + 1) + 2k^2 - 2nk + k \begin{bmatrix} 1_k \\ 1 \end{bmatrix}
\]

Thus, \( \lambda = \frac{2k + 1 - n}{2} + \sqrt{\frac{(2k + 1 - n)^2 + 4(2k^2 - 2nk + k)}{2}} \)

and
\[
\lambda = \frac{2k + 1 - n}{2} - \sqrt{\frac{(2k + 1 - n)^2 + 4(2k^2 - 2nk + k)}{2}} \]

are the eigenvalues with multiplicity of at least one.

Thus partial complement Seidel spectrum of complete graph is
\[
\begin{bmatrix} 0 & \frac{k-1}{n-k-1} \\ \frac{2k + 1 - n}{2} & \frac{(2k + 1 - n)^2 + 4(2k^2 - 2nk + k)}{2} \end{bmatrix}
\]

So,
\[
SE_p(K_n \oplus S) = (n-k-1) + \sqrt{4nk - 4k^2 + n^2 - 2n + 1}.
\]

**Theorem 17.** Let \( K_{n-1} \oplus S \) be the partial complement of star graph with \(|S| = k\) vertices including central vertex. Then \( SE_p(K_{n-1} \oplus S) = (k + n - 3) + \sqrt{(n - 2k + 1)^2 + 4(k + 2k - 2k^2 - 2n + 2)} \).
Proof: $S_p(K_{1,n-1} \oplus S)$ is 
\[
\begin{bmatrix}
1 & J_{(1 \times (n-k))} & -J_{1 \times (n-k)} \\
J_{(n \times (n-k))} & (2I - J)_{n-1} & J_{(n \times (n-k))} \\
-J_{(n \times (n-k))} & J_{(n \times (n-k))} & (J-I)_{n-1}
\end{bmatrix}
\]
where $| \mathbb{R}^{n \times n} |$ is Seidel matrix of partial complements of $K_{1,n-1} \oplus S$. 
On $| | S_p(K_{1,n-1} \oplus S) - \lambda I | |$, applying row operation $R_i' \rightarrow R_i - R_{i+1}$, for $i = 1, 2, \ldots, p+1$, $p = 1, \ldots, k-1, k-p+1, \ldots, M-1, M+1, \ldots, l-p-1$ for the above determinant, we get 
\[
(\lambda - 2)^{k-2}(\lambda + 1)^{n-k-1} \det(A)
\]
de $A$ is of order 3.

Hence $| | S_p(K_{1,n-1} \oplus S) - \lambda I | | = (n-k+2)(n-k+1)n-k+1$.

Therefore Seidel spectrum of $K_{1,n-1} \oplus S$ is 
\[
\begin{pmatrix}
2 & -1 & P & Q \\
1 & -1 & P & Q \\
1 & 0 & -1 & P
\end{pmatrix}
\]
where $P = \frac{n-2k+1}{2}$ and $Q = \frac{\sqrt{(n-2k+1)^2 + 4(k+2kn-2k^2-2n+2)}}{2}$.

Hence $SE_p(K_{1,n-1} \oplus S) = (n-k+2)(n-k+1)n-k+1$.

Theorem 18. Let $K_{1,m} \oplus S$ be partial complement of complete bipartite graph with parts $V_1$ and $V_2$ of $l$ and $m$ vertices respectively and $S$ be an induced subset of $V$ which consists of $p$ vertices of $V_1$ and $k-p$ vertices of $V_2$. Then Seidel energy of $K_{1,m} \oplus S$ is 
\[
SE_p(K_{1,m} \oplus S) = \sqrt{(n-k+2)(n-k+1)n-k+1} + \sqrt{n^2 - 4k^2 + 4kn - 6n + 9}.
\]

Proof: The Seidel characteristic polynomial of $K_{1,m} \oplus S$ is given by $| S_p(K_{1,m} \oplus S) - \lambda I | = \begin{vmatrix} P & Q \\ Q & R \end{vmatrix}_{n \times n}$, where 
\[
P = \begin{vmatrix} (2 - \lambda)I - J \\ J_{p \times p} \end{vmatrix}_p,
\]
\[
Q = \begin{vmatrix} -J_{M \times p} \\ J_{p \times p} \end{vmatrix}_{p \times p},
\]
and 
\[
R = \begin{vmatrix} (J - (\lambda + 1)I) \\ -J_{p \times p} \end{vmatrix}_{M \times p}.
\]

On expansion of $\det(C)$, it reduces to order 4. On further simplification, we get polynomial $(\lambda^2 - 2k^2 + 2kn - 2k^2 - 2n + 2)$.

Hence Seidel characteristic polynomial of $K_{1,m} \oplus S$ is 
\[
(\lambda - 2)^{k-1}(\lambda + 1)^{n-k-1}[\lambda^2 + (2k^2 + 2n - 2k^2 - 2n + 2) - \lambda^2].
\]

Also Seidel spectrum of $K_{1,m} \oplus S$ is 
\[
\begin{pmatrix}
2 & -1 & P & Q & P & Q \\
1 & -1 & P & Q \\
1 & 0 & -1 & P
\end{pmatrix}
\]
where $P = \frac{n-2k+1}{2}$ and 
\[
Q = \frac{\sqrt{(n-k+2)(n-k+1)n-k+1} + \sqrt{n^2 - 4k^2 + 4kn - 6n + 9}}{2}.
\]

Therefore Seidel energy of $K_{1,m} \oplus S$ is 
\[
SE_p(K_{1,m} \oplus S) = (n-k+2)(n-k+1)n-k+1 + \sqrt{n^2 - 4k^2 + 4kn - 6n + 9}.
\]

Theorem 19. Let $S_n \oplus S$ be the partial complement of a crown graph with $|S| = k$.

(i) $SE_p(S_n \oplus S) = 5(n+1) + \sqrt{8n^2 - 28n + 25}$ for $k = n$.

(ii) $SE_p(S_n \oplus S) = 2(3n-4)$ for $k = 2n$.

Proof: (i) Let $S_p = \begin{vmatrix} (2I - J) \end{vmatrix}_n \begin{vmatrix} (2I - J) \end{vmatrix}_n \begin{vmatrix} (J - I) \end{vmatrix}_n$ be the Seidel matrix of $S_n \oplus S$.

Let $W = \begin{vmatrix} X \\ Y \end{vmatrix}$ be an eigenvector of order 2n partitioned conformally with $S_p$.

Consider 
\[
(S_p - \lambda I) \begin{vmatrix} X \\ Y \end{vmatrix} = \begin{vmatrix} (2 - \lambda)I - J \\ (2I - J) \end{vmatrix}_n \begin{vmatrix} (2I - J) \end{vmatrix}_n + (\lambda - 1) \begin{vmatrix} I \end{vmatrix}_n
\]
\[
(2I - J)X + (J - (\lambda + 1)I)Y
\]
\[
= \begin{vmatrix} 2 - \lambda \end{vmatrix}_n \begin{vmatrix} \lambda - 2 \end{vmatrix}_n
\]
\[
X
\]
\[
Y
\]
\[
X
\]
\[
Y
\]

Hence $\lambda = 3$ and $\lambda = -2$ are the eigenvalues with multiplicity of at least $n-1$, as there are $n-1$ eigenvectors of the form $X_j$.

Case 2: Let $Y = 1_n$ and $X = \begin{vmatrix} -\lambda - n + 1 \\ -n \end{vmatrix}_n$, where $\lambda$ is any root of the equation 
\[
\lambda^2 - 2n^2 + 7n - 6 = 0.
\]

From equation (20), 
\[
(2I - J)X + (J - (\lambda + 1)I) \begin{vmatrix} \lambda - 2 \end{vmatrix}_n
\]
\[
X
\]
\[
Y
\]
\[
X
\]
\[
Y
\]
\[
X
\]
\[
Y
\]
\[
X
\]
\[
Y
\]

Hence $\lambda = 3$ and $\lambda = -2$ are the eigenvalues with multiplicity of at least one.

Thus Seidel spectrum of partial complement of crown graph with $|S| = n$ is
Case 1: Let $X = X_j, j = 2, 3, \ldots, n$ and $Y = -\frac{\lambda - 2}{2} X_j$, where $\lambda$ is any root of the equation $\lambda^2 - 4\lambda = 0$.

From equation (21),
\[
(\lambda - 2)X_j + [(\lambda - 2) - \lambda - \frac{2}{2}] X_j = \left[-2 + \frac{(-\lambda - 2)^2}{2}\right] X_j.
\]
Hence $\lambda = 0$ and $\lambda = 4$ are the eigenvalues each with multiplicity of at least $n - 1$, as there are $n - 1$ eigenvectors of the form $X_j$.

Case 2: Let $X = 1_n$ and $Y = \frac{\lambda + n - 1}{n - 2} 1_n$, where $\lambda$ is any root of the equation $\lambda^2 + (2n - 4)\lambda = 0$.

From equation (21),
\[
(\lambda - 2)1_n + [(\lambda - 2) - \lambda - \frac{2}{2}] 1_n = \frac{(4 - 2n)\lambda - \lambda^2}{n - 2} 1_n.
\]
Thus $\lambda = 0$ and $\lambda = 4 - 2n$ are the eigenvalues with multiplicity of at least one. Thus Seidel spectrum of partial complement of crown graph with $|S| = 2n$ is \(\begin{pmatrix} 0 & 4 & 4 - 2n \\ 4 - 2n & n - 1 & 1 \\ 4 - 2n & 1 & n - 1 \end{pmatrix}\) and its Seidel energy is $SE_p(S_n \oplus S) = 6n - 8$.

**Theorem 24.** Let $S(l,m)$ be double star graph of order $l + m + 2$ with $\langle S \rangle = K_{1,1}$. Then $S_p(S(l,m) \oplus S) = (\lambda + 1)^{m-1}(\lambda - 2)^{l-1}[\lambda^4 + (l - m - 2)\lambda^2 + (m - 2l - 2lm - 2)\lambda^2 + (2m - l - 2lm + 3)\lambda + 2l + 2]$.

**Proof:** The Seidel characteristic polynomial of $S(l,m) \oplus S$ is given by
\[
|S_p(S(l,m) \oplus S) - \lambda I| =
\begin{vmatrix}
(2l - J) & J_{j,1} & J_{j,1} & \cdots & J_{j,m} \\
J_{j,1} & (2l - J) & J_{j,1} & \cdots & J_{j,m} \\
J_{j,1} & J_{j,1} & (2l - J) & \cdots & J_{j,m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
J_{j,1} & J_{j,1} & J_{j,1} & \cdots & (2l - J)
\end{vmatrix}_{1 \leq j \leq m + 2}
\]

Step 1: Applying row operation $R_i' \rightarrow R_i - R_{i+1}$, for $i = 1, 2, \ldots, l + m + 1$ for the above determinant, we get $(\lambda - 2)^{l-1}(\lambda + 1)^{m-1} \det(B)$.

Step 2: In $\det(B)$, performing column operations $C_j' \rightarrow C_i + C_{i-1} + \cdots + C_1, i = l, l - 1, \ldots, 2, C_j' \rightarrow C_j + C_{j-1} + \cdots + C_{l+3}, j = l + m + 2, l + m + 1, \ldots, l + 4$, we get $\det(C)$.

On expansion of $\det(C)$, it reduces to order 4. On further simplification, we get polynomial $\lambda^4 + (l - m - 2)\lambda^3 + (m - 2l - 2lm - 2)\lambda^2 + (2m - l + 2lm + 3)\lambda + 2l + 2$.

Hence Seidel characteristic polynomial of $S(l,m) \oplus S$ is $(\lambda + 1)^{m-1}(\lambda - 2)^{l-1}[\lambda^4 + (l - m - 2)\lambda^3 + (m - 2l - 2lm - 2)\lambda^2 + (2m - l + 2lm + 3)\lambda + 2l + 2]$. 

\[\]
Seidel energy of partial complement of amalgamation of $m$ copies of $K_n$

A graph amalgamation is a relationship between two graphs (one graph is an amalgamation of another). Amalgamations can be used to reduce a graph to a simpler graph while retaining some structure.

Definition 25. Let $(G_1, G_2, G_3, \ldots, G_m)$ be a finite collection of graphs and each $G_i$ has a fixed vertex $v_0_i$, called a terminal. The amalgamation $Amal(v_0_i, G_i)$ is formed by taking all the $G_i$’s and identifying their terminals. In particular, if we take $G_i = K_n$ for $i = 1, 2, \ldots, m$ we get amalgamation of $m$ copies of $K_n$, denoted by $Amal(m, K_n)$, $m \geq 2$. For convenience we denote $v_0$ as the vertex of amalgamation and $v_{j2}, v_{j3}, \ldots, v_{jn}$ are the remaining vertices of the $j$th copy of $K_n$, where $1 \leq j \leq m$.

Theorem 26. Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ and $S = \{v_0\}$. Then, $SE_p(\text{Amal}(m, K_n) \oplus S) = 3mn - 2n - 5m + 3 + \sqrt{(mn - 2n - m + 4)^2 + 4(2n - 3)}$.

Proof: Let $S_p = \begin{pmatrix} J_1 & -J_{n-1 \times 1} & -J_{n-1 \times 2} & \cdots & -J_{n-1 \times (n-1)} \\ -J_{n-1 \times 1} & J_{n-1} & -J_{n-1 \times 2} & \cdots & -J_{n-1 \times (n-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -J_{n-1 \times 1} & -J_{n-1 \times 2} & \cdots & J_{n-1} \end{pmatrix}_{m \times (m(n-1)+1)}$

be the Seidel matrix of $Amal(m, K_n) \oplus S$. Here $J$ is matrix of all 1’s and $B$ is the adjacency matrix of complete subgraph.

Step 1: Consider $|\lambda I - S_p|$.

Applying row operation $R_{i \rightarrow j} \rightarrow R_{i \rightarrow j+l, i} = 1, 2, \ldots, m, j = 2, 3, \ldots, n - 1$ and column operation $C \rightarrow C + C_{i \rightarrow j+l, i}$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n - 1$. Then, on $|\lambda I - S_p|$, we get $|\lambda I - (\lambda^m - 3n + 4)\lambda - 2n + 3|$

Hence Seidel spectrum of partial complement of $Amal(m, K_n)$ is

$$|\lambda I - \begin{pmatrix} 1 & 3 - 2n & P + Q \\ m(n-1) & 1 & P - Q \end{pmatrix}|$$

where $P = mn + 4 - 2n - m$ and $Q = \sqrt{(mn + 4 - 2n - m)^2 - 4(3 - 2n)}$.

So $SE_p(\text{Amal}(m, K_n) \oplus S) = mn - 2n - 5m + 3 + \sqrt{(mn - 2n - m + 4)^2 + 4(2n - 3)}$.

Theorem 27. Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ with $S = \{v_0| i = 1, 2, \ldots, m, j = 2, 3, \ldots, n\}$. Then $SE_p(\text{Amal}(m, K_n) \oplus S) = 3mn - 4mn - 2n + 2 + \sqrt{(mn - 2n - m + 4)(2n - 3)}$.

Proof: Let $S_p = \begin{pmatrix} 0_1 & -J_{n-1 \times 1} & -J_{n-1 \times 2} & \cdots & -J_{n-1 \times (n-1)} \\ -J_{n-1 \times 1} & J_{n-1} & -J_{n-1 \times 2} & \cdots & -J_{n-1 \times (n-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -J_{n-1 \times 1} & -J_{n-1 \times 2} & \cdots & J_{n-1} \end{pmatrix}_{(m(n-1)+1)}$

be the Seidel matrix of partial complement of $Amal(m, K_n)$. Repeating Step 1 to Step 3 of Theorem 26, we get the polynomial

$\lambda^m - 2n + 2 + 3 - 2n + 1 \lambda + mn - m + m$.

Hence the Seidel spectrum of partial complement of $Amal(m, K_n)$ is

$$\begin{pmatrix} 0 & 2n - 2 & P + Q \\ (m(n-1) - 1) & 1 & P - Q \end{pmatrix}$$

where $P = mn - 2n + m$ and $Q = \sqrt{(mn - 2n + m)^2 + 4m(n-1)}$.

Therefore, $SE_p(\text{Amal}(m, K_n) \oplus S) = 3mn - 4m - 2n + 2 + \sqrt{(mn - 2n - m + 4)(2n - 3)}$.

SEIDEL ENERGY OF PARTIAL COMPLEMENT OF LADDER RUNG GRAPH

The ladder rung graph $L_{rn}$ is a regular graph of degree one on $2n$ vertices. Let the vertices of $L_{rn}$ be $v_1, v_2, \ldots, v_{2n}$ and the vertex $v_i$ is adjacent to $v_{i+1}$, $i = 1, 3, \ldots, 2n-1$. We obtain $SE_p(L_{rn} \oplus S)$, where $S = \{v_1, v_3, \ldots, v_{2n-1}\}$ in the following theorem.

Theorem 28. Let $L_{rn}$ be the Ladder rung graph with $S = \{v_1, v_3, \ldots, v_{2n-1}\}$. Then, $SE_p(L_{rn} \oplus S) = 5(n-1) + \sqrt{8n^2 - 28n + 25}$.

Proof: Let $S_p = \begin{pmatrix} (2I - J)_n & -(2I - J)_n \\ -(2I - J)_n & -(I - J)_n \end{pmatrix}$ be the Seidel matrix of $L_{rn} \oplus S$.

Let $W = \begin{pmatrix} X \end{pmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $S_p$.

Consider $(\lambda I - S_p)X = [(\lambda - 2)I + J]X + [(1 + \lambda)I - J]Y$.

Hence $X = \begin{pmatrix} 2 - (\lambda + 1)(\lambda - 2) \end{pmatrix}$, where $\lambda$ is any root of the equation $\lambda^2 - \lambda - 6 = 0$.

From equation (29),

$$(2I - J)X = [(1 + \lambda)I - J]X.$$

Hence $\lambda = -2$ and $\lambda = 3$ are the eigenvalues each with multiplicity of at least one, as there are $n - 1$ eigenvectors of the form $X_j$.

Case 2: Let $X = 1_n$ and $Y = \frac{\lambda + n - 2}{2} I_n$, where $\lambda$ is any root of the equation $\lambda^2 - \lambda - 2n^2 + 7n - 6 = 0$.

From equation (29),

$$(2I - J)1_n + [(1 + \lambda)I - J]1_n = 2 - (\lambda + 1 - n)(\lambda + n - 2)1_n, n - 2$$

Thus $\lambda = 1 + 8n^2 - 28n + 25$ and $\lambda = 1 - 8n^2 - 28n + 25$ are the eigenvalues with multiplicity of at least one.

Therefore Seidel spectrum of partial complement of ladder rung graph with respect $(S)$ is

$$\begin{pmatrix} -2 & 3 \\ \frac{1 + 8n^2 - 28n + 25}{2n - 2} & \frac{1 - 8n^2 - 28n + 25}{2n - 2} \end{pmatrix},$$

and its Seidel energy is

$$SE_p(L_{rn} \oplus S) = 5(n-1) + \sqrt{8n^2 - 28n + 25}.$$
IV. CONCLUSION

In this paper, we computed the Seidel energy and Seidel spectrum of partial complements of several graph classes. The Seidel energy of a partial complement of a graph is determined by the subgraph chosen from $G$. We also examined a few properties and established upper and lower bounds for $SE_P(G \oplus S)$.

REFERENCES


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