Sufficient Condition for Synchronization in Complete Networks of Reaction-Diffusion Equations of Hindmarsh-Rose Type with Linear Coupling

Phan Van Long Em

***Abstract*— This paper studies the identical synchronization in a complete network consisting of n nodes. Each node is represented by reaction-diffusion equations of Hindmarsh-Rose type which was simplified from the famous Hodgkin-Huxley model. They are connected by linear coupling. From this complete network, a sufficient condition on the coupling strength is identified to get the synchronization. The result shows that the complete networks synchronize more easily if they have more nodes. The paper also shows this theoretical result numerically and sees that there is a compromise.**

***Index Terms*— complete network, linear coupling, reaction-diffusion equations of Hindmarsh-Rose, synchronization**

I. INTRODUCTION

Synchronization is a ubiquitous feature and studied in many natural systems and nonlinear science. The word "*synchronization*" is of Greek origin, with *syn* as "common" and *chronous* as "time", which means having the same behavior at the same time [1]. Therefore, the synchronization of two dynamical systems usually means that one system copies the movement of the other. When the behaviors of many systems are synchronized, these systems are called *synchronous*. Studies by Aziz-Alaoui [1] and Corson [2] suggested that a phenomenon of synchronization may appear in a network of many weakly coupled oscillators. There are a lot of different applications that have emerged to increase the power of lasers, control oscillations in chemical reactions, encode electronic messages for secure communications, or synchronize the output of electric circuits [1,3].

Synchronization has been extensively studied in many fields and many natural phenomena reflect the synchronization such as the movement of birds forming the cloud, the movement of fishes in the lake, the movement of the parade, the reception and transmission of a group of cells [1,4-8]. Therefore, the study of synchronization is necessary. Specifically, the network of cells is considered in this work.

In the human brain, there are many cells, they connect to form a network. A cellular network is a system of cells that are physiologically linked together. The exchange between them is mainly based on electrochemical processes. This paper studies the sufficient condition of coupling strength to obtain synchronization in a complete network of cells. In which, each cell is described by a system of reaction-diffusion equations of Hindmarsh-Rose type. To make the study easier, a complete network of n neurons interconnected together with linear coupling is investigated.

In 1952, Hodgkin and Huxley presented a four-dimensional mathematical model that could approximate the energizing properties of cell voltage [2,4,7]. Based on this model, many simpler models have been published to describe the cell voltage dynamics. In 1982, Hindmarsh J. L. and Rose R. M. published a new model named Hindmarsh-Rose model (HR) [9] known as a simplified two-dimensional model from Hodgkin-Huxley's famous system of equations [6]. Although the model is simpler, it has many remarkable analytical results and retains the properties and biological significance. It represents the equilibrium, activity, and bursting of the cell voltage. The model has constituted a common form of two equations in the two variables u and v. The first variable is the fast one called excitatory representing the transmembrane voltage. The second one is the slow recovery variable describing several physical quantities, such as the electrical conductivity of ion currents across the membrane. The Hindmarsh-Rose equations (HR) are given by:

$$\begin{cases} \dfrac{du}{dt} = v - u^3 + au^2 + I \\ \dfrac{dv}{dt} = 1 - bu^2 - v \end{cases} \quad (1)$$

where the parameters $a = 3,\ b = 5$ are constants determined by practical experience, I presents the external current.

However, this model is not strong enough to describe the propagation of action potential. To solve this problem, the cable equation is investigated. This mathematical equation is derived from a circuit model of the membrane and its intracellular and extracellular space to provide a quantitative description of current flow and voltage change

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Phan Van Long Em is a lecturer of An Giang University, Vietnam National University, Ho Chi Minh City, VIETNAM; (e-mail: pvlem@agu.edu.vn).

both within and between neurons, allowing a quantitative and qualitative understanding of how neurons function. Hence, the reaction-diffusion equations of Hindmarsh-Rose type (HR) are considered as follows:

$$\begin{cases} \dfrac{du}{dt} = u_t = v - u^3 + au^2 + I + d_u \Delta u \\ \dfrac{dv}{dt} = v_t = 1 - bu^2 - v \end{cases} \tag{2}$$

where $u = u(x,t),\ v = v(x,t),\ (x,t) \in \Omega \times \mathbb{R}^+,\ d_u$ is a positive constant, Δu is the Laplace operator of u, $\Omega \subset \mathbb{R}^N$ is a regular bounded open set and with Neumann zero flux boundary conditions. This system allows the emergence of a variety of patterns and relevant phenomena in physiology. This system consists of two nonlinear partial differential equations of the incomplete parabolic type. The first equation describes the action potential and the second one presents the recovery variable in the whole set of neurons. Besides, the first equation is similar to the so-called cable equation. It describes the distribution of the potential along the axon of a single neuron [6,7].

System (2) is considered as a neural model and from this, a network of *n* coupled systems (2) based on HR type is constructed as follows:

$$\begin{cases} u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i - h(u_i, u_j) \\ v_{it} = 1 - bu_i^2 - v_i \\ i, j = 1, ..., n, i \neq j, \end{cases} \tag{3}$$

where $(u_i, v_i), i = 1, 2, ..., n$ is defined by (2).

Function *h* is the coupling function that determines the type of connection between neurons u_i and u_j. Connections between neurons are essential of two types: chemical connection and electrical connection, where a chemical connection is more abundant than an electrical one. For easy research, this paper only focuses on electrical connection, then the coupling function is linear [2,10,11] and is given by the following formula:

$$h(u_i, u_j) = g_{syn} \sum_{j=1, j\neq i}^{n} c_{ij}(u_i - u_j), \quad i = 1, 2, ..., n.$$

The parameter g_{syn} represents the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n\times n}$, defined by: $c_{ij} = 1$ if u_i and u_j are coupled, $c_{ij} = 0$ if u_i and u_j are not coupled, where $i, j = 1, 2, ..., n, i \neq j.$

In recent years, there have been many research papers on the resonance of the network of cells, but most of them only study cells stimulated by the system of equations of FitzHugh-Nagumo type [10,11] or the system of ordinary differential equations of Hindmarsh-Rose type [2], there is no research related to the system of reaction-diffusion equations of Hindmarsh-Rose type on a complete network of cells. From there, it shows that the research on this issue is meaningful and brings a practical application value to the currently applied mathematics.

II. SYNCHRONIZATION OF A COMPLETE NETWORK

In this paper, the synchronization is investigated in a complete network, i.e, each node connects to all other nodes of the network [10,11]. For example, Figure 1 shows the complete graphs from 3 to 10 nodes. Each node represents a neuron modeled by a system of reaction-diffusion equations of Hindmarsh-Rose type and each edge represents a synaptic connection modeled by a linear coupling function. A network of *n* neurons (2) bi-directionally coupled by the electrical synapses, based on HR, is given as follows:

$$\begin{cases} u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i - g_{syn} \sum_{j=1, j\neq i}^{n} (u_i - u_j) \\ v_{it} = 1 - bu_i^2 - v_i \\ i = 1, ..., n, \end{cases} \tag{4}$$

where g_{syn} is the coupling strength between u_i and u_j.

Definition 1 (Ambrosio and Aziz-Alaoui, 2012**).** *Let* $S_i = (u_i, v_i),\ i = 1, 2, ..., n$ *and* $S = (S_1, S_2, ..., S_n)$ *be a network. We say that* S *is identically synchronous if*

$$\lim_{t\to+\infty} \sum_{i=1}^{n-1} \left(\|u_i - u_{i+1}\|_{L^2(\Omega)} + \|v_i - v_{i+1}\|_{L^2(\Omega)} \right) = 0.$$

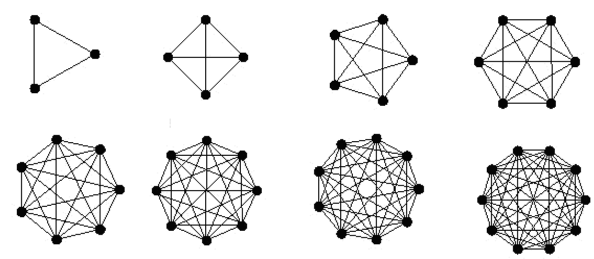

Fig. 1. Complete graphs from 3 to 10 nodes

The system (4) can be rewritten as follows:

$$\begin{cases} u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i - g_{syn} \sum_{j=1, j\neq i}^{n} (u_i - u_j) \\ v_{it} = 1 - bu_i^2 - v_i \\ u_{1t} = v_1 - u_1^3 + au_1^2 + I + d\Delta u_1 - g_{syn} \sum_{j=2}^{n} (u_1 - u_j) \\ v_{1t} = 1 - bu_1^2 - v_1 \\ i = 2, ..., n \end{cases} \tag{5}$$

Let $X = u_i - u_1, Y = v_i - v_1$ and $U = u_i + u_1, i = 2, ..., n.$ We have then the system corresponding to the variables X, Y:

$$\begin{cases} \frac{dX}{dt} = Y - \frac{1}{4}X^3 + X(aU - \frac{3}{4}U^2 - ng_{syn}) + \Delta X \\ \frac{dY}{dt} = -bXU - Y \end{cases} \quad (6)$$

Theorem 1. *If the coupling strength* g_{syn} *verifies the condition:*

$$g_{syn} \geq \max\left\{\frac{a^2}{3n}, \frac{1}{4n\gamma} + \frac{(b-2a)^2}{4n(3-\gamma b^2)}\right\}, \text{ with } 0 < \gamma < \frac{3}{b^2},$$

for all initial conditions $u_i(0), v_i(0), i = 1,2,...,n,$ *the system (5) will synchronize.*

Proof. Let's choose the Lyapunov function as follows:

$$E(X,Y) = \int_\Omega \left(\frac{1}{2}X^2 + \frac{\gamma}{2}Y^2\right)dx,$$

where γ is a positive constant. By taking derivative this Lyapunov function according to t, we have:

$$\frac{dE(X,Y)}{dt} = \int_\Omega \left(-\frac{X^4}{4} - \left(AX^2 - BXY + \gamma Y^2\right)\right)dx,$$

where $A = \frac{3}{4}U^2 - aU + ng_{syn}, B = \gamma bU - 1.$

It can be seen that $AX^2 - BXY + \gamma Y^2 > 0$ if the following two conditions are verified :

(i) Since $A = \frac{3}{4}U^2 - aU + ng_{syn}$, the solutions of the equation $A = 0$ are $U_{1,2} = \frac{2\left(a \pm \sqrt{a^2 - 3ng_{syn}}\right)}{3}$ if $g_{syn} \leq \frac{a^2}{3n}$. Therefore, $A > 0$ if $g_{syn} > \frac{a^2}{3n}$;

(ii) $\gamma A - \frac{B^2}{4} > 0 \Leftrightarrow (3-\gamma b^2)U^2 - 2(a-2b)U + 4ng_{syn} - \frac{1}{\gamma} > 0.$

This condition is satisfied if $g_{syn} > \frac{1}{4n\gamma} + \frac{(b-2a)^2}{4n(3-\gamma b^2)}$ and $\gamma < \frac{3}{b^2}$.

Then, if the coupling strength g_{syn} verifies the condition:

$$g_{syn} \geq \max\left\{\frac{a^2}{3n}, \frac{1}{4n\gamma} + \frac{(b-2a)^2}{4n(3-\gamma b^2)}\right\}, \text{ with } 0 < \gamma < \frac{3}{b^2},$$

we have $AX^2 - BXY + \gamma Y^2 > 0.$

It leads to $\frac{dE(X,Y)}{dt} < 0,$ for all $X, Y.$ It implies that the origin is globally asymptotically stable for $E(X,Y)$ (see [12]). Hence, the neurons of the network (5) is globally asymptotically synchronized. The theorem has been proven.

III. NUMERICAL RESULTS AND DISCUSSION

This research focuses on the minimal values of coupling strength g_{syn} to observe a phenomenon of synchronization between n subsystems modeling the function of neuron networks.

In the following, the paper shows the numerical results obtained by integrating the system (4) with

$$n = 3, a = 3, b = 5, I = 0, d = 1, i = 1,2,3,$$
$$[0;T] \times \Omega = [0;200] \times [0;100] \times [0;100].$$

The integration of the system is realized by using C++ and the results are represented by Gnuplot,

Fig. 2 illustrates the synchronization of the complete network of 3 systems of reaction-diffusion equations of Hindmarsh-Rose type. The simulations show that the system synchronizes from the value $g_{syn} = 0.2$.

Fig. 2(a), 2(b), 2(f), 2(g), 2(k), 2(l), 2(p), 2(q) represent the synchronization errors of the coupled solutions $\left(u_1(x_1,x_2,t), u_2(x_1,x_2,t)\right)$ and $\left(u_2(x_1,x_2,t), u_3(x_1,x_2,t)\right)$, where $t \in [0;T]$ and for all $(x_1,x_2) \in \Omega$.

In Fig. 2(p) and 2(q) with $g_{syn} = 0.2$, the simulation shows that the synchronization errors reach to zero, it means:

$u_1(x_1,x_2,t) \approx u_2(x_1,x_2,t)$ an $u_2(x_1,x_2,t) \approx u_3(x_1,x_2,t)$ for all $(x_1,x_2) \in \Omega$.

Fig. 2(c), 2(d), 2(e), 2(h), 2(i), 2(j), 2(m), 2(n), 2(o), 2(r), 2(s), 2(t) represent the solutions $u_i(x_1,x_2,190), i = 1,2,3,$ of the network from when no synchronization has occurred until they have the same shape, i.e, the synchronization is performed.

Before synchronization with $g_{syn} = 0.05$, Fig. 2(a) represents the synchronization error between u_2 and u_1, for all $(x_1,x_2) \in \Omega$; Fig. 2(b) represents the synchronization error between u_3 and u_2; Fig. 2(c) represents a solution $u_1(x_1,x_2,190)$; similarly, Fig. 2(d) and 2(e) represent the solutions $u_2(x_1,x_2,190)$ and $u_3(x_1,x_2,190)$ when they are coupled together; the results are similarly done for $g_{syn} = 0.1$ (Fig. 2(f), 2(g), 2(h), 2(i), 2(j)), $g_{syn} = 0.15$ (Fig. 2(k), 2(l), 2(m), 2(n), 2(o)) and $g_{syn} = 0.2$ (Fig. 2(p), 2(q), 2(r), 2(s), 2(t)). For $g_{syn} = 0.2$, the synchronization occurs.

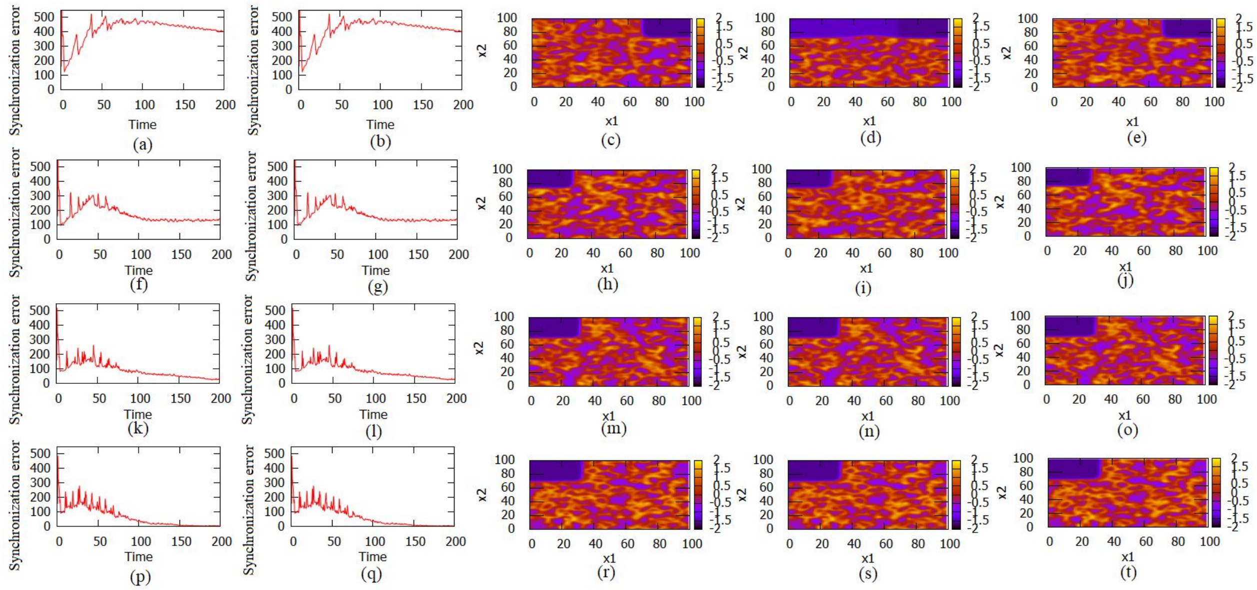

Fig. 2. Synchronization in the complete network of 3 connected cells electrically

From the above result, in the case of three linearly coupled neurons, the coupling strength over or equal to $g_{syn} = 0,2,$ these neurons has synchronous behaviors. By doing similarly for the complete networks of linearly identical coupled neurons, the values of coupling strength according to the number of neurons *n* are reported in Table 1. In Table 1, for each value of *n*, we seek one necessary value of coupling strength to get the synchronization in complete network corresponding to *n* from 3 to 20.

TABLE I
MINIMAL COUPLING STRENGTH NECESSARY TO OBSERVE THE SYNCHRONIZATION

n		3	4	5	6
g_{syn}		0.2	0.1565	0.136	0.124
n	7	8	9	10	11
g_{syn}	0.1133	0.107	0.1	0.099	0.095
n		12	13	14	15
g_{syn}		0.0935	0.0915	0.09	0.0885
n	16	17	18	19	20
g_{syn}	0.0873	0.0855	0.0853	0.0844	0.084

Following these numerical experiments, it is easy to see that the coupling strength required to observe the synchronization of *n* neurons depends on the number of neurons. Indeed, the points in Fig. 3 represent the coupling strength of synchronization according to the number of neurons in complete network from Table 1, and we find a function presenting the relation between the number of neurons *n* and the coupling strength reported in Table 1. This function is as follows:

$$g_{syn} = \frac{0.26}{n-1} + 0.07, \qquad (7)$$

In Fig. 3, the function (7) is represented by a curve where the points corresponding to the coupling strengths are almost on. It means that the coupling strength necessary to obtain the synchronization in complete network follows the law presented by (7). These simulations show that the bigger the number of neurons is, the smaller the coupling strength is. It means that synchronization is easier when the number of neurons in complete networks is bigger.

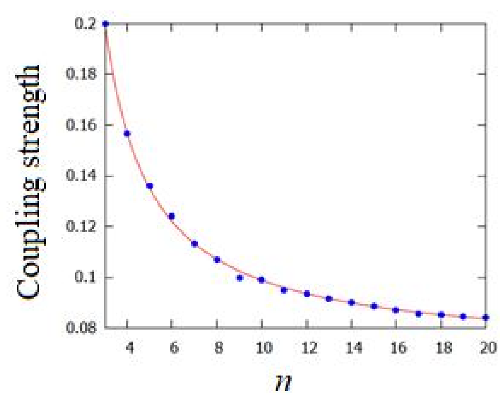

Fig. 3. The evolution of the coupling strength with respect to the number of neurons

IV. CONCLUSION

This study gave a sufficient condition on the coupling strength to achieve the synchronization in the complete network of *n* linearly coupled systems of reaction-diffusion equations of Hindmarsh-Rose type. Theorem 1 shows that the bigger the value of *n* is, the smaller g_n is. Numerically, it displays that the synchronization is stable when the coupling strength exceeded a certain threshold and depends on the number of neurons in graphs. The bigger the number of neurons is, the easier the phenomenon of synchronization will be obtained. Then, a compromise between the theoretical and numerical results can be reached. In addition, it is necessary to conduct further studies on the different synchronization regimes in free networks coupled with chemical synapses.

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