# Qualitative and Bifurcation Analysis of a Single Species Logistic Model with Allee Effect and Feedback Control

Kan Fang    Jiangbin Chen*    Zhengliang Zhu    Fengde Chen    Xiaoying Chen

*Abstract*—**We consider the dynamical behaviors of the single species logistic model considering Allee effect together with feedback control in this paper. By using the method of formal power series of Poincaré, we put forward sufficient conditions for the positive equilibrium to be a stable (unstable) fine-focus with order at 1 or a unstable fine-focus with order at 2. Qualitative analysis of the model indicates the ultimate boundedness of the positive solutions to the system, and when the positive equilibrium is unstable, we obtain sufficient conditions that sustain a unique stable limit cycle. Finally, numerical simulations are conducted to validate the major conclusions.**

*Index Terms*—**Single species Logistic model, Allee Effect, Feedback control, Bifurcation, Limit cycle.**

## I. Introduction

As widely acknowledged, the single species Logistic model (1) admits a unique positive equilibrium $x = 1$ which is globally stable.

$$\dot{x} = rx(1-x). \tag{1}$$

Gopalsamy and Weng[1] added feedback control to system (1) and studied the dynamic behaviors of system (2).

$$\begin{aligned} \dot{x} &= rx(1-x) - axu, \\ \dot{u} &= -bu + cx. \end{aligned} \tag{2}$$

They showed there existed a unique positive equilibrium $(\frac{br}{ac+br}, \frac{br}{ac+br})$ which was also globally stable. Many studies regarding systems with feedback control have been performed recently[2–9]. In the natural world, cooperation, in its broad sense, can be as important as competition. Based on experiments, Allee [10] defined Allee effect after researching positive and negative correlations among species, which is an ecological concept that could be traced back at least to the 1920s. A population where individuals have lower fitness when the population is small or sparse can be said to "have an Allee effect". This effect can be caused by difficulties in mating [11], inbreeding depression [12], defense to avoid predators, social dysfunction at a low-densities [13]. In the past decades, Allee effect had been used by mathematicians and ecologists in the explanation of some important biological phenomena [14–23].

Lin [24] added Allee effect to system (2) and got the following system (3)

$$\begin{aligned} \dot{x} &= rx(1-x)\frac{x}{x+A} - axy, \\ \dot{y} &= -by + cx. \end{aligned} \tag{3}$$

It was shown in the paper that there existed a unique positive equilibrium $E(\frac{br-acA}{ac+br}, \frac{c(br-acA)}{b(ac+br)})$ which was locally asymptotically stable under the conditions that (1) $A < \frac{br^2}{ac(ac+2br)}$, or (2) $\frac{br^2}{ac(ac+2br)} < A < \frac{br}{ac}$ and $b > r$. However, global stability, bifurcation and limit cycles have not been studied in [24] yet.

In this paper, we continue to prove the ultimate boundedness and the sufficient conditions for system (3) to be globally asymptotically stable in section 1. The Hopf bifurcation and the order of the fine-focus are examined in section 2. We put forward the sufficient conditions for system to be locally unstable as well as existence and uniqueness of limit cycle in subsequent section 3. Numerical examples and simulations are presented in Section 4 to illustrate our results. Finally, discussions and conclusions are presented in Section 5.

## II. Global Stability

In this section, our major objective is to confirm the global stability of $E^*(x^*, y^*) = (\frac{br-acA}{ac+br}, \frac{c(br-acA)}{b(ac+br)})$. To discuss the global stability, we first prove the ultimate boundedness of system (3) using the following Lemma.

**Lemma 2.1.**[25] Suppose $c$, $d > 0$, and $\frac{dx}{dt} \le x(t)(c - dx(t))$ with $x(t) > 0$, then $\limsup_{t\to+\infty} x(t) \le \frac{c}{d}$ ; Suppose $c$, $d > 0$, and $\frac{dx}{dt} \ge x(t)(c - dx(t))$ with $x(t) > 0$, then $\liminf_{t\to+\infty} x(t) \ge \frac{c}{d}$.

**Proposition 2.1.**

(1) solutions $(x(t), y(t))$ to system (3) satisfying $t \ge 0$ are all positive .

(2) Supposing $x(0) > 0$, $y(0) > 0$, solutions $(x(t), y(t))$ to system (3) satisfying $t \ge 0$ are all ultimately bounded.

**Proof.** (1) It follows from the first equation of system (3) that $x(t) = 0$ is an invariant set. This shows that $x(t) > 0$ for all $t \ge 0$ if $x(0) > 0$. Therefore, we have $\dot{y} > 0$ when

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Kan Fang is a lecturer of Zhicheng College, FuZhou University, Fuzhou, Fujian 350002, China (e-mail: chrisfang517@sina.com).

Jiangbin Chen (corresponding author) is a Phd candidate of the Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China, and an associate professor of Zhicheng College, FuZhou University, Fuzhou, Fujian 350001, China(e-mail: jbchen2021@163.com).

Zhengliang Zhu is a Phd candidate of the College of Mathematics and Computer Science, FuZhou University, Fuzhou, Fujian 350116, China (e-mail: N170320020@fzu.edu.cn).

Fengde Chen is a professor of the College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350116, China (e-mail: fdchen@fzu.edu.cn).

Xiaoying Chen is an associate professor of Zhicheng College, FuZhou University, Fuzhou, Fujian 350001, China (e-mail: chrisfang517@sina.com).

we substitute $y(t) = 0$ into the second equation of system (3), which implies that $y(t) > 0$ for all $t \geq 0$ if $y(0) > 0$. Therefore, any trajectory starting at the first quadrant can not cross either x-axis or y-axis. This proves (1).

(2) We can easily obtain according to system (3) that

$$\dot{x} = \frac{rx^2(1-x)}{A+x} - axy \quad \leq \quad rx(1-x). \tag{4}$$

From Lemma 2.1, we have $\limsup_{t\to+\infty} x(t) \leq 1$. Therefore, there exists a positive constant $M^*$ such that $x(t) \leq M^*$ for all $t \geq 0$. Furthermore, we know

$$\dot{y} = -by + cx \quad \leq \quad cM^*. \tag{5}$$

To summarize, solutions to system (3) satisfying $x(0) > 0$, $y(0) > 0$ are all ultimately bounded.

**Theorem 2.1.** Positive equilibrium $E^*(x^*, y^*)$ of system (3) is globally asymptotically stable, when the following conditions is satisfied:

$$\frac{br^2}{ac(ac+2br)} < A < \frac{br}{ac}, \quad b > r. \tag{6}$$

**Proof.** From Lin [24], positive equilibrium $E^*(x^*, y^*)$ of system (3) is locally stable under the above conditions (6). Let us examine the Dulac function $u(x, y) = \frac{1}{x}$, and we have:

$$\begin{aligned} D &= \frac{\partial(u(x,y)f(x,y))}{\partial x} + \frac{\partial(u(x,y)g(x,y))}{\partial y} \\ &= \frac{r(-x^2-2Ax+A)}{(A+x)^2} - \frac{b}{x} \\ &< \frac{r(-x^2-2Ax+A)}{(A+x)^2} - \frac{r}{A+x}(b>r) \\ &= -\frac{r(x^2+2Ax+x)}{(A+x)^2} \\ &< 0. \end{aligned} \tag{7}$$

Therefore, by Dulac-Bendixson theorem [26], closed trajectory won't appear in the first quadrant. Meanwhile, solutions to system (3) satisfying $x(0) > 0$, $y(0) > 0$ are all ultimately bounded from Proposition 2.1, so the only positive equilibrium $E^*(x^*, y^*)$ for system (3) is globally asymptotically stable. This complete the proof.

## III. Hopf Bifurcation

In this section, we first study Hopf bifurcation around $\bar{E}$ and prove that no less than one stable limit cycle appears around $\bar{E}$. For simplicity purpose, we carry out the following coordinate transformation (inspired by [27]):

$$\bar{x} = \frac{x}{x^*}, \quad \bar{y} = \frac{y}{y^*}, \quad \bar{A} = \frac{A}{x^*}, \quad \bar{K} = \frac{1}{x^*}, \quad \bar{a} = ay^*.$$

Removing bar, system (3) becomes:

$$\begin{aligned} \dot{x} &= \frac{rx^2(K-x)}{(A+x)K}) - axy, \\ \dot{y} &= -by + bx. \end{aligned} \tag{8}$$

Make a time scaling $d\tau = \frac{1}{(A+x)}dt$, system (8) changes to:

$$\begin{aligned} \dot{x} &= rx^2(1-\frac{x}{K}) - axy(A+x), \\ \dot{y} &= -by(A+x) + bx(A+x)(\beta_0 + x). \end{aligned} \tag{9}$$

Since system (9) has equilibrium $\bar{E}(1, 1)$ corresponding to $E^*(x^*, y^*)$ in system (3), then $a = \frac{r(K-1)}{K(A+1)} > 0$. So we have

$$\begin{aligned} \dot{x} &= rx^2(1-\frac{x}{K}) - \frac{r(K-1)}{K(A+1)}xy(A+x), \\ \dot{y} &= -by(A+x) + bx(A+x). \end{aligned} \tag{10}$$

To make sure system (3) is topologically equivalent to system (10), parameters in system (9) satisfy

$$r > 0, \quad K > 1, \quad A > \tfrac{1}{K+2}. \tag{11}$$

Define $b_* = \frac{rKA-2rA-r}{K(A+1)^2} > 0(A > \frac{1}{K+2})$, we then arrive at the following conclusions.

**Theorem 3.1.** Under the conditions (11), we have:

$\bar{E}$ is a sink if $b > b_*$, and a source if $b < b_*$,

$\bar{E}$ is a weak focus or a center if $b = b_*$.

**Proof.**

$$J_{\bar{E}} = \begin{bmatrix} \frac{rKA-2rA-r}{K(A+1)} & -\frac{r(K-1)}{K} \\ b(A+1) & -b(A+1) \end{bmatrix}.$$

The determinant of $\bar{E}$ is

$$detJ_{\bar{E}} = \frac{br(K+A)}{K},$$

and the trace of $\bar{E}$

$$trJ_{\bar{E}} = \frac{rKA-2rA-r}{K(A+1)} - b(A+1).$$

We say that $detJ_{\bar{E}} > 0$, and $trJ_{\bar{E}} = 0(> 0 \text{ or } < 0)$ if $b = b_*(b < b_* \text{ or } b > b_*)$. The proof is completed.

When $b = b_*$, we may consider Hopf bifurcation and the exact multiplicity of weak focus of $\bar{E}$.

**Theorem 3.2.** Suppose that $b = b_*$ and $A > \frac{1}{K+2}$, we have:

(1) If $KA^2 + KA - 2A^2 - K - 4A - 1 > 0$, system (10) generates a supercritical Hopf bifurcation. The positive equilibrium $\bar{E}$ is a stable fine-focus of order at 1, and one stable limit cycle appears around $\bar{E}$ after perturbation.

(2) If $KA^2 + KA - 2A^2 - K - 4A - 1 < 0$, system (10) generates a subcritical Hopf bifurcation. The positive equilibrium $\bar{E}$ is a unstable fine-focus of order at 1, and one unstable limit cycle appears around $\bar{E}$ after disturbance.

(3) If $K = \frac{2A^2+4A+1}{A^2+A-1}$, system (10) undergoes a degenerate Hopf bifurcation. The positive equilibrium $\bar{E}$ is a stable fine-focus of order at 2, and there exist exactly two stable limit cycles around $\bar{E}$ after perturbation, where the exterior limit cycle is stable while the interior one is unstable.

**Proof.** It is easy to check the assumptions under which Hopf bifurcation occurs.

$$\frac{d}{d_s}trJ_{\bar{E}}|_{b_*} = -(A+1) < 0, \tag{12}$$

then the stability of $\bar{E}$ will change as $b$ changes and at least one limit cycle will come out around $\bar{E}$.

Now we will determine the stability and exact multiplicity of limit cycle. Firstly, translating $\bar{E}(1,1)$ to the origin $(0,0)$ using the translation $(\hat{x},\hat{y}) = (x-1, y-1)$ when $b = b_*$, system (10) can be written as

$$\begin{cases} \dot{\hat{x}} = \hat{a}_{10}\hat{x} + \hat{a}_{01}\hat{y} + \hat{a}_{20}\hat{x}^2 + \hat{a}_{11}\hat{x}\hat{y} + \hat{a}_{21}\hat{x}^2\hat{y} + \hat{a}_{30}\hat{x}^3, \\ \dot{\hat{y}} = \hat{b}_{10}\hat{x} + \hat{b}_{01}\hat{y} + \hat{b}_{20}\hat{x}^2 + \hat{b}_{11}\hat{x}\hat{y}, \end{cases} \tag{13}$$

where

$\hat{a}_{10} = \frac{r(KA-2A-1)}{K(A+1)}$, $\hat{a}_{01} = -\frac{r(K-1)}{K}$, $\hat{a}_{02} = 0$,

$\hat{a}_{11} = -\frac{r(K-1)(A+2)}{K(A+1)}$, $\hat{a}_{20} = \frac{r(KA-3A-2)}{K(A+1)}$, $\hat{a}_{30} = -\frac{r}{K}$,

$\hat{a}_{21} = -\frac{r(K-1)}{K(A+1)}$, $\hat{a}_{12} = 0$, $\hat{a}_{03} = 0$, $\hat{b}_{10} = \frac{r(KA-2A-1)}{K(A+1)}$,

$\hat{b}_{01} = -\frac{r(KA-2A-1)}{K(A+1)}$, $\hat{b}_{20} = \frac{r(KA-2A-1)}{K(A+1)^2}$, $\hat{b}_{02} = 0$,

$\hat{b}_{11} = -\frac{r(KA-2A-1)}{K(A+1)^2}$, $\hat{b}_{21} = \hat{b}_{12} = \hat{b}_{03} = \hat{b}_{30} = 0$.

We move on to make another transformation $\hat{u} = -\hat{x}$, $\hat{v} = \frac{1}{\sqrt{\Delta}}(\hat{a}_{10}\hat{x} + \hat{a}_{01}\hat{y})$, and $\mathrm{d}t = -\sqrt{\Delta}\mathrm{d}\tau$,where $\Delta = \frac{r^2(A+K)(KA-2A-1)}{K^2(A+1)^2}$ and $\hat{a}_{10} + \hat{b}_{01} = 0$. Then system (13) becomes (still denoting $\tau$ as t)

$$\begin{cases} \dot{\hat{u}} = -\hat{v} + f(\hat{u},\hat{v}), \\ \dot{\hat{v}} = \hat{u} + g(\hat{u},\hat{v}), \end{cases} \tag{14}$$

where

$$f(\hat{u},\hat{v}) = \hat{c}_{20}\hat{u}^2 + \hat{c}_{11}\hat{u}\hat{v} + \hat{c}_{30}\hat{u}^3 + \hat{c}_{21}\hat{u}^2\hat{v},$$
$$g(\hat{u},\hat{v}) = \hat{d}_{20}\hat{u}^2 + \hat{d}_{11}\hat{u}\hat{v} + \hat{d}_{30}\hat{u}^3 + \hat{d}_{21}\hat{u}^2\hat{v},$$

$\hat{c}_{20} = -\frac{A\sqrt{A+K}}{(A+1)\sqrt{KA-2A-1}}$, $\hat{c}_{11} = -\frac{A+2}{A+1}$, $\hat{c}_{21} = \frac{1}{A+1}$,

$\hat{c}_{30} = \frac{A\sqrt{A+K}}{(A+1)\sqrt{KA-2A-1}}$, $\hat{d}_{20} = 1$, $\hat{d}_{11} = -\frac{\sqrt{KA-2A-1}}{\sqrt{A+K}}$,

$\hat{d}_{30} = -\frac{A}{A+1}$, $\hat{d}_{21} = -\frac{\sqrt{KA-2A-1}}{(A+1)\sqrt{A+K}}$,

$\hat{d}_{12} = \hat{c}_{02} = \hat{c}_{12} = \hat{c}_{03} = \hat{d}_{02} = \hat{d}_{03} = 0$.

Citing the method of formal power series of Poincare in [28] and calculating the first Lyapunov coefficient with the help of MAPLE, the expression can be described as

$$l_1 = \frac{A(KA^2 + KA - 2A^2 - K - 4A - 1)}{(A+1)^2\sqrt{(KA-2A-1)(A+K)}}.$$

(1) If $KA^2 + KA - 2A^2 - K - 4A - 1 > 0$, $l_1 > 0$. Considering the negative time transformation, system (10) generates a supercritical Hopf bifurcation. $\bar{E}$ is a stable fine-focus of order at 1, and there exists one stable limit cycle around $\bar{E}$ after perturbation.

(2) If $KA^2 + KA - 2A^2 - K - 4A - 1 < 0$, $l_1 < 0$. Considering the negative time transformation, system (10) generates a subcritical Hopf bifurcation. Positive equilibrium $\bar{E}$ is a unstable fine-focus of order at 1, and there exists one unstable limit cycle around $\bar{E}$ after perturbation.

(3) If $K = \frac{2A^2+4A+1}{A^2+A-1}$, $l_1 = 0$. System (10) generates a degenerate Hopf bifurcation. Calculate the second Lyapunov coefficient, we have

$$l_2 = \frac{A^2(3A+4)}{24(A+1)^3\sqrt{A+1}} > 0.$$

Considering the negative time transformation, positive equilibrium $\bar{E}$ is a stable fine-focus of order at 2, and there exist exactly two stable limit cycles around $\bar{E}$ after perturbation, where the exterior limit cycle is stable while the interior one is unstable.

## IV. Existence and uniqueness of limit cycles

In this section, we first investigate the existence of the limit cycles.

**Lemma 4.1.** Positive equilibrium $E^*(x^*, y^*)$ of system (3) is unstable, if $A_1 < A < A_2, a > a_1, b < r$. Where

$a_1 = \frac{b^2 r}{c(r-b)}$, $A_1 = \frac{b\sqrt{r(b^2+ac+br)}}{\sqrt{ac(acr-abc-b^2r)}}$, $A_2 = \frac{br}{ac}$.

**Proof.** By applying simple computation, we know the determinant and trace of Jacobian matrix at $E^*(x^*, y^*)$ are

$$Det(J(x^*, y^*)) = \frac{(acA - br)^2)}{rb(A+1)} > 0,$$

$$tr(J(x^*, y^*)) = \frac{\tau(x)}{rb^2(A+1)(ac+br)},$$

where $\tau(x) = -a^2c^2(ac+2br)A^2 + br(a^2c^2 + 3abcr - ab^2c - b^3r)A - b^3r(r^2 + ac + br)$. Since $A < \frac{br}{ac}$, we have $\tau(x) > abc(acr - abc - b^2r)A^2 - b^3r(r^2 + ac + br)$. Therefore, under the conditions of Lemma 4.1, $\tau(x) > 0$, thus $tr(J(x^*, y^*)) > 0$. The discussions above substantiate that positive equilibrium $E^*(x^*, y^*)$ for system (3) is unstable.

**Theorem 4.1.** Under the conditions $A_1 < A < A_2$, $a > a_1$ and $b < r$, no less than one limit cycle appears in $\Omega^*$ for system (3) .

**Proof.** Under system (3), we create a Bendixson enclosed area $\widehat{OPQRO}$ encompassing the positive equilibrium $E^*(x^*, y^*)$. Let the region enclosed by the curve $\widehat{OPQRO}$ be denoted as $\Omega^*$. As shown in Fig.1, we define $\overline{RO}$ as the line segment on the line $L_1 : x = 0$ , which is the solution of system (3), and $\overline{OP}$ as the line segment on the line $L_2 : y = 0$. Define $\overline{PQ}$ as as the line segment on the line $L_3 : x = 1$, and we have $\dot{x}|_{L_3} = -ay < 0$, so the trajectory of system (3) go through from the exterior of $\widehat{PQ}$ into the inner part of the $\Omega^*$ . Define $\overline{QR}$ as the line segment on the line $L_4 : y = \frac{c}{b} + 1$, and we have $\dot{y}|_{L_4} = -c(1-x) - b < 0 (0 < x < 1)$, so the trajectory of system (3) go through from the exterior of $\widehat{QR}$ into the inner part of the $\Omega^*$. Moreover, $E^*(x^*, y^*)$ is unstable under the assumptions of Theorem 4.1. Using Poincaré-Bendixson Theorem, no less than one limit cycle appears in $\Omega^*$ for system (3). This finishes the proof of Theorem 4.1.

Next, we move on to validate the unique existence of the limit cycles, and present at first the following uniqueness theorem of limit cycles (Lemma 4.2) [29], modified by [30]:

**Lemma 4.2.** Suppose functions $r(u), q(u)$ are continuous and derivable on the open interval $(u_1, u_2)$, $u_1 < u_2$ , and suppose function $\Phi(v)$ is a continuous and derivable on $\Omega$ and consider the system:

$$\begin{aligned} \dot{u} &= \Phi(v) - \int_{u^*}^{u} r(s)\,\mathrm{d}s, \\ \dot{v} &= -q(u), \end{aligned} \tag{15}$$

such that

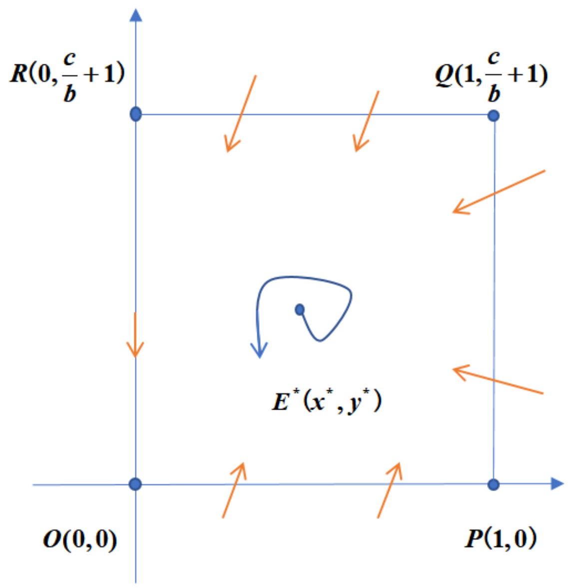


Fig. 1. Bendixson ring and existence of limit cycles.

(1) $\frac{d\Phi(v)}{dv} > 0$,

(2) $u^* \in (u_1, u_2)$ is a unique value satisfying $(u - u^*)q(u) > 0$, for $u \neq u^*$, and $q(u^*) = 0$,

(3) $r(u^*)\frac{d\frac{r(u)}{q(u)}}{du} < 0$, for $u \neq u^*$.

Then system (15) has no more than one limit cycle. If the limit cycle exists, it is hyperbolic.

**Theorem 4.2.** Under the conditions $A_1{}^* < A < A_2$, $a > a_2$ and $r > 8b$, system (3) has no more than one hyperbolic stable limit cycle in $\Omega^*$.

$a_2 = \frac{br(5b+r)}{2c(r-3b)}$, $\quad A_1{}^* = \max\{A_1, A_3, A_4, A_5, A_6\}$,

$A_3 = \frac{2acr-2abc-b^2r-\sqrt{\Delta_1}}{2abc}$, $\quad A_4 = \frac{2acr-4abc-b^2r-\sqrt{\Delta_2}}{4abc}$,

$A_5 = \frac{br}{br+2ac}$, $\quad A_6 = \frac{6abc-2acr+5b^2r+br^2+\sqrt{\Delta_3}}{4acr}$,

$\Delta_1 = b^4r^2 + 4a^2c^2r^2 - 8a^2bc^2r - 8ab^2cr^2$,

$\Delta_2 = b^4r^2 + 4a^2c^2r^2 - 8a^2bc^2r - 12ab^2cr^2$,

$\Delta_3 = 36a^2b^2c^2 + 25b^4r^2 + 4a^2c^2r^2 + 60ab^3cr - 24a^2bc^2r - 16ab^2cr^2 + abcr^3 + 10b^3r^3 + b^2r^4$.

**Proof.** Under the assumptions of Theorem 4.2, the positive equilibrium is unstable. In order to change system (3) to the Lienard equation, we firstly do a time scaling $d\tau = bdt$ (still denoting $\tau$ as t), system (3) changes to:

$$\begin{aligned} \dot{x} &= q_0(x) - q_1(x)y, \\ \dot{y} &= \frac{c}{b}x - y, \end{aligned} \tag{16}$$

where

$$q_0(x) = \frac{rx^2(1-x)}{b(A+x)}, \quad q_1(x) = \frac{a}{b}x > 0(x > 0).$$

By taking another time transformation $d\tau = -q_1(x)dt$ (still denoting $\tau$ as t), system (16) is changed into the following system:

$$\begin{aligned} \dot{x} &= y - \frac{q_0(x)}{q_1(x)}, \\ \dot{y} &= \frac{y - \frac{c}{b}x}{q_1(x)}. \end{aligned} \tag{17}$$

After that, we make a series of substitution in variables,

$$u = x, \quad v = y - \int_{x^*}^{x} \frac{1}{q_1(s)} ds$$

system (17) is transformed into the following Lienard equation:

$$\begin{aligned} \dot{x} &= y - R(x), \\ \dot{y} &= -q(x), \end{aligned} \tag{18}$$

and we have

$$R(x) = \frac{q_0(x)}{q_1(x)} - \int_{x^*}^{x} \frac{1}{q_1(s)} ds = \frac{rx(1-x)}{a(A+x)} - \frac{b}{a}\ln\frac{x}{x^*},$$

$$r(x) = R'(x) = \frac{-rx^3 - (2rA+b)x^2 + (rA-2bA)x - bA^2}{ax(A+x)^2},$$

$$q(x) = \frac{\frac{c}{b}xq_1(x) - q_0(x)}{(q_1(x))^2} = \frac{(ac+br)x + (acA-br)}{a^2(A+x)}.$$

Obviously, the series of transformations above are nonsingular when $x > 0$, and we now check whether system (18) satisfies the conditions of Lemma 4.2.

From the discussions above, we know that functions $r(x), q(x)$ are continuous and derivable on the open interval $(0, 1)$, and $\Phi(y) = y$ satisfies:

$$\Phi'(y) = 1 > 0. \tag{19}$$

Meanwhile, we have $q(x^*) = 0$, and $q'(x) = ac+br > 0$. So when $x > (<)x^*$, we have $q(x) > (<)0$, which is equivalent to say that there exists a unique $x^*$ satisfying:

$$x \neq x^*, \ (x - x^*)q(x) > 0, \ q(x^*) = 0, \tag{20}$$

then $r(x^*) = \frac{r(A+1)TrJ_A}{(ac+br)^3}$. We can get from Lemma 4.1 that when $A_1 < A < A_2$, $a > a_1$ and $b < r$, we have

$$r(x^*) > 0. \tag{21}$$

Finally, we need to prove that $\frac{d\frac{r(x)}{q(x)}}{dx} < 0$. Applying a few computational steps, we get that

$$\frac{d\frac{r(x)}{q(x)}}{dx} = \frac{ar(x)}{(A+x)^2x^2((ac+br)x + acA - br)^2}, \tag{22}$$

where

$p(x) = p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0$,

$p_4 = b(r^2A + r^2 + ac + br)$,

$p_3 = 2A(2abc + 2b^2r - acrA - acr)$,

$p_2 = A(6abcA + 5b^2rA + br^2A + br^2 - 2acrA^2 - 2acrA - b^2r)$, $p_1 = 2bA^2(2acA + brA - br)$, $p_0 = bA^3(acA - br)$.

Define $a_2 = \frac{br(5b+r)}{2c(r-3b)}$, $\quad A_1{}^* = \max\{A_1, A_3, A_4, A_5, A_6\}$,

$A_3 = \frac{2acr-2abc-b^2r-\sqrt{\Delta_1}}{2abc}$, $\quad A_4 = \frac{2acr-4abc-b^2r-\sqrt{\Delta_2}}{4abc}$,

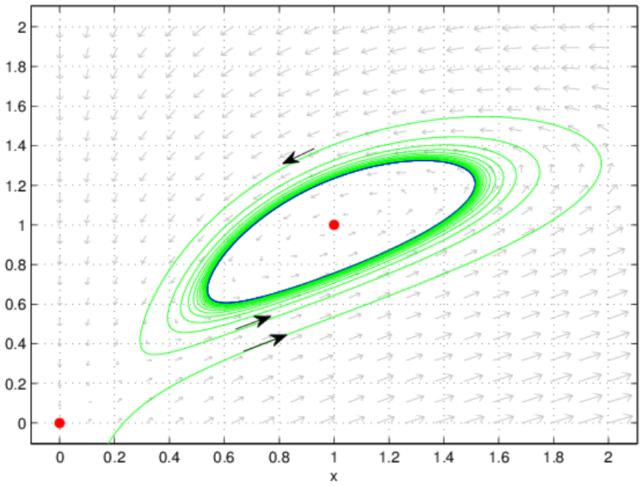


Fig. 2. A stable limit cycle coming out after perturbation.

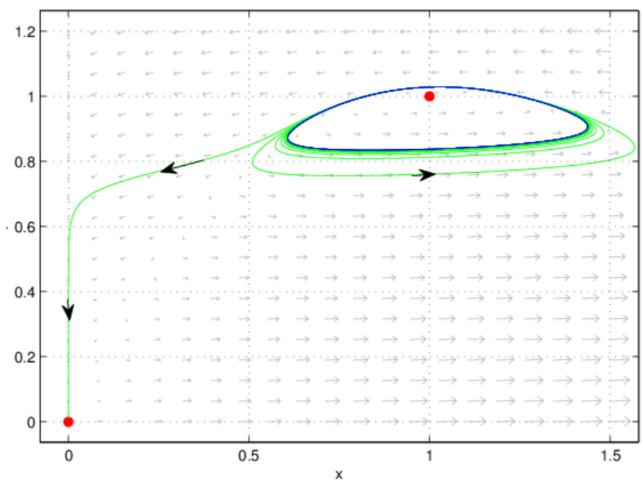


Fig. 3. A unstable limit cycle coming out after perturbation.

$A_5 = \frac{br}{br+2ac}$, $A_6 = \frac{6abc-2acr+5b^2r+br^2+\sqrt{\Delta_3}}{4acr}$,
$\Delta_1 = b^4r^2 + 4a^2c^2r^2 - 8a^2bc^2r - 8ab^2cr^2$,
$\Delta_2 = b^4r^2 + 4a^2c^2r^2 - 8a^2bc^2r - 12ab^2cr^2$,
$\Delta_3 = 36a^2b^2c^2 + 25b^4r^2 + 4a^2c^2r^2 - 24a^2bc^2r - 16ab^2cr^2 + abcr^3 + 10b^3r^3 + b^2r^4$.

When $A_1{}^* < A < A_2$, $a > a_2$ and $r > 8b$, we have $p(x) > 0$, and it is to say we prove that

$$\frac{\mathrm{d}\frac{r(x)}{q(x)}}{\mathrm{d}x} < 0. \tag{23}$$

Therefore, when $A_1{}^* < A < A_2$, $a > a_2$ and $r > 8b$, system (18) holds (19), (20), (21), (23), which shows that system (18) equivalent to system (3) satisfies all conditions of uniqueness theorem of limit cycle in Lemma 4.2, thus system (3) has no more than one limit cycle. The proof of Theorem 4.2 is now completed.

To sum up, system (3) contains a unique stable limit cycle which is hyperbolic.

## V. Numerical Simulations

In this section, examples with numerical simulations we provide to support our results.

Examing system (24) (a special case of system (10)) firstly, we have $r = 1$, $A = 2$, $K = 9$, $b = \frac{13}{81}$.

**Example 5.1**

$$\begin{aligned} \dot{x} &= x^2(1 - \frac{x}{9}) - \frac{8}{27}xy(2+x), \\ \dot{y} &= -\frac{13}{81}y(2+x) + \frac{13}{81}x(2+x) \end{aligned} \tag{24}$$

It is easy to see that $l_1 > 0$. Considering the negative time transformation, system (10) generates a supercritical Hopf bifurcation, and $\bar{E}$ is a stable fine-focus of order at 1. We take $b = \frac{13}{81} - 0.005$, there is one stable limit cycle coming out near $\bar{E}$ after perturbation. (refer to Fig.2).

Secondly, we discuss system (25) (a special case of system (10)) where we have $r = 1$, $A = 2$, $K = \frac{49}{19}$, $b = \frac{1}{147}$.

**Example 5.2**

$$\begin{aligned} \dot{x} &= x^2(1 - \frac{19}{49}x) - \frac{10}{49}xy(2+x), \\ \dot{y} &= -\frac{1}{147}y(2+x) + \frac{1}{147}x(2+x) \end{aligned} \tag{25}$$

It is easy to see that $l_1 < 0$. Considering the negative time transformation, system (10) undergoes a subcritical Hopf bifurcation, and positive equilibrium $\bar{E}$ is a unstable fine-focus of order at 1. We take $b = \frac{1}{147} + 0.0008$, there is one unstable limit cycle around $\bar{E}$ after perturbation. (see Fig.3).

Thirdly, we discuss system (26) (a special case of system (10)) where we have $r = 1$, $A = 2$, $K = \frac{17}{5}$, $b = \frac{1}{17}$.

**Example 5.3**

$$\begin{aligned} \dot{x} &= x^2(1 - \frac{5}{17}x) - \frac{4}{17}xy(2+x), \\ \dot{y} &= -\frac{1}{17}y(2+x) + \frac{1}{17}x(2+x) \end{aligned} \tag{26}$$

It is easy to see that $l_1 = 0$, $l_2 > 0$, system (10) undergoes a degenerate Hopf bifurcation. Considering the negative time transformation, positive equilibrium $\bar{E}$ is a stable fine-focus of order at 2. We take $b = \frac{1}{17} - \frac{4920823}{173457851}, K = \frac{17}{5} - \frac{4893}{9500}$, there exist exactly two stable limit cycles around $\bar{E}$ after perturbation, where the exterior limit cycle is stable while the inner part one is unstable. (see Fig.4).

Finally, we discuss system (27) (a special case of system (3)) where we have $r = 10, A = 0.85, a = 1, b = 0.5, c = 5$.

**Example 5.4**

$$\begin{aligned} \dot{x} &= 10x^2(1-x)\frac{1}{x+0.85} - xy, \\ \dot{y} &= -0.5y + 5x \end{aligned} \tag{27}$$

The conditions in Theorem 4.2 are satisfied, then system (3) generates a unique stable limit cycle which is hyperbolic. (see Fig.5).

## VI. Conclusions

We study the qualitative and bifurcation of single species Logistic model considering Allee effect together with feedback control (3) in this paper. Lin [24] showed there existed

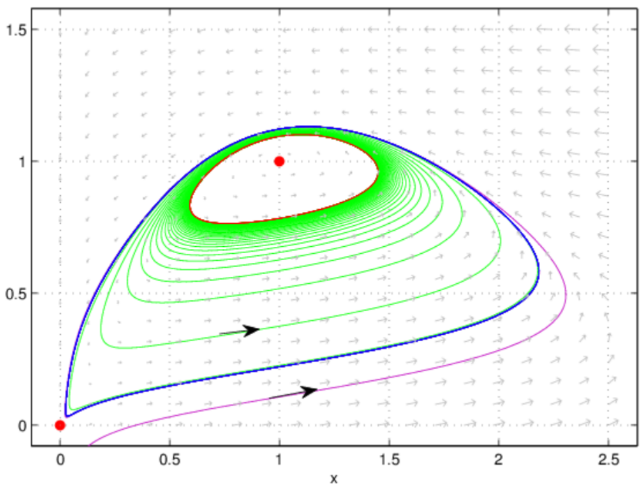


Fig. 4. Two limit cycles coming out after perturbation.

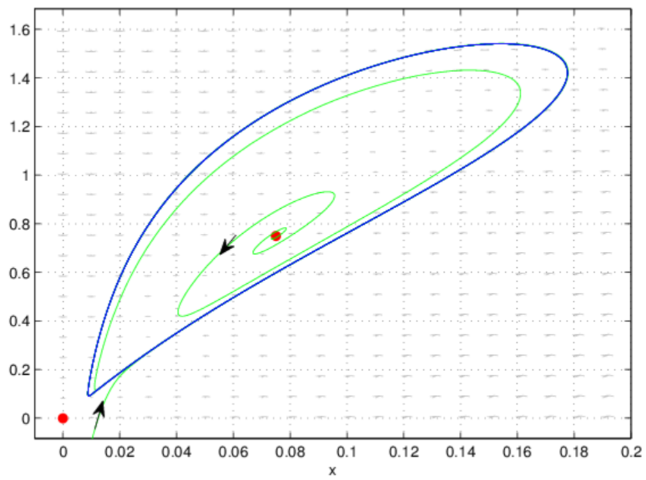


Fig. 5. A unique stable limit cycle which is hyperbolic.

only one locally asymptotically stable positive equilibrium under certain conditions, we continue to propose the sufficient conditions for the system to be globally asymptotically stable. By using formal power series of Poincaré, we find out sufficient conditions for the positive equilibrium being a stable (unstable) fine-focus with order at 1 or a unstable fine-focus with order at 2. Qualitative analysis indicates the ultimate boundedness of the positive solutions to the system, and when the positive equilibrium is unstable, we obtain sufficient conditions that sustain a unique stable limit cycle. Finally, numerical simulations are conducted to validate the major conclusions.

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