Array Cyclic (3*, 4)-Cycle Design

Khaled Matarneh, Ahmad Abubaker, and Suha Al-Shaikh

Abstract — This paper aims to propose a new design known as array cyclic (k*, 4)-cycle design. The development of this design is by forming a combination between a cyclic (k1, k2)-cycle system and a near-two-factor that will be constructed. To do so, we need to introduce a new cycle system, namely a (3*, 4)-cycle system. Thereafter, we present an analysis for the case, v ≡ 8, 4 (mod 12).

Index Terms — Near-two-factor, simple cyclic, difference set, cyclic (k1, ..., kr)-cycle system.

I. INTRODUCTION

Throughout this paper, all graphs will be finite and undirected. A k-cycle, written Ck = (b0, b1, ..., bk−1), consists of k distinct vertices b0, b1, ..., bk−1 and k edges [bibi+1], 0 ≤ i ≤ k − 2 and [b0, bk−1]. Let k1, ..., kr be integers greater than or equal to two, a (k1, ..., kr)-cycle is the union of edge-disjoint ki-cycles for 1 ≤ i ≤ r. A (k1, ..., kr)-cycle system of a graph G is a pair (T, C), where T is the vertex set of G and C is a collection of (k1, ..., kr)-cycles whose edges partition the edges of G. If G = Kν, the complete graph with ν vertices, then such a (k1, ..., kr)-cycle system is called a (k1, ..., kr)-cycle system of order ν. In particular, if k1 = ... = kr = k, this is known as a k-cycle system of a graph G, or (G, Ck)-design. A k-cycle system is Hamiltonian if k = |V|. A trivial counting show that the number of cycles of a Hamiltonian cycle system of Kν is (ν − 1)/2. Hence, a necessary condition for its existence is that ν must be odd [1, 2, 3].

Given a k-cycle Ck = (b0, b1, ..., bk−1), by Ck + j we mean (b0 + j, b1 + j, ..., bk−1 + j), where j ∈ Zν. Analogously, if ℂ = {Ck1, ..., Ckr} is a (k1, ..., kr)-cycle system, we use ℂ + j instead of {Ck1 + j, ..., Ckr + j}. A (k1, ..., kr)-cycle system of order ν, (Tν, ℂν), is said to be m-cyclic if Tν = Zν and for m ∈ Zν, C + m ∈ ℂ whenever C ∈ ℂ and is said to be simple when its cycles are all distinct. In particular, if m = 1, then it is simply called cyclic. A cyclic (k1, ..., kr)-cycle system, of course, is also an m-cyclic (k1, ..., kr)-cycle system for m ∈ Zν. A set of cycles that generates the cyclic (k1, ..., kr)-cycle system of Kν by repeated addition of 1 modular ν is called a starter set.

The existence problem of k-cycle systems of the complete multigraph λKν, a graph where any two vertices are joined by λ distinct edges, has received much attention in recent years. For the important case of λ = 1, this existence problem has been completely solved by Alspach and Gavlas [4] for k odd, by Sajna [5] for k even; and by Alspach et al. [6] for the case λ = 2. The necessary and sufficient conditions for the existence of a k-cycle system of λKν have been established by Bryant et al. in [7] for all values of λ. More general results, such as the existence problem for decomposing λKν into cycles of varying lengths, have been presented in [8, 9]. Furthermore, the necessary and sufficient conditions for the existence of cyclic ν-cycle system of λKν and for the existence of simple cyclic p-cycle system of λKp, where p is a prime, have been provided by Buratti et al. [10].

A k-factor in a graph G is a subgraph of G each of whose vertices has degree k, while a near-k-factor is a subgraph of G in which all but one vertex has degree k with the remaining vertex having degree 0 (isolated vertex). Note that an almost 2-regular graph is equivalent to a near-2-factor [11].

The partition of an edge set of a graph G into k-factor (respectively, near-k-factor) called a k-factorisation (respectively, near-k-factorisation). The decomposition of λKν into near-λ-factor for λ ∈ {2, 4} and ν ≡ 2, 9, 10(mod 12) has been recently constructed in [12, 13, 14].

The main concern of the literature is limited to the existence problem of cyclic k-cycle system of λKν with λ > 1, which lacks a complete solution given by Colbourn and Colbourn [15] for the very special case of k = 3. In this paper, we propose a new design that is called an array cyclic (3*, 4)-cycle design denoted by ACC(3*, 4), 2Kν), which is obtained by merging an m-cyclic (k1, ..., kr)-cycle system of a graph G = 2Kν for k1 = 4, except for k1 = 3 and near-two-factor. In addition, ACC(3*, 4, 2) is an (ν × 2) array design that satisfies the following conditions:

1. The cycles in row r form a near-two-factor with focus r.
2. The cycle associated with the rows contains no repetitions.

II. PRELIMINARY DEFINITIONS

The main results of this paper will be obtained by using the method of difference set that we are going to explain in this section.

Let G be a group of order ν, with the operation +. A k-subset D of G, is a (ν, k, λ) difference set of G if each non-identity element g ∈ G can be written in precisely λ different ways in the form of x − y for x, y ∈ D, where λ is constant.
In this section, we provide some definitions and results of $ACC((3^*, 4), 2K_v)$, which will be needed in the sequel.

Definition 1 A $(3^*, 4)$-cycle system of a graph $2K_v$ is an $m$-cyclic $(k_1, ..., k_r)$-cycle system of a graph $G = 2K_v$ for $k_i = 4$ if $i = 2, ..., \frac{v}{4}$ and $k_3 = 1$.

Definition 2 Let $B_i = \{(b_{ij}, b_{ij+1}, b_{ij+2}, b_{ij+3}) | i = 2, ..., \frac{v}{4}\}$ and $B^* = \{(b_{ij}^*, b_{ij+1}^*, b_{ij+2}^*) | i = 2, ..., \frac{v}{4}\}$ be cycles with vertices in $Z_v$, the list of differences from $B_i$ and $B^*$ is the multi-set.

Definition 3 Let $F = \{B_1, B_2, ..., B_{\frac{v}{4}}\}$ be a set of cycles of $\lambda K_v$ for $\lambda$ and $v$ are even, $F$ is called a $(\lambda K_v, F)$-difference set $(D(Z_v))$, if the multi-set $D(F) = (U_i \cup D(B_i) \cup D(B^*))$ covers each non-zero element of $Z_v$ exactly $\lambda$ times except for the middle difference $v/2$ which appears $\frac{1}{2}$ times.

The following lemma is a consequence of the theory developed in [16]. Accordingly, it will be crucial to prove our main results.

Lemma 1 Let $F$ be a multi-set of cycles of $\lambda K_v$ for $\lambda$ and $v$ are even. Then, $F$ is a starter of cycles $(k_1, ..., k_r)$-cycle system of $\lambda K_v$, if and only if $F$ is a $(\lambda K_v, F)$-difference set.

Definition 4 The array cyclic $(3^*, 4)$-cycle design of a graph $G = 2K_v$, denoted by $ACC((3^*, 4), 2K_v)$, is an $\left(\frac{v}{4} \times \frac{v}{4}\right)$ array design that satisfies the following conditions:

1. The cycles in row $r$ form a near-two-factor with focus $r$.
2. The set of cycles in the first row that generates all the cycles in $\left(\frac{v}{4} \times \frac{v}{4}\right)$ array by repeated addition of $1$ modular($v$).
3. The cycle associated with the rows contains no repetitions.

Now, we will present the following example to illustrate the construction of $ACC((3^*, 4), 2K_v)$, when $v = 4$.

Example 1 Let $G = 2K_4$, $B^* = \{(2, 3, 4)\}$ and $B_1 = \emptyset$, the list of differences from $B_1$ and $B^*$ is the multi-set $\Delta B_i = \emptyset$, $\Delta B^* = \{\pm (b_i^* - b_i^*-1) | i = 1, ..., 4\}$ and $D(B^*) = \{\min\{|b_i^* - b_i^*-1|, v - |b_i^* - b_i^*-1| | i = 1, ..., 4\}$ where $b_i^* = b_i^*-1 = 2$.

$F = B^* = \{(2, 3, 4)\}$, $\Delta F = \Delta B^* = \{1, 3, 1, 3, 2, 2\}$ and $D(B^*) = \{d_{ij}^* | d_{ij}^* = \min\{1, 3, 1, 3, 2, 2\} | i = 1, 2\}$ see Table I.

<table>
<thead>
<tr>
<th>-</th>
<th>3</th>
<th>2</th>
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<tbody>
<tr>
<td>3</td>
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<td>1</td>
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<tr>
<td>2</td>
<td>3</td>
<td>0</td>
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</table>

Difference set $(1, 1, 2)$

Starter cycles $(2, 3, 4)$

It can be seen from Table I that each non-zero integer $1, 2, 3$ in $Z_4$ occurs exactly twice in the off-diagonal position. In addition, $D(F)$ covers each non-zero element of $Z_v$ exactly twice except for the middle difference $(4/2) = 2$ which appears once.

Now consider the graph $G = 2K_4$ of $4$ vertices and one is focus. The starter on Table I which is a near-two-factor, has a $C_4$ cycle and any difference in $D(F)$ appears twice in the cycle edges, except for the middle difference $(2)$ which appears once. It follows then, that a $ACC((3^*, 4), 2K_4)$ is a $(4 \times 1)$ array design and the starter cycles $(2, 3, 4)$ in the first row generate all the cycles in the $(4 \times 1)$ array by repeated addition of $1$ modular(4) (see Table II).

<table>
<thead>
<tr>
<th>Focus</th>
<th>$ACC((3^*, 4), 2K_4)$</th>
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<tbody>
<tr>
<td>$i = 1$</td>
<td>$(2, 3, 4)$</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>$(3, 4, 1)$</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>$(4, 1, 2)$</td>
</tr>
<tr>
<td>$i = 4$</td>
<td>$(1, 2, 3)$</td>
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</tbody>
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In Table II, we can see that every edge in $K_4$ appears two times and is able to generate all cycles by addition of modular $4$. In the next part, we will be able to find the solution for a general case of $v = 12n + 4$.

Lemma 2 There exists a $ACC((3^*, 4), 2K_{12n+4})$.

Proof: Let the starter cycles be $ACC((3^*, 4), 2K_{12n+4})$, as shown in Fig. 1:
Let us breakdown the proof into five parts as follows:

**Part 1:** We will calculate the vertices.

\[ v_j = \begin{cases} \left\lfloor \frac{3n-1}{2} \right\rfloor, & \text{if } j = \{1, 2, 3, 4\} \\ \left\lceil \frac{3n}{2} \right\rceil, & \text{v(j), if } j = \{5, 6, 7, 8\} \end{cases} \]

and \( v^*_j : j = 1, ..., 7 \) then

- \( v_1 = \{12n + 3, 12n + 1, ..., 9n + 7\} \) if \( n \) even
- \( \{12n + 3, 12n + 1, ..., 9n + 6\} \) if \( n \) odd
- \( v_2 = \{3, 5, 3n - 1\} \) if \( n \) even
- \( \{3, 5, 3n\} \) if \( n \) odd
- \( v_3 = \{6, 10, 6n - 2\} \) if \( n \equiv 0(\text{mod } 4) \)
- \( \{6, 10, 6n - 2\} \) if \( n \equiv 1(\text{mod } 4) \)
- \( v_4 = \{12n - 4, 9n + 4, ..., 6n + 8\} \) if \( n \equiv 0(\text{mod } 4) \)
- \( \{12n - 4, 9n + 4, ..., 6n + 6\} \) if \( n \equiv 1(\text{mod } 4) \)
- \( v_5 = \{12n + 2, 12n - 2, ..., 9n + 6, ..., 6n + 6\} \) if \( n \equiv 0(\text{mod } 4) \)
- \( \{12n + 2, 12n - 2, ..., 9n + 6, ..., 6n + 6\} \) if \( n \equiv 1(\text{mod } 4) \)
- \( v_6 = \{4, 8, 3n, ..., 6n\} \) if \( n \equiv 0(\text{mod } 4) \)
- \( \{4, 8, 3n, ..., 6n\} \) if \( n \equiv 0(\text{mod } 4) \)
- \( v_7 = \{6n + 3, ..., 9n + 3\} \) if \( n \) even
- \( \{6n + 3, ..., 9n + 3\} \) if \( n \) odd
- \( v_8 = \{6n + 3, ..., 9n + 3\} \) if \( n \) even
- \( \{6n + 3, ..., 9n + 3\} \) if \( n \) odd
- \( v_9 = \{6n + 3\}, v^*_9 = \{6n + 2\}, v_9^* = \{12n + 4\}, v^*_9 = (2) \),

**Part 2:** We need to prove that the \( (U^j_{j=1}(v_j)) \cup (U^j_{j=1}(v_j)) \) covers all vertices in \( K_{12n+4} \), except for the focus one.

\[ v_0 \cup v_1 \cup v_2 \cup v_3 \cup v_4 \cup v_5 \cup v_6 = \{2, 4, ..., 3n + 1, 6n + 2, 9n + 4, 12n + 4\} \]

if \( n \) even

\[ \{3, 5, ..., 3n + 1, ..., 9n + 3\} \] if \( n \) odd

We will use (1) and (2)

\[ (U^j_{j=1}(v_j)) \cup (U^j_{j=1}(v_j)) = \{2, 3, ..., 12n + 3, 12n + 4\} \]

**Part 3:** We will check for the difference \( D = \{d(j,j) : j = 1, ..., 8\} \) and \( \{d^j : j = 1, ..., 7\} \)

\[ d(1,1) = \min\{v(1,1) - v(3,0), 12n + 4 - |v(1,1) - v(2,0)|\} \]

since \( (1) \leq i \leq \frac{3n-1}{2} \) then \( d(1,1) = (12n + 4) - (12n - 4i + 4) = 4i. \)

\[ d(2,2) = \min\{v(3,1) - v(3,0), 12n + 4 - |v(3,1) - v(2,1)|\} \]

since \( (1) \leq i \leq \frac{3n-1}{2} \) then \( d(2,2) = (21 + 1) \).

\[ d(3,3) = \min\{v(4,1) - v(3,0), 12n + 4 - |v(4,1) - v(2,0)|\} \]

Since \( (1) \leq i \leq \frac{3n-1}{2} \) then

\[ d(3,3) = \begin{cases} 8i + 2 & \text{if } i \leq \frac{3n}{4} \\ 12n - 8i + 2 & \text{if } i > \frac{3n}{4} \end{cases} \]

\[ d(4,4) = \min\{v(5,1) - v(4,0), 12n + 4 - |v(5,1) - v(4,0)|\} \]

since \( (1) \leq i \leq \frac{3n-1}{2} \) then \( d(4,4) = (2i + 1) \).

\[ d(5,5) = \min\{v(6,1) - v(5,0), 12n + 4 - |v(6,1) - v(5,0)|\} \]

since \( (1) \leq i \leq \frac{3n}{2} \) then

\[ d(5,5) = \begin{cases} \frac{9n + 4}{2} & \text{if } i \leq \frac{3n}{4} \\ 12n - 8i + 2 & \text{if } i > \frac{3n}{4} \end{cases} \]

\[ d(6,6) = \min\{v(7,1) - v(6,0), 12n + 4 - |v(7,1) - v(6,0)|\} \]

since \( (1) \leq i \leq \frac{3n}{2} \) then

\[ d(6,6) = \begin{cases} 8i + 2 & \text{if } i \leq \frac{3n}{4} \\ 12n - 8i + 2 & \text{if } i > \frac{3n}{4} \end{cases} \]
\[d(5,1) = \begin{cases} 8i - 2 & \text{if } i \leq \frac{3n+2}{4} \\ 12n - 8i + 6 & \text{if } i > \frac{3n+2}{4} \end{cases} = \min(8i - 2, 12n - 8i + 6).\]

- \(d(6,5) = \min[v(\gamma_3) - v(\delta_6)], 12n + 4 - |v(\gamma_3) - v(\delta_6)|\)
  - since \(1 \leq i \leq \frac{3n+1}{2}\) then \(d(6,5) = (6n - 2i + 3).\)
- \(d(7,6) = \min[v(\gamma_3) - v(\delta_5)], 12n + 4 - |v(\gamma_3) - v(\delta_5)|\)
  - since \(1 \leq i \leq \frac{3n+1}{2}\) then \(d(7,6) = (6n - 2i + 3).\)
- \(d(8,6) = \min[v(\gamma_3) - v(\delta_4)], 12n + 4 - |v(\gamma_3) - v(\delta_4)|\)
  - since \(1 \leq i \leq \frac{3n+1}{2}\) then \(d(8,6) = (4i).\)
- \(d_1^* = \min[v_1^* - v_3^*], 12n + 4 - |v_1^* - v_3^*| = 1.\)
- \(d_2^* = \min[v_2^* - v_3^*], 12n + 4 - |v_2^* - v_3^*| = 1.\)
- \(d_3^* = \min[v_3^* - v_3^*], 12n + 4 - |v_3^* - v_3^*| = 2.\)
- \(d_4^* = \min[v_3^* - v_3^*], 12n + 4 - |v_3^* - v_3^*| = 2.\)
- \(d_5^* = \min[v_3^* - v_3^*], 12n + 4 - |v_3^* - v_3^*| = 2.\)
- \(d_6^* = \min[v_3^* - v_3^*], 12n + 4 - |v_3^* - v_3^*| = 2.\)
- \(d_7^* = \min[v_3^* - v_3^*], 12n + 4 - |v_3^* - v_3^*| = 1.\)
- \(d_8^* = \min[v_3^* - v_3^*], 12n + 4 - |v_3^* - v_3^*| = 1.\)

**Part 4:** We will calculate the difference \(D = \{d(j,i) : j = 1, \ldots, 8\} \cup \{d_j^* : j = 1, \ldots, 7\}\).

Suppose \(d_j = \bigcup_{i=1}^{3n+1} d(j,i)\) if \(j = \{1, 2, 3, 4\}\)
then
\- \(d_1 = \{(4,8,6n-4) : n \equiv 0, 2^{(mod\ 4)}\}\)
\- \(d_2 = \{(3,5,3n-1) : n \equiv 1, 3^{(mod\ 4)}\}\)
\- \(d_3 = \{(10,18,6n+2) \cup \{6n-6,18,10\} : n \equiv 0^{(mod\ 4)}\}\)
\- \(d_4 = \{(3,5,3n-3) : n \equiv 1, 3^{(mod\ 4)}\}\)
\- \(d_5 = \{(6,14,6n-2) \cup \{6n-2,14,16\} : n \equiv 0^{(mod\ 4)}\}\)
\- \(d_6 = \{(6n+1,6n-1,3n+3) : n \equiv 0, 2^{(mod\ 4)}\}\)
\- \(d_7 = \{(4,8,12,6n) : n \equiv 1, 3^{(mod\ 4)}\}\)
\- \(d_8 = \{(6n+1,6n-1,3n+3) : n \equiv 0, 2^{(mod\ 4)}\}\)

**Part 5:** We need to prove that each difference in \(Z_{6n+3}^* = \{1, 2, \ldots, 6n + 1, 6n + 2\}\) appears twice in \(D = \bigcup_{j=1}^{3n+1} \{d(j,i), U_j\bigcup(d_j^*)\}\), except for the middle difference \((6n + 2)\) which appears once.

**Lemma 3** There exists an \(ACC(3,4,2K_{12n-4}).\)

**Proof:** Let the starter cycles be \(ACC(3,4,2K_{12n-4}), as shown in Fig. 2.

If we break the proof into five parts as follows:

**Part 1:** We will calculate the vertices \(\{v(j,i) : j = 1, \ldots, 8\}\)
\- Suppose \(v_j = \left\{\begin{cases} \{j \in \{1, 2, 3, 4\}\} & \text{if } j = \{1, 2, 3, 4\} \\ \{j \in \{5, 6, 7, 8\}\} & \text{if } j = \{5, 6, 7, 8\} \end{cases}\right.\)
  - \(v_1 = \{j \in \{1, 2, 3, 4\}\} \text{ if } j = \{1, 2, 3, 4\} \text{ then } \left\{\begin{cases} \{12n - 4, 12n - 6, \ldots, 9n + 2, 9n\} & \text{if } n \text{ even} \\ \{12n - 4, 12n - 6, \ldots, 9n + 1, 9n - 1\} & \text{if } n \text{ odd} \end{cases}\right.\)
  - \(v_2 = \{j \in \{5, 6, 7, 8\}\} \text{ if } j = \{5, 6, 7, 8\} \text{ then } \left\{\begin{cases} \{24,6, \ldots, 3n - 4, 3n - 2\} & \text{if } n \text{ even} \\ \{24,6, \ldots, 3n - 3, 3n - 2\} & \text{if } n \text{ odd} \end{cases}\right.\)
  - \(v_3 = \{j \in \{5, 6, 7, 8\}\} \text{ if } j = \{5, 6, 7, 8\} \text{ then } \left\{\begin{cases} \{3,5,7, \ldots, 3n - 3, 3n - 1\} & \text{if } n \text{ even} \\ \{3,5,7, \ldots, 3n - 2, 3n\} & \text{if } n \text{ odd} \end{cases}\right.\)
  - \(v_4 = \{j \in \{5, 6, 7, 8\}\} \text{ if } j = \{5, 6, 7, 8\} \text{ then } \left\{\begin{cases} \{6n + 2,6n + 4, \ldots, 9n - 5,9n - 3\} & \text{if } n \text{ even} \\ \{6n + 2,6n + 4, \ldots, 5n - 9,9n - 3\} & \text{if } n \text{ odd} \end{cases}\right.\)
  - \(v_5 = \{j \in \{5, 6, 7, 8\}\} \text{ if } j = \{5, 6, 7, 8\} \text{ then } \left\{\begin{cases} \{6n - 4,6n - 6, \ldots, 3n + 3, 3n + 1\} & \text{if } n \text{ even} \\ \{6n - 4,6n - 6, \ldots, 3n + 3, 3n + 1\} & \text{if } n \text{ odd} \end{cases}\right.\)
  - \(v_6 = \{j \in \{5, 6, 7, 8\}\} \text{ if } j = \{5, 6, 7, 8\} \text{ then } \left\{\begin{cases} \{6n + 1,6n + 3, \ldots, 9n - 5,9n - 3\} & \text{if } n \text{ even} \\ \{6n + 1,6n + 3, \ldots, 6n - 9,9n - 4\} & \text{if } n \text{ odd} \end{cases}\right.\)
  - \(v_7 = \{j \in \{5, 6, 7, 8\}\} \text{ if } j = \{5, 6, 7, 8\} \text{ then } \left\{\begin{cases} \{6n - 3,6n - 5, \ldots, 3n + 3, 3n + 1\} & \text{if } n \text{ even} \\ \{6n - 3,6n - 5, \ldots, 4n + 3, 3n + 1\} & \text{if } n \text{ odd} \end{cases}\right.\)
  - \(v_8 = \{j \in \{5, 6, 7, 8\}\} \text{ if } j = \{5, 6, 7, 8\} \text{ then } \left\{\begin{cases} \{6n - 1\}, v_2 = (6n - 2), v_3 = (6n) \end{cases}\right.\)
Fig. 2.  ACC\((3^*, 4), 2K_{12n-4}\)

**Part 2:** We need to prove that \(\bigcup_{j=1}^{3^*} (v_j) \cup (U_{3^*}^j (v_j))\) covers all vertices in \(K_{12n-4}\) except for the focus one.

\[
\begin{align*}
(v_2 \cup v_4) \cup (v_3^* \cup v_5^*) \cup (v_3 \cup v_4^*) = & \{24, \ldots, 3n, \ldots, 6n - 2, 6n, 6n + 2, \ldots, 9n, \ldots, 12n - 4\} \text{ if } n \text{ even,}\nonumber
\end{align*}
\]

\[
\begin{align*}
(v_2 \cup v_3) \cup (v_3^* \cup v_5^*) \cup (v_3 \cup v_4^*) = & \{24, \ldots, 3n + 1, \ldots, 6n - 2, 6n, 6n + 2, \ldots, 9n - 1, \ldots, 12n - 4\} \text{ if } n \text{ odd.}\nonumber
\end{align*}
\]

We will use (5) and (6)

\[
\bigcup_{j=1}^{3^*} (v_j) \cup (U_{3^*}^j (v_j)) = \{2, 3, \ldots, 12n - 5, 12n - 4\}\}

**Part 3:** We will check for the difference \(D = \{d(j,i) : j = 1, \ldots, 3^*\} \cup \{d_{ij} : j = 1, \ldots, 7\}\)

- \(d_{1,1} = \min \{|v_{1,j} - v_{2,j}|, 12n - 4 - |v_{1,1} - v_{2,j}|\}\) since \(1 \leq i \leq \frac{3n-1}{2}\) then \(d_{1,1} = (12n - 4) - (12n - 2 - 4i) = 4i - 2\).

- \(d_{1,2} = \min \{|v_{1,j} - v_{2,j}|, 12n - 4 - |v_{1,2} - v_{2,j}|\}\) since \(1 \leq i \leq \frac{3n-1}{2}\) then \(d_{1,2} = (12n - 4) - (12n - 3 - 4i) = 4i - 1\).

- \(d_{1,3} = \min \{|v_{1,j} - v_{4,j}|, 12n - 4 - |v_{1,3} - v_{4,j}|\}\) since \(1 \leq i \leq \frac{3n-1}{2}\) then \(d_{1,3} = (12n - 4) - (12n - 4 - 4i) = 4i\).

- \(d_{1,4} = \min \{|v_{1,j} - v_{4,j}|, 12n - 4 - |v_{1,4} - v_{4,j}|\}\) since \(1 \leq i \leq \frac{3n-1}{2}\) then \(d_{1,4} = (12n - 4) - (12n - 3 - 4i) = 4i - 1\).

- \(d_{1,5} = \min \{|v_{1,j} - v_{5,j}|, 12n - 4 - |v_{1,5} - v_{5,j}|\}\) since \(1 \leq i \leq \frac{3n-2}{2}\) then \(d_{1,5} = 4i + 2\).

- \(d_{1,6} = \min \{|v_{1,j} - v_{6,j}|, 12n - 4 - |v_{1,6} - v_{6,j}|\}\) since \(1 \leq i \leq \frac{3n-2}{2}\) then \(d_{1,6} = 4i + 1\).

- \(d_{1,7} = \min \{|v_{1,j} - v_{7,j}|, 12n - 4 - |v_{1,7} - v_{7,j}|\}\) since \(1 \leq i \leq \frac{3n-2}{2}\) then \(d_{1,7} = 4i\).

- \(d_{2,1} = \min \{|v_{2,j} - v_{3,j}|, 12n - 4 - |v_{2,1} - v_{3,j}|\}\) since \(1 \leq i \leq \frac{3n-2}{2}\) then \(d_{2,1} = 4i + 2\).

- \(d_{2,2} = \min \{|v_{2,j} - v_{3,j}|, 12n - 4 - |v_{2,2} - v_{3,j}|\}\) since \(1 \leq i \leq \frac{3n-2}{2}\) then \(d_{2,2} = 4i + 1\).

**Part 4:** We will calculate the difference \(d_j = \bigcup_{i=1}^{\frac{3n-1}{2}} d_{(j,i)}\) if \(j = 1, 2, 3, 4\) then

\[
\begin{align*}
d_1 &= \bigcup_{i=1}^{\frac{3n-1}{2}} d_{(1,i)} = \{2, 6, 10, \ldots, 6n - 10, 6n - 6\} \text{ if } n \text{ even,}\nonumber
\end{align*}
\]

\[
\begin{align*}
d_2 &= \bigcup_{i=1}^{\frac{3n-1}{2}} d_{(2,i)} = \{2, 6, 10, 6n - 8, 6n - 4\} \text{ if } n \text{ odd.}\nonumber
\end{align*}
\]

**Part 5:** We must prove that each difference in \(Z_{6n-1}^* = \{1, 2, \ldots, 6n - 3, 6n - 2\}\) appears two times in 
\[
D = \bigcup_{j=1}^{3^*} (v_j) \cup (U_{3^*}^j (v_j))
\]
except for the middle difference \((6n - 2)\) which appears once.

\[
\begin{align*}
&\text{if } n \text{ even,}\nonumber
\end{align*}
\]

\[
\begin{align*}
&\text{if } n \text{ odd.}\nonumber
\end{align*}
\]

\[
\begin{align*}
&\text{if } n \text{ even,}\nonumber
\end{align*}
\]

\[
\begin{align*}
&\text{if } n \text{ odd.}\nonumber
\end{align*}
\]
We will use (3) and (4).

\[
\{ d_1^+ U d_2^+ U d_3^+ U d_4^+ U d_5 U d_8 \cup (d_2^+ U d_4^+ U d_1 U d_3 U d_8) \ U \{ i, 6n - 3, 6n - 2 \} \} =
\begin{cases} 
\{ i, 2n, ...\} & \text{if } n \text{ even} \\
\{ i, 2, ...\} & \text{if } n \text{ odd}
\end{cases}
\]

It can then be seen that every difference in \(Z_{6n-1}\) appears twice in \(D = \left( \bigcup_{j=1}^{n} \{ d_j^+ \} \right) \cup \left( \bigcup_{j=1}^{n} \{ d_j^+ \} \right)\), except for the middle difference \((6n - 2)\) which appears once.

**Example 2** let \(G = 2K_8\). By Lemma 3, \(B^* = \{(4,5,6)\}\) and \(B_1 = \{(8,2,7,3)\}\), the list of differences set from \(B_1\) and \(B^*\) is the multi-set \(D(B^*) = \{1,2,3\}\) and \(D(B_1) = \{(2,3,4,3)\}\).

\[
F = B^* U B_1 = \{(4,5,6),(2,3,4,3)\}, D(F) = \{1,1,2,2,3,4,3\}, \text{see Table III.}
\]

### Table III

<table>
<thead>
<tr>
<th>(\Delta(F)) and (D(F)) of (2K_8)</th>
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<tr>
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<tr>
<td>4 6 0</td>
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<td>3 3 0</td>
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</table>

This presents an analysis for array cyclic \((k^*, 4)\)-cycle design for case \(v \equiv 8,4 \text{ (mod 12)}\). Furthermore, several definitions and concepts were formulated to construct \(ACC((k^*, 4), 2K_g)\). The algorithm proposed in Lemma 2 and Lemma 3 will be a basis for further research in developing designs for \(v = 12n\). However, we are unable find a method to construct \(ACC((k^*, 4), 2K_g)\) in general.

### References


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