Toughness and Fractional Covered Graph

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***Abstract*—In computer network, the structure of network is modelled by a graph, and toughness, as a graph parameter, is employed to detect the vulnerability of the network. A fractional k-factor exists to satisfy $h(e) = 1$, if for arbitrary $e \in E(G)$, graph G is fractional k-covered. This work studies the mutual influence of toughness and fractional fractional k-covered graph, and obtains the result that a graph G is fractional k-covered if $t(G) \geq k - \frac{1}{k}$. This conclusion implies that the best toughness bound for the existence of the fractional k-factor is also the tight $t(G)$ condition of the fractional k-coverage graph. Finally, some extended conclusions are presented.**

***Index Terms*—graph, fractional factor, fractional covered graph, toughness.**

I. Introduction

As an illustrious network model, the graph is used to represent its topological structure. In this work, we only take the simple graphs (finite, having no loops and multiple edges) into consideration. All the notations and terminologies follow from standard graph theory which can be referred to [1].

Let g and f be two non-negative integer-valued functions on $V(G)$ satisfying $g(x) \leq f(x)$ for all $x \in V(G)$. A fractional (g, f)-factor is a function $h : E(G) \to [0, 1]$ such that $g(x) \leq \sum_{e \in E(x)} h(e) \leq f(x)$ for arbitrary $x \in V(G)$. If $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$, then a fractional (g, f)-factor is a fractional $[a, b]$-factor. If $g(x) = f(x) = k$ for arbitrary $x \in V(G)$, then a fractional (g, f)-factor is a fractional k-factor.

If for any $e \in E(G)$, A graph G is a fractional (g, f)-covered graph, there exists a fractional (g, f)-factor h such that $h(e) = 1$. The fractional $[a, b]$-covered graph and fractional k-covered graph can be defined similarly. A graph G is fractional (g, f, n)-critical covered graph (resp. fractional (a, b, n)-critical covered graph or fractional (k, n)- critical covered graph) if any n vertices from G is deleted, the resulting subgraph remains as a fractional (g, f)-covered graph (resp. fractional $[a, b]$-covered graph or fractional k-covered graph). In a data transmission network, data is divided into several small pieces and transmit through different channels, and the fractional factor corresponds to whether the data transmission is feasible at the same time in the network. The meaning of the fractional covered graph is that a certain channel must be fully utilized when performing a certain transmission task. Such situations are quite common in computer networks, and the feasibility of the fractional factor in this setting is worthy of being studied.

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Li et al. [2] presented the sufficient and necessary condition on which a graph can be fractional (g, f)-covered. We set $f(x) = g(x) = k$ for any vertex $x \in V(G)$, then the result determined in [2] is degenerated to the condition characterizing fractional k-covered graphs which are stated as follows.

Lemma 1: Let G and k be a graph and an integer, respectively. Then, G is a fractional k-covered graph if and only if

$$k|S| - k|T| + d_{G-S}(T) \geq \varepsilon_1$$

holds for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$, where

$$\varepsilon_1 = \begin{cases} 2, & \text{if } S \text{ is dependent,} \\ 1, & \text{if } S \text{ is independent and an edge} \\ & \text{exists to join } S \text{ and } V(G) \setminus (S \cup T), \text{ or} \\ & \text{an edge } e = uv \text{ exists to join } S \text{ and } T \\ & \text{such that } v \in T, d_{G-S}(v) = k, \\ 0, & \text{otherwise,} \end{cases}$$

By simply modifying the parameter ε_1 and subset T, we yield the following equivalent version.

Lemma 2: Let G and k be a graph and an integer, respectively. Then, G is a fractional k-covered graph if and only if

$$k|S| - k|T| + d_{G-S}(T) \geq \varepsilon_2$$

holds for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k - 1\}$, where

$$\varepsilon_2 = \begin{cases} 2, & \text{if } S \text{ is dependent,} \\ 1, & \text{if } S \text{ is independent and an edge} \\ & \text{exists to join } S \text{ and } V(G) \setminus (S \cup T), \\ 0, & \text{otherwise,} \end{cases}$$

Li et al. [3] studied the relationship between isolated toughness (to minimize the ratio $|S| \setminus i(G - S)$ such that the denominator part is at least 2) and fractional k-covered graphs, and determined that a graph having $\delta(G) \geq k + 1$ is fractional k-covered if $I(G) > k$. Zhou [4] considered the neighborhood union bound for fractional (a, b, k)-critical covered graphs. Zhou et al. [5] investigated the graph parameter condition for fractional ID-$[a, b]$-factor-critical covered graphs (after removing any independent set, the resulting subgraph is still a fractional $[a, b]$-covered graph). More results on a graph to be fractional (a, b, k)-critical covered can be referred to Zhou [6].

Toughness was introduced by Chvátal [7] which is formulated by: $t(G) = +\infty$ for complete graph; otherwise $t(G)$ is obtained by minimizing the ratio $|S| \setminus \omega(G - S)$ with $\omega(G - S) \geq 2$. In network security, toughness is a parameter to evaluate the sturdiness and vulnerability of network. It is clear that the complete graph has the most sturdy network structure, and on the contrary, some graphs are very crisp such as star networks. Consider $G = K_{1,n}$, then $t(G) = \frac{1}{n}$. Hence, toughness is used to measure the inner characteristics of networks from the perspective of graph cut. In reality, engineering always seeks the balance point for constructing specific networks. If toughness is too large, then the cost of building such networks will be high, and if toughness is too small, the network structure is not stable. Therefore, it is valuable to determine the marginal toughness value to ensure that the graph admits fractional factor.

In fact, in very early years, the possibility of relevance between toughness and existence of k-factors has been found by researchers, see [8], [9], [10] and [11] for details. About 10 years ago, scholars realized that toughness is also in close relevance with the existence of fractional factor, as well as its extended concepts such as fractional critical graph and fractional deleted graph. Zhou et al. [14] raised a toughness condition for fractional (k, m)-deleted graphs, and this bound was improved by Gao et al. [15]. Gao et al. [12] determined the toughness bound for a graph to make it fractional (g, f, n)- critical. Gao and Wang [13] did research on the toughness condition for fractional critical deleted graphs. Gao et al. [16] presented two independent set conditions for a graph to be fractional (g, f, m)-deleted.

The fractional critical graph measures the failure of the network to work when some sites are damaged or under attack, as long as the number of damaged sites does not exceed a given bound, the remaining network can still work normally. When the channel is attacked or damaged by other reasons, as long as the number of inoperable channels does not exceed a given value, the remaining network can still work normally; the final fractional critical cover graph measures that after both the channel and the site are destroyed at the same time, as long as the number of damaged channels and stations is controlled within a certain range, and the remaining sub-networks can still work normally, which is a fusion of fractional critical graph and fractional cover graph.

Although there are rich advances in toughness and fractional factors in various settings, toughness and fractional covered graphs still keep open relationships. Even there is no conclusion discussed on toughness bounds for fractional k-covered graph which is the simplest form of fractional covered graphs. It motivated us to consider this topic, and our main conclusion is declared as follows.

Theorem 1: Let G be a graph with $|V(G)| \geq k+2$ and $k \geq 2$ be an integer. If $t(G) \geq k - \frac{1}{k}$, G is fractional k-covered graph.

The rest sections are organized as follows: some useful lemmas are introduced in next section and the specific proof is manifested in the third section; several extended conclusions are manifested and some prospects are discussed in the conclusion section.

II. Some Useful Lemmas

In this section, we list the following lemmas which will support the proof of Theorem 1.

Lemma 3: (Chvátal [7]) If a graph G is non-complete, $t(G) \leq \frac{1}{2}\delta(G)$.

Lemma 4: (Liu and Zhang [17]) Take G as a graph and let $\zeta = G[T]$ such that $1 \leq d_G(x) \leq k-1$ for every $x \in V(\zeta)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of ζ and make it satisfy $d_G(x) = j$ for each $x \in T_j$ in which some T_j are allowed as empty. If a vertex of degree can be found in each component of ζ at most $k-2$ in G, ζ has a maximal independent set Ξ and a covering set $\Upsilon = V(H) - \Xi$ such that

$$\sum_{j=1}^{k-1}(k-j)\varsigma_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)\iota_j,$$

where $\varsigma_j = |\Upsilon \cap T_j|$ and $\iota_j = |\Xi \cap T_j|$ for $j = 1, \ldots, k-1$.

Lemma 5: (Liu and Zhang [17]) Let G be a graph and let $\zeta = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(\zeta)$ and there aren't any components of ζ isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. An independent set Ξ and the covering set $\Upsilon = V(\zeta) - \Xi$ of ζ exist to satisfy

$$|V(\zeta)| \leq \sum_{i=1}^{k}(k-i+1)|\Xi^{(i)}| - \frac{|\Xi^{(1)}|}{2}$$

and

$$|\Upsilon| \leq \sum_{i=1}^{k}(k-i)|\Xi^{(i)}| - \frac{|\Xi^{(1)}|}{2}$$

where $\Xi^{(i)} = \{x \in \Xi, d_\zeta(x) = k-i\}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k}|\Xi^{(i)}| = |\Xi|$.

The above two lemmas describe the characteristics between independent sets and covering sets under certain conditions, and are the basis for the proof of the main theorem.

III. Proof of Theorem 1

If G is complete, then the result is yielded by $|V(G)| \geq k+2$. Hence, we suppose G as non-complete.

Assume that G satisfies the hypothesis of Theorem 1, but is not fractional k-covered. In light of Lemma 2 and $\varepsilon_2 \leq 2$, there exist $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k-1\}$ satisfying

$$k|S| - k|T| + \sum_{x\in T} d_{G-S}(x) \leq 1. \qquad (1)$$

If $S = \emptyset$, $G - S = G$ and $T = \{x : x \in V(G), d_G(x) \leq k-1\}$. Using Lemma 3, we acquire $\delta(G) \geq 2t(G) \geq 2k - \frac{2}{k} \geq 2k-1$, which implies $T = \emptyset$ and $\varepsilon_2 = 0$. Hence, we get $0 > 0$ by Lemma 2, a contradiction. Therefore, $S \neq \emptyset$.

Take l as the number of the K_k components in $\zeta' = G[T]$ and set $T_0 = \{x \in V(\zeta') | d_{G-S}(x) = 0\}$. Take ζ as the subgraph yielded from $\zeta' - T_0$ by removing all K_k components.

If $|V(\zeta)| = 0$, according to (1) we get

$$k|S| \leq k|T_0| + kl + 1$$

or

$$1 \leq |S| \leq |T_0| + l + \frac{1}{k}.$$

If $|T_0| + l = 0$, then $T = \emptyset$ and $2 \leq k|S| \leq 1$ by (1), a contradiction. If $\omega(G-S) = |T_0| + l > 1$, then $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{k(|T_0|+l)+1}{k(|T_0|+l)} < \frac{k+1}{k}$, which contradicts to $t(G) \geq k - \frac{1}{k}$ and $k \geq 2$. Suppose $\omega(G-S) = |T_0| + l = 1$. By Lemma 3, $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq 2t(G)$, and thus

$$2k - \frac{2}{k} \leq 2t(G) \leq k - 1 + |S| \leq k + \frac{1}{k}.$$

It reveals that $k^2 \leq 3$ which contradicts to $k \geq 2$. Therefore, we deduce $|V(\zeta)| \geq 1$.

Let $\zeta = \Phi_1 \cup \Phi_2$ where Φ_1, the union of components of ζ, satisfies that $d_{G-S}(x) = k-1$ for arbitrary vertex $x \in V(\Phi_1)$ and $\Phi_2 = \zeta - \Phi_1$. Considering Lemma 5, Φ_1 has a maximum independent set Ξ_1 and the covering set $\Upsilon_1 = V(\Phi_1) - \Xi_1$ meet

$$|V(\Phi_1)| \leq \sum_{i=1}^{k}(k-i+1)|\Xi^{(i)}| - \frac{|\Xi^{(1)}|}{2}, \tag{2}$$

and

$$|\Upsilon_1| \leq \sum_{i=1}^{k}(k-i)|\Xi^{(i)}| - \frac{|\Xi^{(1)}|}{2}, \tag{3}$$

where $\Xi^{(i)} = \{x \in \Xi_1, d_{\Phi_1}(x) = k-i\}$ for $1 \leq i \leq k$ and $\sum\limits_{i=1}^{k}|\Xi^{(i)}| = |\Xi_1|$. Meanwhile, let $T_j = \{x \in V(\Phi_2)|d_{G-S}(x) = j\}$ for $1 \leq j \leq k-1$. In view of the definitions of ζ and Φ_2 it's also confirmed that a vertex of degree exists in each component of Φ_2 at most $k-2$ in $G-S$. In light of Lemma 5, Φ_2 has a maximal independent set Ξ_2 and the covering set $\Upsilon_2 = V(\Phi_2) - \Xi_2$ such that

$$\sum_{j=1}^{k-1}(k-j)\varsigma_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)\iota_j, \tag{4}$$

where $\varsigma_j = |\Upsilon_2 \cap T_j|$ and $\iota_j = |\Xi_2 \cap T_j|$ for each $j = 1, \ldots, k-1$. Let $W = V(G) - S - T$ and $\Gamma = S \cup \Upsilon_1 \cup (N_G(\Xi_1) \cap W)) \cup \Upsilon_2 \cup (N_G(\Xi_2) \cap W)$. It's derived that

$$\begin{aligned}
& |\Upsilon_2| + |N_G(\Xi_2) \cap W| \\
= \ & |V(\Upsilon_2)| - |\Xi_2| + |N_{G-S-T}(\Xi_2)| \\
= \ & |V(\Upsilon_2)| - |\Xi_2| + |N_{G-S}(\Xi_2)| - |N_T(\Xi_2)| \\
= \ & (|V(\Upsilon_2)| - |\Xi_2| - |N_T(\Xi_2)|) + |N_{G-S}(\Xi_2)| \\
\leq \ & (|V(\Upsilon_2)| - |\Xi_2| - |N_{\Upsilon_2}(\Xi_2)|) + |N_{G-S}(\Xi_2)| \\
\leq \ & 0 + \sum_{j=1}^{k-1} j\iota_j = \sum_{j=1}^{k-1} j\iota_j.
\end{aligned}$$

Moreover, we infer

$$|\Gamma| \leq |S| + |\Upsilon_1| + \sum_{j=1}^{k-1} j\iota_j + \sum_{i=1}^{k}(i-1)|\Xi^{(i)}| \tag{5}$$

and

$$\omega(G-\Gamma) \geq t_0 + l + |\Xi_1| + \sum_{j=1}^{k-1}\iota_j, \tag{6}$$

where $t_0 = |T_0|$. When $\omega(G-\Gamma) > 1$, we have

$$|\Gamma| \geq t\omega(G-\Gamma), \tag{7}$$

If $\omega(G-\Gamma) = 1$, then

$$t\omega(G-\Gamma) = t \leq \frac{\delta(G)}{2} \leq \frac{|S| + d_{G-S}(x)}{2} \leq \frac{\Gamma}{2}.$$

Hence, (7) also establishes if $\omega(G-\Gamma) = 1$.

By (5)-(7), we acquire

$$\begin{aligned}
& |S| + |\Upsilon_1| && (8) \\
\geq \ & \sum_{j=1}^{k-1}(t-j)\iota_j + t(t_0+l) + t|\Xi_1| - \sum_{i=1}^{k}(i-1)|\Xi^{(i)}|.
\end{aligned}$$

According to $k|T| - d_{G-S}(T) \geq k|S| - 1$, we verify

$$kt_0 + kl + |V(\Phi_1)| + \sum_{j=1}^{k-1}(k-j)\iota_j + \sum_{j=1}^{k-1}(k-j)\varsigma_j \geq k|S| - 1.$$

Combined with (8), we deduce

$$\begin{aligned}
& |V(\Phi_1)| + \sum_{j=1}^{k-1}(k-j)\varsigma_j + k|\Upsilon_1| \\
\geq \ & \sum_{j=1}^{k-1}(kt-kj-k+j)\iota_j + (kt-k)(t_0+l) \\
& + kt|\Xi_1| - k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}| - 1. && (9)
\end{aligned}$$

Using (2) and (3), we acquire

$$\begin{aligned}
& |V(\Phi_1)| + k|\Upsilon_1| && (10) \\
\leq \ & \sum_{i=1}^{k}(k^2-ki+k-(i-1))|\Xi^{(i)}| - \frac{(k+1)|\Xi^{(1)}|}{2}.
\end{aligned}$$

By means of (4), (9) and (10), we obtain

$$\begin{aligned}
& \sum_{j=1}^{k-1}(k-2)(k-j)\iota_j && (11) \\
& + \sum_{i=1}^{k}(k^2-ki+k-(i-1))|\Xi^{(i)}| \\
\geq \ & \sum_{j=1}^{k-1}(kt-kj-k+j)\iota_j + (kt-k)(t_0+l) + kt|\Xi_1| \\
& + \frac{(k+1)|\Xi^{(1)}|}{2} - k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}| - 1.
\end{aligned}$$

The discussion below is composed of two circumstances in light of the value of $t_0 + l$.

Case 1. $t_0 + l \geq 1$. Here, we obtain $k(t-1)(t_0+l) - 1 \geq 0$. Thus (11) becomes

$$\begin{aligned}
& \sum_{j=1}^{k-1}(k-2)(k-j)\iota_j && (12) \\
& + \sum_{i=1}^{k}(k^2-ki+k-(i-1))|\Xi^{(i)}| \\
\geq \ & \sum_{j=1}^{k-1}(kt-kj-k+j)\iota_j + kt|\Xi_1| + \frac{(k+1)|\Xi^{(1)}|}{2} \\
& - k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}|.
\end{aligned}$$

Therefore, at least one subcase below must hold.

Subcase 1.1. $\sum\limits_{j=1}^{k-1}(k-2)(k-j)\iota_j \geq \sum\limits_{j=1}^{k-1}(kt-kj-k+j)\iota_j$.

There exists j satisfying

$$(k-2)(k-j) \geq kt - kj - k + j,$$

which implies

$$kt \leq (k-2)(k-j) + kj + k - j = k(k-2) + j + k \leq k^2 - 1.$$

By $t(G) \geq k - \frac{1}{k}$, we get $\sum_{j=1}^{k-2}\iota_j = 0$, which has conflicts with the definition of Φ_2 and the selecting of Ξ_2 (see the proof of Lemma 6 in [17] satisfying $\sum_{j=1}^{k-2}\iota_j \neq 0$).

Subcase 1.2. $\sum_{i=1}^{k}(k^2-ki+k-(i-1))|\Xi^{(i)}| \geq kt|\Xi_1| + \frac{(k+1)|\Xi^{(1)}|}{2} - k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}|$.

If $t_0+l\geq 2$ or $k\geq 3$, by $(kt-k)(t_0+l)-1\geq 1$, we have

$$\begin{aligned} & \sum_{i=1}^{k}(k^2-ki+k-(i-1))|\Xi^{(i)}| \\ \geq\ & kt|\Xi_1| + \frac{(k+1)|\Xi^{(1)}|}{2} - k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}| + 1 \\ \geq\ & (k^2-1)|\Xi_1| + \frac{(k+1)|\Xi^{(1)}|}{2} \\ & -k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}| + 1. \end{aligned}$$

That is,

$$|\Xi^{(1)}|(-\frac{k}{2}+\frac{1}{2}) + \sum_{i=2}^{k}(-i+2)|\Xi^{(i)}| \geq 1,$$

which leads to a contradiction.

We have $t_0+l=1$ and $k=2$. In this case, $(kt-k)(t_0+l)-1\geq 0$ and hence

$$|\Xi^{(1)}|(-\frac{k}{2}+\frac{1}{2}) + \sum_{i=2}^{k}(-i+2)|\Xi^{(i)}| \geq 0.$$

It implies that $|\Xi^{(1)}|=0$ and $|\Xi_1|=|\Xi^{(2)}|$.

Let $Y=N_G(\Xi_1)\cup W$. Then we ome to the two subcases below.

Subcase 1.2.1. There exists $y\in Y$ such that $|N_G(y)\cap \Xi_1|=1$.

Let $\Gamma=S\cup\Upsilon_1\cup(N_G(\Xi_1)\cap(W-\{y\}))$. Then, we infer $|\Gamma|\leq|S|+|\Xi_1|(k-1)-1<\frac{|\Xi_1|(k-1)+2}{k}+|\Xi_1|(k-1)-1=\frac{|\Xi_1|+2}{2}+|\Xi_1|-1=\frac{3}{2}|\Xi_1|$. Hence,

$$\frac{3}{2}\leq t(G)\leq\frac{|\Gamma|}{\omega(G-\Gamma)}\leq\frac{\frac{3}{2}|\Xi_1|}{|\Xi_1|+1}<\frac{3}{2},$$

a contradiction.

Subcase 1.2.2. Any vertex in Y has at least two neighborhoods in Ξ_1.

Let $\Gamma=S\cup\Upsilon_1\cup(N_G(\Xi_1)\cap W)$. Then, we deduce $|\Gamma|\leq|S|+|\Xi_1|(k-2)+\frac{|\Xi_1|}{2}<\frac{|\Xi_1|(k-1)+2}{k}+|\Xi_1|(k-2)+\frac{|\Xi_1|}{2}=\frac{3|\Xi_1|+2}{2}$. Therefore,

$$\frac{3}{2}\leq t(G)\leq\frac{|\Gamma|}{\omega(G-\Gamma)}\leq\frac{\frac{3|\Xi_1|+2}{2}}{|\Xi_1|+1}<\frac{3}{2},$$

a contradiction.

Case 2. $t_0+l=0$.

In this circumstance, using (11) we get

$$\begin{aligned} & \sum_{j=1}^{k-1}(k-2)(k-j)\iota_j \\ & +\sum_{i=1}^{k}(k^2-ki+k-(i-1))|\Xi^{(i)}| \\ \geq\ & \sum_{j=1}^{k-1}(kt-kj-k+j)\iota_j + kt|\Xi_1| + \frac{(k+1)|\Xi^{(1)}|}{2} \\ & -k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}| - 1. \end{aligned} \tag{13}$$

Now, the following circumstances are considered on the basis of whether Ξ_1 or Ξ_2 is empty.

Subcase 2.1. $|\Xi_1|=0$.

In this case, (13) becomes

$$\sum_{j=1}^{k-1}((k-2)(k-j)-(kt-kj-k+j))\iota_j+1\geq 0.$$

Let

$$\begin{aligned} \Theta &= (k-2)(k-j)-(kt-kj-k+j) \\ &= k^2+j-k-kt\leq j-k+1. \end{aligned}$$

Then $\max\{\Theta_j\}=\Theta_{k-1}=0$ and the second largest value of Θ_j is $\Theta_{k-2}=-1$. By analyzing the proof process of Lemma 4 in Liu and Zhang [17], it's ensured that Φ_2 is connected, any vertex in Ξ_2 has degree $k-1$ in $G-S$ with the exception that one vertex has degree $k-2$ in $G-S$. It reveals that

$$|\Upsilon_2|\leq(k-2)+(|\Xi_2|-1)(k-1-1)=|\Xi_2|(k-2),$$
$$|T|\leq|\Xi_2|(k-1),$$

and

$$|S|\leq|\Xi_2|+\frac{1-|\Xi_2|}{k}.$$

If $|\Xi_2|=1$, then $|S|\leq 1$, $\delta(G)\leq|S|+(k-1)\leq k$, which is contradictory to $\delta(G)\geq 2t(G)>k$. Therefore, $|\Xi_2|\geq 2$ and

$$\begin{aligned} & k-\frac{1}{k}\leq t(G) \\ \leq\ & \frac{|\Gamma|}{\omega(G-\Gamma)}\leq\frac{\frac{1-|Xi_2|}{k}+|\Xi_2|+|\Xi_2|(k-2)}{|\Xi_2|} \\ =\ & (k-1-\frac{1}{k})+\frac{1}{k|\Xi_2|}. \end{aligned}$$

This reveals $1\leq\frac{1}{k|\Xi_2|}$, which contradicts to $k\geq 2$ and $|\Xi_2|\geq 2$.

Subcase 2.2. $|\Xi_2|=0$.

Here, (13) changes to

$$\begin{aligned} & \sum_{i=1}^{k}(k^2-ki+k-(i-1))|\Xi^{(i)}|-kt|\Xi_1| \\ & -\frac{(k+1)|\Xi^{(1)}|}{2}+k\sum_{i=1}^{k}(i-1)|\Xi^{(i)}|+1\geq 0. \end{aligned}$$

This implies

$$\sum_{i=2}^{k}(-i+2)|\Xi^{(i)}|+(-\frac{1}{2}k+\frac{1}{2})|\Xi^{(1)}|+1\geq 0.$$

We get $\sum_{i=4}^{k}|\Xi^{(i)}|=0$, $|\Xi^{(3)}|\le 1$ and $|\Xi^{(1)}|\le 2$.

Subcase 2.2.1. $|\Xi^{(1)}|=1$. Here, we have $\sum_{i=3}^{k}|\Xi^{(i)}|=0$. In light of the proof tricks of Lemma 2.2 in Liu and Zhang [17], we verify $|\Xi_1|\ge 2$,

$$|T|\le (k-1)+(|\Xi_1|-1)(k-1)=|\Xi_1|(k-1),$$
$$|S|\le\frac{|T|+1}{k}\le\frac{|\Xi_1|(k-1)+1}{k},$$

and

$$\begin{aligned}|\Gamma| &\le |S|+|\Upsilon_1|+\sum_{i=1}^{k}(i-1)|\Xi^{(i)}|\\ &\le \frac{|\Xi_1|(k-1)+1}{k}+|\Xi_1|(k-1)\\ &\quad -|\Xi_1|+(|\Xi_1|-1)\\ &= \frac{|\Xi_1|(k-1)+1}{k}+|\Xi_1|(k-1)-1.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{k^2-1}{k}&\le t(G)\le\frac{|\Gamma|}{\omega(G-\Gamma)}\\ &\le \frac{\frac{|\Xi_1|(k-1)+1}{k}+|\Xi_1|(k-1)-1}{|\Xi_1|}.\end{aligned}$$

This implies $0\le\frac{1-k}{|\Xi_1|}$, a contradiction.

Subcase 2.2.2. $|\Xi^{(1)}|=2$. Here, we obtain $\sum_{i=3}^{k}|\Xi^{(i)}|=0$, and a contradiction will be similar to Subcase 2.2.1.

Subcase 2.2.3. $|\Xi^{(1)}|=0$. Here, we confirm $\sum_{i=4}^{k}|\Xi^{(i)}|=0$ and $|\Xi^{(3)}|\le 1$. If $|\Xi_1|=1$, then $|S|\le 1$. Thus, we get

$$k=1+k-1\ge k-1+|S|\ge\delta(G)\ge 2t(G)\ge 2k-\frac{2}{k},$$

a contradiction. Thus, $|\Xi_1|\ge 2$. Let $Y=N_G(\Xi_1)\cap W$.

If there exists a vertex $y\in Y$ with $|N_G(y)\cap\Xi_1|=1$. Reset

$$\Gamma=S\cup\Upsilon_1\cup(N_G(\Xi_1)\cap(W-\{y\})).$$

Thus, we derive

$$|\Gamma|\le|S|+|\Xi_1|(k-1)-1\le\frac{|\Xi_1|(k-1)+1}{k}+|\Xi_1|(k-1)-1.$$

In light of $|\Xi_1|\ge 2$, we obtain

$$\begin{aligned}\frac{k^2-1}{k}&\le t(G)\le\frac{|\Gamma|}{\omega(G-\Gamma)}\\ &\le \frac{\frac{|\Xi_1|(k-1)+1}{k}+|\Xi_1|(k-1)-1}{|\Xi_1|}.\end{aligned}$$

This implies $0\le\frac{1-k}{|\Xi_1|}$, a contradiction.

If any vertex in Y is adjacent to at least two vertices in Ξ_1. We determine

$$\begin{aligned}|\Gamma|&\le|S|+|\Xi_1|(k-2)+\frac{|\Xi_1|}{2}\\ &\le \frac{|\Xi_1|(k-1)+1}{k}+|\Xi_1|(k-2)+\frac{|\Xi_1|}{2},\end{aligned}$$

where $\Gamma=S\cup\Upsilon_1\cup(N_G(\Xi_1)\cap W)$. Using $|\Xi_1|\ge 2$, we get

$$\begin{aligned}\frac{k^2-1}{k}&\le t(G)\le\frac{|\Gamma|}{\omega(G-\Gamma)}\\ &\le \frac{\frac{|\Xi_1|(k-1)+1}{k}+|\Xi_1|(k-2)+\frac{|\Xi_1|}{2}}{|\Xi_1|}.\end{aligned}$$

This implies $0\le(\frac{1}{|\Xi_1|}-\frac{k}{2})$, which contradicts to $k\ge 2$ and $|\Xi_1|\ge 2$.

Subcase 2.3. $|\Xi_1|\ne 0$ and $|\Xi_2|\ne 0$. According to the discussion presented in Subcase 2.1, we yield $\sum_{j=1}^{k-1}(k-2)(k-j)i_j\le\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+1$. Then, we yield

$$\begin{aligned}&\sum_{i=1}^{k}(k^2-ki+k-(i-1))|I^{(i)}|\\ \ge\;& kt|I_1|+\frac{(k+1)|I^{(1)}|}{2}-k\sum_{i=1}^{k}(i-1)|I^{(i)}|.\end{aligned}$$

This implies

$$\sum_{i=2}^{k}(-i+2)|I^{(i)}|+(-\frac{1}{2}k+\frac{1}{2})|\Xi^{(1)}|\ge 0.$$

Hence, we verify $\sum_{i=4}^{k}|\Xi^{(i)}|=0$, $|\Xi^{(3)}|\le 1$ and $|\Xi^{(1)}|\le 2$ by what we have argued in Subsection 1.2. We only discuss the circumstances of $|\Xi^{(1)}|=0$, and the other two situations for $|\Xi^{(1)}|=1$ and $|\Xi^{(1)}|=2$ can be done in terms of the same fashion.

Using $|\Xi^{(1)}|=0$, we derive $\sum_{i=4}^{k}|\Xi^{(i)}|=0$, $|\Xi^{(3)}|\le 1$,

$$|T|\le|\Xi_1|(k-1)+|\Xi_2|(k-1)=(k-1)(|\Xi_1|+|\Xi_2|),$$

and

$$|S|\le\frac{|T|+1}{k}\le\frac{(k-1)(|\Xi_1|+|\Xi_2|)+1}{k}.$$

In view of $|\Xi_1|+|\Xi_2|\ge 2$, we infer

$$\begin{aligned}\frac{k^2-1}{k}&\le t(G)\le\frac{|\Gamma|}{\omega(G-\Gamma)}\\ &\le \frac{|S|+|\Xi_2|(k-2)+|\Xi_1|(k-1)}{|\Xi_1|+|\Xi_2|}.\end{aligned}$$

Hence,

$$\begin{aligned}&(k^2-1)(|\Xi_1|+|\Xi_2|)\\ \le\;&(k-1)(|\Xi_1|+|\Xi_2|)+1+(k^2-2k)(|\Xi_1|+|\Xi_2|)\\ &+k|\Xi_1|,\end{aligned}$$

which reveals $0\le 1-k|\Xi_2|$, a contradiction.

Thus, Theorem 1 is proved.

IV. The Extended Results

Let $I(G)$ be the isolated toughness of graph G. In the theorems below, it's usually assumed that a,b are positive integers, g,f are two non-negative integer-valued functions which are defined on $V(G)$ meeting $a\le g(x)\le f(x)\le b$ for all $x\in V(G)$, n be a non-negative integer and $\Delta=b-a$. Using the tricks presented in Gao et al. [18]-[19] and this paper, we get the following extended results, and the specific proving is skipped.

Theorem 2: Let G be a graph, $b\ge 2$ and $(a,b)\ne(1,2)$. Assume $|V(G)|\ge n+b+2$ if G is complete. If $t(G)\ge\frac{b^2-\Delta+bn-1}{a}$, G is fractional (g,f,n)-critical covered.

Theorem 3: Let G be a graph and $2\le a\le b$. Assume $|V(G)|\ge n+b+2$ if G is complete. If $t(G)\ge\frac{ab-b+a-\Delta+bn-1}{a}$, G is fractional (a,b,n)-critical covered.

Theorem 4: Let G be a graph, $b \geq 2$ and $(a, b) \neq (1, 2)$. Assume $|V(G)| \geq n + b + 2$ if G is complete. If $t(G) \geq \frac{b^2-\Delta-1}{a} + n$, G is fractional (g, f, n)-critical covered.

Theorem 5: Let G be a graph, $b \geq 2$ and $(a, b) \neq (1, 2)$. Assume $|V(G)| \geq n + b + 2$ if G is complete. If $I(G) \geq \frac{b^2-\Delta+bn}{a}$ and $\delta(G) \geq n + b + 1$, G is fractional (g, f, n)-critical covered.

Theorem 6: Let G be a graph and $2 \leq a \leq b$. Assume $|V(G)| \geq n+b+2$ if G is complete. If $I(G) \geq \frac{ab-b+a-\Delta+bn}{a}$ and $\delta(G) \geq n + b + 1$, then G is fractional (a, b, n)-critical covered.

Theorem 7: Let G be a graph, $b \geq 2$ and $(a, b) \neq (1, 2)$. Assume $|V(G)| \geq n + b + 2$ if G is complete. If $I(G) \geq \frac{b^2-\Delta}{a} + n$ and $\delta(G) \geq n + b + 1$, G is fractional (g, f, n)-critical covered.

V. Conclusion and Discussion

In computer network, the network flow problem is transformed into the fractional factor existence problem, and the resource scheduling algorithm and its feasibility are based on the existence of graph fractional factor. In real work, we are often faced with two work scenarios: upgrading the existing network to meet new business requirements; or rebuilding new network facilities according to business needs. Therefore, different scenarios need to correspond to the existence of fractional factors under different settings, and the scenarios corresponding to fractional critical graphs and coverage graphs are that some networks cannot be used normally due to attacks on the network; or there are some channels that must pass through as network data streams or service streams.

This paper mainly contributes to determining the toughness bounds for fractional covered and fractional critical covered graph in different settings. All toughness bounds presented in our article are tight in some sense because these bounds are exactly the tight bounds for the existence of corresponding fractional factor. For example, $t(G) \geq k - \frac{1}{k}$ is a sharp bound for a graph admits fractional k-factor (see Liu and Zhang [17]), and hence the toughness condition presented in Theorem 3 is sharp as well. Furthermore, it also reveals that after the network toughness parameter reaching the predetermined critical value, not only can the network have a fractional factor, but also a fractional factor that must pass through a certain channel can be guaranteed.

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