A Note on Somos's Identities of Level 6

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***Abstract*—Michael Somos conjectured thousands of theta-functions of various levels with the help of computer and not provided the proof. These theta-functions highly matches Ramanujan's recordings. The objective of our work in this paper is to prove two theta-functions of level 6 conjectured by Somos and to give an alternate proof of the identities proved by B. R. Srivatsa Kumar and G. Sharath. Further, as an application of this, we offer colored partition identities for the same.**

***Index Terms*—Colored partitions, Dedekind eta-function, Modular equations, Theta functions.**

I. Introduction

Ramanujan documented many identities which involve $f(-q), f(-q^2), f(-q^n)$ and $f(-q^{2n})$ in his second and 'Lost' notebook [7, 8]. For example [5, p.206], if

$$P = \frac{f(-q)}{q^{1/6}f(-q^5)} \text{ and } Q = \frac{f(-q^2)}{q^{1/3}f^2(-q^{10})}$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3$$

or

$$f_1^4 f_2^4 f_5^2 f_{10}^2 + 5 f_1^2 f_2^2 f_5^4 f_{10}^4 = f_2^6 f_5^6 + f_1^6 f_{10}^6$$

where

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n)$$

Subsequently after publishing [5], numerous authors along with C. Adiga et al. [1], N. D. Baruah [2, 3] and K. R. Vasuki [14, 15] and many mathematicians found several new type of modular equations of the above type and employed for the evaluation of continued fractions, weber class invariants and many more. Recently, Michael Somos [9] conjectured/discovered nearly 6200 Dedekind η-function identities of numerous levels by GP/PARI scripts using computer and offered no proof of these. Many authors have given the proof of his identities and for the wonderful work one may refer [10-13, 16, 17]. The Somos identities of level 6 contains arguments in $f(-q), f(-q^2), f(-q^3)$ and $f(-q^6)$ namely $-q, -q^2, -q^3$ and $-q^6$ all have exponents dividing 6, which is thus call it as level of the identity 6. The aim of our work in this paper is to prove two of new η-function identities of level 6 conjectured by Somos by using modular equations given by Ramanujan. The present work is classified as follows. In Section 2, we prove two Somos's theta-function identities of level 6 and give an alternate proof of the identities proved by B. R. Srivatsa Kumar and G. Sharath in [14]. Further as an application of this in Section 3 we demonstrate colored partition identities and verify the same with an example. Prior to pursue to demostrate Somos's identities, we have conscript at the beginning to review some theta-functions and modular equation identities which will be required in future. All through the paper, we shall use the typical q-notation. For $|q| < 1$, the q-shifted factorial is defined as

$$(a;q)_\infty := \prod_{n=1}^{\infty} \left(1 - aq^{n-1}\right).$$

If $|mn| < 1$, Ramanujan's general theta function $f(m,n)$ is stated as follows:

$$f(m,n) = \sum_{j=-\infty}^{\infty} m^{\frac{j(j+1)}{2}} n^{\frac{j(j-1)}{2}}.$$

By Jacobi's triple product identity [4, p.35], we have

$$f(m,n) = (-m, -n, mn; mn)_\infty.$$

The meaningful particular cases of $f(m,n)$ [4, p.36], are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty^2$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty,$$

$$\chi(-q) := (-q;q)_\infty.$$

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For any complex number τ, if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ is the classical Dedekind η-function with $Im(\tau) > 0$ and is defined as

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$
$$= e^{-\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-q^{2n\pi i\tau}\right).$$

After Ramanujan, we define

$$\chi(q) := \left(-q;q^2\right)_\infty.$$

Also, one can easily see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2},\ \chi(q) = \frac{f_2^2}{f_1 f_4},\ \chi(-q) = \frac{f_1}{f_2}, \quad (1)$$

where $f_k = (q^k;q^k)_\infty$. A modular equation of n^{th} degree

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

is an equation relating α and β where

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, |z|<1.$$

stands for an ordinary hypergeometric function with

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

and generally it is said that β is of n^{th} degree over α and call the ratio

$$m := \frac{z_1}{z_n}$$

as the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

II. Main Results

Theorem 1. *We have*

$$2\frac{\varphi^3(q)}{\varphi(q^3)} + 6q\frac{\varphi^3(q^3)}{\varphi(q)} = 3\frac{\varphi^3(-q^3)}{\varphi(-q)} - \frac{\varphi^3(-q)}{\varphi(-q^3)}.$$

Proof: If β has degree 3 over α, Ramanujan documented the following modular equation [9] [4, pp. 230-238 Entry 5]

$$\frac{1}{P} - P = \frac{1}{2\sqrt{2}}\left(\frac{1}{Q}+Q\right) \quad (2)$$

where $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $Q = \{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\}^{1/4}$. Suppose if $y = \pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)}$ and $z = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)$, then from Entry 10(i) and 12(v)[4, pp. 122-124], we have

$$\varphi(q) := \sqrt{z} \quad (3)$$

and

$$\chi(q) := 2^{1/6}\{x(1-x)q\}^{-1/24} \quad (4)$$

where $q = e^{-y}$. On transforming (2) by using (4), we obtain

$$\frac{m^6}{n^6} + \frac{n^6}{m^6} = m^3n^3 - \frac{8}{m^3n^3} \quad (5)$$

where

$$m := m(q) = q^{-1/24}\chi(q)$$

and

$$n := n(q) = q^{-1/8}\chi(q^3)$$

Now on multiplying (5) throughout by $2/n^{12}$, we obtain

$$16\frac{m^3}{n^9} + 2 - 2\frac{m^9}{n^3} + 2\frac{m^{12}}{n^{12}} = 0.$$

Equivalently,

$$\left(2+\frac{m^9}{n^3}\right)\left(1-4\frac{m^3}{n^9}\right) - \frac{m^8}{n^8}\left(2\frac{m^4}{n^4}+mn^5\right)\left(4\frac{n^3}{m^9}-1\right) = 0. \quad (6)$$

Also from Entry 5[5, pp. 230-238], we have

$$m = \frac{1-2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1-2(\alpha\beta)^{1/4}}$$

and

$$\frac{3}{m} = \frac{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}-1}{1-2(\alpha\beta)^{1/4}}$$

which implies

$$\frac{m^2}{3} = \frac{1-2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}-1}.$$

On rewriting the above into theta function using (3) and (4), we obtain

$$\frac{\varphi^4(q)}{3\varphi^4(q^3)} = \frac{1-4q\frac{m^3}{n^9}}{4\frac{n^3}{m^9}-1}. \quad (7)$$

Employing (7) in (6), we deduce

$$\left(2+\frac{n^9}{m^3}\right)\frac{\varphi^4(q)}{\varphi^4(q^3)} - \frac{m^8}{n^8}\left(2\frac{m^4}{n^4}+mn^5\right) = 0. \quad (8)$$

From (1) it is observed that

$$\frac{\varphi(q)}{\varphi(q^3)} = \frac{m^2}{n^2}\frac{f_2}{f_6}. \tag{9}$$

Using (9) in (8), we obtain

$$2 - 3\left[2\frac{m^4}{n^4} + mn^5\right]\left(\frac{f_6}{f_2}\right)^4 + \frac{m^9}{n^3} = 0.$$

On letting $q \to -q$ in the above, rewriting $m(-q)$ and $n(-q)$ in terms of f_n by employing (1) and after multiplying throughout by $f_1^3 f_2^3 f_3^3 f_6^3$ and on simplifying (1), we obtain the desired result. ∎

Theorem 2. *We have*

$$81q\frac{\varphi^4(q^3)}{\varphi^4(q)}\frac{\varphi^4(-q^3)}{\varphi^4(-q)} = \left(1 - \frac{\chi^9(-q)}{\chi^3(-q^3)}\right)\left(1 + 8\frac{\chi^3(-q^3)}{\chi^9(-q)}\right).$$

Proof: On multiplying (5) throughout by $n^{-24}\left(16m^{12} - m^9n^9 - 16m^3n^3 - 8n^{12}\right)$, we obtain

$$80\frac{m^3}{n^9} - 16\frac{m^{12}}{n^{12}} + 17\frac{m^{21}}{n^{15}} - 7\frac{m^9}{n^3} - 112\frac{m^{15}}{n^{21}} - \frac{m^{18}}{n^6} - 16\frac{m^{24}}{n^{24}} + 128\frac{m^6}{n^{18}} + 8 = 0.$$

which is equivalent to

$$9\frac{m^{21}}{n^{15}}\left(4\frac{n^3}{m^9} - 1\right)^2 + \left(7\frac{m^9}{n^3} + \frac{m^{18}}{n^6} - 8\right)\left(1 - 4\frac{m^3}{n^9}\right)^2 = 0.$$

Employing (7) in the above, we see that

$$81\frac{m^5}{n}\left(\frac{f_6}{f_2}\right)^8 + 7\frac{m^9}{n^3} + \frac{m^{18}}{n^6} - 8 = 0.$$

On letting $q \to -q$ in the above, rewriting $m(-q)$ and $n(-q)$ in terms of f_n by employing (1) and after multiplying throughout by $f_2^{18} f_3^7 f_6$ and on simplifying (1), we obtain the desired result. ∎

Theorem 3. *We have*

$$f_1 f_2^3 f_3^9 + q f_1^4 f_6^9 - f_2^8 f_3^4 f_6 = 0.$$

Proof: On dividing Theorem 3 by $f_1 f_2^3 f_3^9$, we obtain

$$1 + q\frac{f_1^3 f_6^9}{f_2^3 f_3^9} - \frac{f_2^5 f_6}{f_1 f_3^5} = 0. \tag{10}$$

From [16, Theorem 3.4(i)] if

$$A = \frac{f_1}{q^{1/24} f_2} \qquad \text{and} \qquad B = \frac{f_3}{q^{1/8} f_6},$$

we have

$$(AB)^3 + \frac{8}{(AB)^3} = \left(\frac{B}{A}\right)^6 - \left(\frac{A}{B}\right)^6. \tag{11}$$

Also from [5, p. 204, Entry 51], if

$$P = \frac{f_1^2}{q^{1/12} f_3^2} \qquad \text{and} \qquad Q = \frac{f_2^2}{q^{1/6} f_6^2}$$

we have

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{12}$$

Employing A, B, P and Q in (10), reduces to

$$(AB)^3 = PQ - \frac{P^3}{Q^3}.$$

On employing the above in (11) and then factorizing, we obtain

$$Q(P^6 - P^4Q^4 - 9P^2Q^2 + Q^6) = 0.$$

But the second factor is nothing but (12) and it verifies Theorem 3. ∎

Theorem 4. *We have*

$$f_1^9 f_3 f_6^3 + 9q f_1^4 f_2 f_6^8 - f_2^9 f_3^4 = 0.$$

Proof: On dividing Theorem 4 by $f_1^9 f_3 f_6^3$, we obtain

$$1 + 9q\frac{f_2 f_6^5}{f_1^5 f_3} - \frac{f_2^9 f_3^3}{f_1^9 f_6^3} = 0.$$

On using the definitions of A, B, P and Q from (11) and (12) in the above and after simplification, we obtain

$$(AB)^3 = \frac{Q^3}{P^3} - \frac{9}{PQ}.$$

Employing the above in (11) and then factorizing, we obtain (12) and it verifies Theorem 4. ∎

III. Application to Partitions

S. S. Huang [6] introduced the theory of colored partitions. We use the notation

$$(a_1, a_2, ...a_n; q)_\infty := \prod_{i=1}^{\infty}(a_j; q)_\infty,$$

Also for clarity, we write

$$(q_{a_1}^{m_1}, q_{a_2}^{m_2}, ..., q_{a_k}^{m_k}; q^n)_\infty = (q^{m_1}; q^n)_\infty^{a_1}(q^{m_2}; q^n)_\infty^{a_2} \cdots \times (q^{m_k}; q^n)_\infty^{a_k}(q^{n-m_1}; q^n)_\infty^{a_1}(q^{n-m_2}; q^n)_\infty^{a_2} \cdots \times (q^{n-m_k}; q^n)_\infty^{a_k}.$$

For example $(q_4^{1\pm}; q^6)_\infty = (q_4^1, q_4^5; q^6)_\infty = (q_4^1; q^6)_\infty (q_4^5; q^6)_\infty$.

Now we define the colored partition as defined in the literature.

"A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into colored parts are called colored partitions".

Example: Suppose 1, 2 and 3 are allowed, having two colors, then the achievable partitions of 3 are $3_i, 3_r, 2_i + 2_i, 2_r + 2_r, 2_r + 2_i, 1_r + 1_r + 1_r, 1_i + 1_i + 1_i, 1_i + 1_r + 1_r, 1_r + 1_i + 1_i$, where we employed (indigo) and (red) as the indices to distinguish two colors 1, 2 and 3. Also the generating function for the partitions of n is given by

$$\frac{1}{(q^a;q^b)_\infty^m} = \sum_{n=0}^{\infty} \alpha(n) q^n,$$

for $\alpha(0) = 1$ and the entire summands are $\equiv a(\text{mod } b)$ with m colors.

Theorem 5. *We have for any $n \geq 1$*

$$2p_1(n) + 6p_2(n-1) - 3p_3(n) + p_4(n) = 0,$$

where $p_1(n)$: absolute sum of partitions of n splits into many summands which are $\equiv \pm 1, \pm 2(mod\ 6)$ with nine colors and $+3(mod\ 6)$ with fourteen colors. $p_2(n)$: total number of partitions of n splits into many summands $\equiv \pm 1(mod\ 6)$ having five colors, $+2(mod\ 6)$ having thirteen colors and $+3(mod\ 6)$ with fourteen colors. $p_3(n)$: absolute sum of partitions of n splits into many summands $\equiv \pm 1(mod\ 6)$ having eight colors, $+2(mod\ 6)$ having thirteen colors and $+3(mod\ 6)$ with eight colors. $p_4(n)$: unrestricted sum of partitions of n splits into many summands $+2(mod\ 6)$ having nine colors and $+3(mod\ 6)$ with fourteen colors.

Proof: On rephrasing Theorem 1 to the base q^6 throughout, we obtain

$$\frac{2}{(q_9^{1\pm}, q_9^{2\pm}, q_{14}^{3}; q^6)_\infty} + \frac{6q}{(q_5^{1\pm}, q_{13}^{2\pm}, q_{14}^{3}; q^6)_\infty} - \frac{3}{(q_8^{1\pm}, q_{13}^{2\pm}, q_8^{3}; q^6)_\infty} + \frac{1}{(q_9^{2\pm}, q_8^{3}; q^6)_\infty} = 0.$$

The quotients of the above equation are the generating functions of $p_1(n), p_2(n), p_3(n)$ and $p_4(n)$ respectively. Hence we have an equivalent identity as

$$2\sum_{n=0}^{\infty} p_1(n)q^n + 6q\sum_{n=0}^{\infty} p_2(n)q^n - 3\sum_{n=0}^{\infty} p_3(n)q^n + \sum_{n=0}^{\infty} p_4(n)q^n = 0,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. On equating the coefficient of q^n in the above, we achieve the desired identity. ■

Example: Below Table I verifies the above theorem for $n = 2$.

TABLE I

THE COLORED PARTITIONS FOR $n = 2$

$p_1(2) = 54:$	$1_w+1_w, 1_r+1_r, 1_g+1_g, 1_b+1_b, 1_{bl}+1_{bl}, 1_y+1_y, 1_o+1_o, 1_v+1_v, 1_i+1_i, 1_w+1_r, 1_w+1_g, 1_w+1_b, 1_w+1_{bl}, 1_w+1_y, 1_w+1_o, 1_w+1_v, 1_w+1_i, 1_r+1_g, 1_r+1_b, 1_r+1_{bl}, 1_r+1_y, 1_r+1_o, 1_r+1_v, 1_r+1_i, 1_g+1_b, 1_g+1_{bl}, 1_g+1_y, 1_g+1_o, 1_g+1_v, 1_g+1_i, 1_b+1_{bl}, 1_b+1_y, 1_b+1_o, 1_b+1_v, 1_b+1_i, 1_{bl}+1_y, 1_{bl}+1_o, 1_{bl}+1_v, 1_{bl}+1_i, 1_y+1_o, 1_y+1_v, 1_y+1_i, 1_o+1_v, 1_o+1_i, 1_v+1_i, 2_w, 2_r, 2_g, 2_b, 2_{bl}, 2_y, 2_o, 2_v, 2_i.$
$p_2(1) = 5:$	$1_w, 1_r, 1_g, 1_b, 1_{bl}$
$p_3(2) = 49:$	$1_w+1_w, 1_r+1_r, 1_g+1_g, 1_b+1_b, 1_{bl}+1_{bl}1_y+1_y, 1_o+1_o, 1_v+1_v, 1_w+1_r, 1_w+1_g, 1_w+1_b, 1_w+1_{bl}, 1_w+1_y, 1_w+1_o, 1_w+1_v, 1_r+1_g, 1_r+1_b, 1_r+1_{bl}, 1_r+1_y, 1_r+1_o, 1_r+1_v, 1_g+1_b, 1_g+1_{bl}, 1_g+1_y, 1_g+1_o, 1_g+1_v, 1_b+1_{bl}, 1_b+1_y, 1_b+1_o, 1_b+1_v, 1_{bl}+1_y, 1_{bl}+1_o, 1_{bl}+1_v, 1_y+1_o, 1_y+1_v, 1_o+1_v, 2_w, 2_r, 2_g, 2_b, 2_{bl}, 2_y, 2_o, 2_v, 2_i, 2_p, 2_{br}, 2_m, 2_{pr}.$
$p_4(2) = 9:$	$2_w, 2_r, 2_g, 2_b, 2_{bl}, 2_y, 2_o, 2_v, 2_i$

Theorem 6. *We have for any $n \geq 1$*

$$81p_1(n-1) - 8p_2(n) + p_3(n) + 7p_4(n) = 0,$$

where $p_1(n)$: unrestricted sum of partitions of n splits into parts $\equiv \pm 1(mod\ 6)$ with thirteen colors, $\pm 2(mod\ 6)$ with twenty six colors and $+3(mod\ 6)$ with thirteen colors. $p_2(n)$: absolute sum of parts of n into several parts $\equiv \pm 1, \pm 2(mod\ 6)$ having eighteen colors, and $+3(mod\ 6)$ with nineteen colors. $p_3(n)$: total sum of partitions of n splits into parts $\equiv \pm 2(mod\ 6)$ having eighteen colors and $+3(mod\ 6)$ with seven colors. $p_4(n)$: unrestricted sum of partitions of n splits into parts $\equiv \pm 1(mod\ 6)$ having nine colors, $+2(mod\ 6)$ having eighteen colors and $+3(mod\ 6)$ with thirteen colors.

Proof: On rephrasing Theorem 2, to the base q^6 throughout, we obtain

$$\frac{81q}{(q_{13}^{1\pm}, q_{26}^{2\pm}, q_{13}^{3}; q^6)_\infty} - \frac{8}{(q_{18}^{1\pm}, q_{18}^{2\pm}, q_{19}^{3}; q^6)_\infty} + \frac{1}{(q_{18}^{2\pm}, q_7^{3}; q^6)_\infty} + \frac{7}{(q_9^{1\pm}, q_{18}^{2\pm}, q_{13}^{3}; q^6)_\infty} = 0.$$

The quotients of the above represents the generating functions of $p_1(n), p_2(n), p_3(n)$ and $p_4(n)$ respectively. Hence the above identity can be rewritten as

$$81q\sum_{n=0}^{\infty} p_1(n)q^n - 8\sum_{n=0}^{\infty} p_2(n)q^n + \sum_{n=0}^{\infty} p_3(n)q^n + 7\sum_{n=0}^{\infty} p_4(n)q^n = 0,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. On comparing the coefficient of q^n in the above, we achieve the desired identity. ■

Example: For $n = 2$, the Table-II verifies the above theorem.

TABLE II

THE COLORED PARTITIONS FOR $n = 2$

$p_1(1) = 13$:	$1_w, 1_r, 1_g, 1_b, 1_{bl}, 1_y,$ $1_o, 1_v, 1_i, 1_p, 1_{br}, 1_m, 1_{pe}.$
$p_2(2) = 189$:	$1_w + 1_w, 1_r + 1_r, 1_g + 1_g,$ $1_b + 1_b, 1_{bl} + 1_{bl}, 1_y + 1_y, 1_o + 1_o,$ $1_v + 1_v, 1_i + 1_i$ and similar 180 combinations exists
$p_3(2) = 18$:	$2_w, 2_r, 2_g, 2_b, 2_{bl}, 2_y, 2_o, 2_v, 2_i, 2_p,$ $2_{br}, 2_m, 2_{pr}, 2_{go}, 2_{pe}, 2_a, 2_c, 2_{gr}$
$p_4(2) = 63$:	$1_w + 1_w, 1_r + 1_r, 1_g + 1_g, 1_b + 1_b,$ $1_{bl} + 1_{bl}, 1_y + 1_y, 1_o + 1_o, 1_v + 1_v,$ $1_i + 1_i, 1_w + 1_r, 1_w + 1_g, 1_w + 1_{bl},$ $1_w + 1_b, 1_w + 1_y, 1_w + 1_o, 1_w + 1_v,$ $1_w + 1_i, 1_r + 1_g, 1_r + 1_b, 1_r + 1_{bl},$ $1_r + 1_y, 1_r + 1_o, 1_r + 1_v, 1_r + 1_i,$ $1_g + 1_b, 1_g + 1_{bl}, 1_g + 1_y, 1_g + 1_o,$ $1_g + 1_v, 1_g + 1_i, 1_b + 1_{bl}, 1_b + 1_y,$ $1_b + 1_o, 1_b + 1_v, 1_b + 1_i, 1_{bl} + 1_y,$ $1_{bl} + 1_o, 1_{bl} + 1_v, 1_{bl} + 1_i, 1_y + 1_o,$ $1_y + 1_v, 1_y + 1_i 1_o + 1_v, 1_o + 1_i,$ $1_v + 1_i, 2_w, 2_r$ and 16 colors of the similar type exists.

IV. Conclusion

In the present work, we prove Somos's identities of level 6 by employing Ramanujan's modular equation of degree 3 and and also obtained an alternate proof for the identities proved by B. R. Srivatsa Kumar and G. Sharath [14]. Further, as an application of this one can establish the partition identities for the same. By using Ramanujan's modular equations of various degrees, we claim that Somos's Dedekind-η function identities of different levels can be obtained.

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