Existence and Stability Results for a Coupled System of p-Laplacian Fractional Langevin Equations with Anti-periodic Boundary Conditions

Jinbo Ni*, Jifeng Zhang and Wei Zhang

***Abstract*—In this study, a coupled system of Langevin equations was studied with a p-Laplacian operator and subject to anti-periodic boundary conditions. The existence, uniqueness, and stability in the sense of Ulam were established for the proposed system. Finally, an example was presented to illustrate the application of the obtained results.**

***Index Terms*—Fractional Langevin equation, Coupled system, p-Laplacian operator, Anti-periodic boundary condition.**

I. Introduction

FRACTIONAL differential equations appear extensively in studying the fields of sciences such as physics, biology, control theory and circuits [1-7]. For example, Lutz, Burov et al. [6,7] constructed the following fractional Langevin equation by using fractional differential operator:

$$x'' + \gamma^C D_{0+}^{\alpha} x = \Gamma(t),\ 0 < \alpha < 1,$$

where ${}^C D_{0+}^{\alpha}$ is Caputo fractional derivative of order α.

The Langevin equation (first formulated by Langevin in 1908) effectively describes the evolution of physical phenomena in fluctuating environments [8]. In recent years, the study on solutions of initial and boundary value problems for fractional Langevin differential equations has attracted considerable attention [9-12].

For instance, Zhou et al. [12] considered the following Langevin differential equations with anti-periodic boundary conditions by using Leray-Schaefer's fixed point theorem

$$\begin{cases} D_{0+}^{\beta}\phi_p[(D_{0+}^{\alpha}+\lambda)x(t)]=f(t,x(t),D_{0+}^{\alpha}x(t)), 0 \le t \le 1, \\ x(0)=-\,x(1), \quad D_{0+}^{\alpha}x(0)=-\,D_{0+}^{\alpha}x(1), \end{cases}$$

where $0 < \alpha, \beta \le 1, \lambda \ge 0, 1 < \alpha + \beta \le 2, 0 < q < 1$, and $\phi_p(s) = |s|^{p-2}s, 1 < p \le 2$. The operators D_{0+}^{α} and D_{0+}^{β} are Caputo fractional derivatives.

Recently, many results concerning the coupled system of nonlinear fractional Langevin equations associated with several types of boundary conditions have been established [13-20].

Matar et al. [19] considered the coupled system of nonlinear Langevin equations involving Caputo-Hadamard fractional derivative with non-periodic boundary conditions

$$\begin{cases} D^{\alpha_1}(m_1 D^{\beta_1}+\lambda_1)r_1(t)=\eta_1(t,r_1(t),r_2(t)), t \in J=[1,T], \\ D^{\alpha_2}(m_2 D^{\beta_2}+\lambda_2)r_2(t)=\eta_2(t,r_1(t),r_2(t)), t \in J=[1,T], \\ r_1(1)=\xi_1 r_1(T),\ \ r_1'(1)=0=r_1'(T),\ \ \xi_1 \ne 1, \\ r_2(1)=\xi_2 r_2(T),\ \ r_2'(1)=0=r_2'(T),\ \ \xi_2 \ne 1, \end{cases}$$

where $\alpha_1, \alpha_2 \in (0,1], \beta_1, \beta_2 \in (1,2], m_1, m_2 > 0$, and $D^{\nu}(\nu \in \{\alpha_1, \alpha_2, \beta_1, \beta_2\})$ is the Caputo-Hadamard type fractional derivative. Based on the Banach fixed-point theorem, Krasnoselskii's fixed-point theorem, and Urs's stability approach, the existence, uniqueness and stability results were obtained herein.

Baghani, Alzabut and Nieto [20] studied coupled system of fractional Langevin differential equations associated with anti-periodic boundary conditions of the form

$$\begin{cases} D^{\eta_1}(D^{\xi_1} + \chi_1)z(t) = \psi(t, z(t), w(t)),\ t \in (0,1), \\ D^{\eta_2}(D^{\xi_2} + \chi_2)w(t) = \Phi(t, z(t), w(t)),\ t \in (0,1), \\ z(0) + z(1) = 0,\ D^{\xi_1}z(0) + D^{\xi_1}z(1) = 0, \\ D^{2\xi_1}z(0) + D^{2\xi_1}z(1) = 0,\ w(0) + w(1) = 0, \\ D^{\xi_2}w(0) + D^{\xi_2}w(1) = 0,\ D^{2\xi_2}w(0) + D^{2\xi_2}w(1) = 0, \end{cases}$$

where $0<\xi_1, \xi_2 \le 1$, $1<\eta_1, \eta_2 \le 2$, $D^v (v \in \{\eta_1, \xi_1, \eta_2, \xi_2\})$ is a Caputo type fractional derivative. $D^{m\xi_i}(m,\ i = 1, 2)$ are the sequential fractional derivatives, $\psi, \Phi : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ are given continuous functions and $\chi_1, \chi_2 \in \mathbf{R}$. By using the Banach fixed point theorem, the existence and uniqueness results were proved in this study.

Noteworthy, only few relevant studies have been conducted on coupled systems of p-Laplacian fractional Langevin equations. Moreover, in the present study, a coupled system of fractional Langevin equations was investigated with p-Laplacian operator subject to a coupled anti-periodic boundary conditions given by

$$\begin{cases} {}^C D_{0+}^{\beta_1}\phi_p[({}^C D_{0+}^{\alpha_1}+\lambda_1)x_1(t)]=f_1(t,x_2(t)), t \in (0,1), \\ {}^C D_{0+}^{\beta_2}\phi_p[({}^C D_{0+}^{\alpha_2}+\lambda_2)x_2(t)]=f_2(t,x_1(t)), t \in (0,1), \\ x_1(0)=-x_1(1), {}^C D_{0+}^{\alpha_1}x_1(0)=-{}^C D_{0+}^{\alpha_1}x_1(1), \\ x_2(0)=-x_2(1), {}^C D_{0+}^{\alpha_2}x_2(0)=-{}^C D_{0+}^{\alpha_2}x_2(1), \end{cases} \tag{1}$$

where ${}^C D_{0+}^{\theta}(\theta=\alpha_1, \alpha_2, \beta_1, \beta_2)$ is Caputo fractional derivative, $\alpha_i, \beta_i \in (0,1]$, $i=1,2$, $\lambda_1, \lambda_2 > 0$. Furthermore, $f_1, f_2 : [0,1] \times \mathbf{R} \to [0,1]$ are continuous functions. ϕ_p represents the p-Laplacian operator such that $\phi_p(s)=s|s|^{p-2}$, $p>1$ and $\phi_q = \phi_p^{-1}$ denotes the inverse of p-Laplacian, where $(1/p) + (1/q) = 1$. The existence, uniqueness, and stability of the system (1) were discussed. An example was provided to state the main results.

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II. Preliminaries

Definition 2.1 ([21]). The Riemann-Liouville fractional integral of the order $\alpha > 0$ of function $f : [0, \infty) \to \mathbf{R}$ is defined by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2.2 ([21]). The Caputo derivative of the order $\alpha{>}0$ of function $f : [0, \infty) \to \mathbf{R}$ is defined by

$${}^{C}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $t > 0, n-1 < \alpha < n, n = [\alpha] + 1$.

Lemma 2.1 ([21]). Let $\alpha{>}0$ and function $f(t){\in}AC^n[0, \infty)$, then

$$I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbf{R},\ i = 0, 1, 2, \cdots, n-1, n = [\alpha] + 1$.

Lemma 2.2 ([22]). Let ϕ_p is a nonlinear p-Laplacian operator, then

(i) If $1 < p \leq 2,\ xy > 0$, and $|x|, |y| \geq m > 0$, then

$$|\phi_p(x) - \phi_p(y)| \leq (p-1) m^{p-2} |x - y|.$$

(ii) If $p > 2$, and $|x|, |y| \leq M$, then

$$|\phi_p(x) - \phi_p(y)| \leq (p-1) M^{p-2} |x - y|.$$

Definition 2.3 ([23]). The spectral radius of a matrix $U \in C^{n\times n}$ is defined by

$$\Upsilon(U) = max\{|\beta_1|, |\beta_2|, \cdots, |\beta_n|\},$$

where $\beta_1, \beta_2, \cdots, \beta_n$ are the eigenvalues. A matrix U converges to zero if the spectral radius satisfies $\Upsilon(U) < 1$.

Theorem 2.1 ([23]). Assuming the operator $T_1, T_2 : X \times X \to X$ for the operator system

$$\begin{aligned} T_1(x_1, x_2) &= x_1, \\ T_2(x_1, x_2) &= x_2, \end{aligned} \tag{2}$$

for all $x_i, \bar{x}_i \in X, i = 1, 2$, holds the following system of inequations

$$\begin{aligned} ||T_1(x_1, x_2) - T_1(\bar{x}_1, \bar{x}_2)|| &\leq l_1 ||x_1 - \bar{x}_1|| + l_2 ||x_2 - \bar{x}_2||, \\ ||T_2(x_1, x_2) - T_2(\bar{x}_1, \bar{x}_2)|| &\leq l_3 ||x_1 - \bar{x}_1|| + l_4 ||x_2 - \bar{x}_2||, \end{aligned}$$

where $(x_1, x_2), (\bar{x}_1, \bar{x}_2) \in X \times X$ are exact and approximate solutions, respectively, and $l_1, l_2, l_3, l_4 > 0$. If the matrix

$$U = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix} \to 0,$$

then the fixed points of (2) are stable in the sense of Ulam-Hyers.

III. Main results

Denoting $X \times X{=}C([0, 1], \mathbf{R}) \times C([0, 1], \mathbf{R})$, the Banach space equipped with the norm $||(x_1, x_2)|| = ||x_1|| + ||x_2||$, and the topological norm $||\cdot|| = \max_{t\in[0,1]} |\cdot|$. The operator $T : X \times X \to X \times X$ is defined as follows:

$$T(x_1, x_2)(t) := (T_1(x_1, x_2)(t), T_2(x_1, x_2)(t)),$$

$$\begin{aligned} T_1(x_1, x_2)(t) &= -\frac{\lambda_1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} x_1(s) ds \\ &+ \frac{\lambda_1}{2\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} x_1(s) ds \\ &+ \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \phi_q\Big(\frac{1}{\Gamma(\beta_1)} \int_0^s (s-\tau)^{\beta_1-1} \\ &\times f_1(\tau, x_2(\tau)) d\tau - \frac{1}{2\Gamma(\beta_1)} \int_0^1 (1-\tau)^{\beta_1-1} \\ &\times f_1(\tau, x_2(\tau)) d\tau\Big) ds - \frac{1}{2\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} \\ &\times \phi_q\Big(\frac{1}{\Gamma(\beta_1)} \int_0^s (s-\tau)^{\beta_1-1} f_1(\tau, x_2(\tau)) d\tau \\ &- \frac{1}{2\Gamma(\beta_1)} \int_0^1 (1-\tau)^{\beta_1-1} f_1(\tau, x_2(\tau)) d\tau\Big) ds, \end{aligned}$$

$$\begin{aligned} T_2(x_1, x_2)(t) &= -\frac{\lambda_2}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} x_2(s) ds \\ &+ \frac{\lambda_1}{2\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} x_2(s) ds \\ &+ \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \phi_q\Big(\frac{1}{\Gamma(\beta_2)} \int_0^s (s-\tau)^{\beta_2-1} \\ &\times f_2(\tau, x_1(\tau)) d\tau - \frac{1}{2\Gamma(\beta_2)} \int_0^1 (1-\tau)^{\beta_2-1} \\ &\times f_2(\tau, x_1(\tau)) d\tau\Big) ds - \frac{1}{2\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \\ &\times \phi_q\Big(\frac{1}{\Gamma(\beta_2)} \int_0^s (s-\tau)^{\beta_2-1} f_2(\tau, x_1(\tau)) d\tau \\ &- \frac{1}{2\Gamma(\beta_2)} \int_0^1 (1-\tau)^{\beta_2-1} f_2(\tau, x_1(\tau)) d\tau\Big) ds. \end{aligned}$$

To obtain the main results, the following hypotheses are required.

(H_1) There exist continuous non-negative functions $a_i(t)$, $b_i(t) \in C([0, 1], \mathbf{R}^+), i = 1, 2$, such that for $x_1, x_2 \in X$,

$$\begin{aligned} |f_1(t, x_2(t))| &\leq \phi_p(a_1(t) + b_1(t)|x_2|), \\ |f_2(t, x_1(t))| &\leq \phi_p(a_2(t) + b_2(t)|x_1|). \end{aligned}$$

(H_2) There exist positive constants L_1, L_2 such that for x_1, x_2, $\bar{x}_1$, $\bar{x}_2 \in X$,

$$\begin{aligned} |f_1(t, x_2(t)) - f_1(t, \bar{x}_2(t))| &\leq L_1 ||x_2 - \bar{x}_2||, \\ |f_2(t, x_1(t)) - f_2(t, \bar{x}_1(t))| &\leq L_2 ||x_1 - \bar{x}_1||. \end{aligned}$$

(H_3) Existing functions $g_1(t), g_2(t)$ satisfy $\int_0^1 (1-s)^{\beta_1-1} \times g_1(s) ds := M_{f_1} > 0$, $\int_0^1 (1-s)^{\beta_2-1} g_2(s) ds := M_{f_2} > 0$, respectively, and for all $t \in [0, 1]$, $x_1, x_2 \in X$, such that

$$|f_1(t, x_2(t))| \leq g_1(t), |f_2(t, x_1(t))| \leq g_2(t).$$

Lemma 3.1 Let $F_1, F_2 \in C([0,1],\mathbf{R})$ and $x_1, x_2 \in X$. Then the solutions of the following coupled system of equations

$$\begin{cases} {}^C D_{0+}^{\beta_1}\phi_p[({}^C D_{0+}^{\alpha_1}+\lambda_1)x_1(t)] = F_1(t), t\in(0,1), \\ {}^C D_{0+}^{\beta_2}\phi_p[({}^C D_{0+}^{\alpha_2}+\lambda_2)x_2(t)] = F_2(t), t\in(0,1), \\ x_1(0)=-x_1(1), {}^C D_{0+}^{\alpha_1}x_1(0) = -{}^C D_{0+}^{\alpha_1}x_1(1), \\ x_2(0)=-x_2(1), {}^C D_{0+}^{\alpha_2}x_2(0) = -{}^C D_{0+}^{\alpha_2}x_2(1). \end{cases} \tag{3}$$

are given by

$$\begin{aligned} x_1(t) &= -\lambda_1 I_{0+}^{\alpha_1}x_1(t) + \frac{1}{2}\lambda_1 I_{0+}^{\alpha_1}x_1(t)|_{t=1} \\ &+ I_{0+}^{\alpha_1}\phi_q(I_{0+}^{\beta_1}F_1(t) - \frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1}) \\ &-\frac{1}{2}I_{0+}^{\alpha_1}\phi_q(I_{0+}^{\beta_1}F_1(t) - \frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1})|_{t=1}, \end{aligned}$$

$$\begin{aligned} x_2(t) &= -\lambda_2 I_{0+}^{\alpha_2}x_2(t) + \frac{1}{2}\lambda_2 I_{0+}^{\alpha_2}x_2(t)|_{t=1} \\ &+ I_{0+}^{\alpha_2}\phi_q(I_{0+}^{\beta_2}F_2(t) - \frac{1}{2}I_{0+}^{\beta_2}F_2(t)|_{t=1}) \\ &-\frac{1}{2}I_{0+}^{\alpha_2}\phi_q(I_{0+}^{\beta_2}F_2(t) - \frac{1}{2}I_{0+}^{\beta_2}F_2(t)|_{t=1})|_{t=1}. \end{aligned}$$

Proof. Applying the integral operator $I_{0+}^{\beta_1}$ on the first equation of the system (3) and using Lemma 2.1, the following equation is obtained:

$$\phi_p[({}^C D_{0+}^{\alpha_1}+\lambda_1)x_1(t)] = I_{0+}^{\beta_1}F_1(t) + c_0,\ c_0\in\mathbf{R}, \tag{4}$$

since $\phi_p^{-1}(\cdot) = \phi_q$, Eq. (4) is equivalent to the following equation

$${}^C D_{0+}^{\alpha_1}x_1(t) + \lambda_1 x_1(t) = \phi_q(I_{0+}^{\beta_1}F_1(t)+c_0), \tag{5}$$

by using $x_1(0)=-x_1(1)$ and ${}^C D_{0+}^{\alpha_1}x_1(0) = -{}^C D_{0+}^{\alpha_1}x_1(1)$ in Eq. (5), we obtain

$${}^C D_{0+}^{\alpha_1}x_1(0) + \lambda_1 x_1(0) = \phi_q(c_0),$$

$${}^C D_{0+}^{\alpha_1}x_1(1) + \lambda_1 x_1(1) = \phi_q(I_{0+}^{\beta_1}F_1(t)|_{t=1}+c_0).$$

Simultaneously, considering the equations mentioned above, the following relation can be obtained:

$$c_0 = -\frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1}.$$

Thus Eq. (5) becomes

$${}^C D_{0+}^{\alpha_1}x_1(t)+\lambda_1 x_1(t)=\phi_q(I_{0+}^{\beta_1}F_1(t)-\frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1}), \tag{6}$$

after applying the operator $I_{0+}^{\alpha_1}$ on both sides of Eq. (6). With the help of Lemma 2.1, Eq. (7) is obtained

$$\begin{aligned} x_1(t) &= I_{0+}^{\alpha_1}\phi_q(I_{0+}^{\beta_1}F_1(t) - \frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1}) \\ &-\lambda_1 I_{0+}^{\alpha_1}x_1(t) + c_1,\ c_1\in\mathbf{R}. \end{aligned} \tag{7}$$

Using the boundary condition $x_1(0) = -x_1(1)$ in Eq. (7), we obtain the following equation

$$\begin{aligned} c_1 &= \frac{1}{2}\lambda_1 I_{0+}^{\alpha_1}x_1(t)|_{t=1} - \frac{1}{2}I_{0+}^{\alpha_1}\phi_q(I_{0+}^{\beta_1}F_1(t) \\ &-\frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1})|_{t=1}. \end{aligned}$$

Putting the value of c_1 in Eq. (7), the following solution is derived:

$$\begin{aligned} x_1(t) &= -\lambda_1 I_{0+}^{\alpha_1}x_1(t) + \frac{1}{2}\lambda_1 I_{0+}^{\alpha_1}x_1(t)|_{t=1} \\ &+ I_{0+}^{\alpha_1}\phi_q(I_{0+}^{\beta_1}F_1(t) - \frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1}) \\ &-\frac{1}{2}I_{0+}^{\alpha_1}\phi_q(I_{0+}^{\beta_1}F_1(t) - \frac{1}{2}I_{0+}^{\beta_1}F_1(t)|_{t=1})|_{t=1}. \end{aligned}$$

Similarly, the following relation is obtained by repeating the same step for the second equation of the system of equations (3).

$$\begin{aligned} x_2(t) &= -\lambda_2 I_{0+}^{\alpha_2}x_2(t) + \frac{1}{2}\lambda_2 I_{0+}^{\alpha_2}x_2(t)|_{t=1} \\ &+ I_{0+}^{\alpha_2}\phi_q(I_{0+}^{\beta_2}F_2(t) - \frac{1}{2}I_{0+}^{\beta_2}F_2(t)|_{t=1}) \\ &-\frac{1}{2}I_{0+}^{\alpha_2}\phi_q(I_{0+}^{\beta_2}F_2(t) - \frac{1}{2}I_{0+}^{\beta_2}F_2(t)|_{t=1})|_{t=1}. \end{aligned}$$

Theorem 3.1 If $f_1, f_2 : X\times X \to X\times X$ are continuous functions satisfying (H_1) and

$$\begin{aligned} &\frac{3\lambda_1}{2\Gamma(\alpha_1+1)} + \frac{3^q||b_2||}{2^q\Gamma(\alpha_2+1)(\Gamma(\beta_2+1))^{q-1}} < 1, \\ &\frac{3\lambda_2}{2\Gamma(\alpha_2+1)} + \frac{3^q||b_1||}{2^q\Gamma(\alpha_1+1)(\Gamma(\beta_1+1))^{q-1}} < 1. \end{aligned} \tag{8}$$

Then the system (1) has at least one solution.

Proof. In the first step, the operator $T : X\times X \to X\times X$ is proven to be a continuous operator. In fact, for arbitrary constants $M_1, M_2 > 0$, two open bounded subsets are defined as follows:

$$\Omega_1 = \{x_1\in X : ||x_1|| \le M_1\},$$
$$\Omega_2 = \{x_2\in X : ||x_2|| \le M_2\}.$$

By the continuity of f_1, f_2, there exist constants L_{f_1}, L_{f_2}, such that

$$\begin{aligned} &\Big|\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^s (s-\tau)^{\beta_1-1}f_1(\tau,x_2(\tau))d\tau \\ &-\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-\tau)^{\beta_1-1}f_1(\tau,x_2(\tau))d\tau\Big)\Big| \le L_{f_1}, \\ &\Big|\phi_q\Big(\frac{1}{\Gamma(\beta_2)}\int_0^s (s-\tau)^{\beta_2-1}f_2(\tau,x_1(\tau))d\tau \\ &-\frac{1}{2\Gamma(\beta_2)}\int_0^1 (1-\tau)^{\beta_2-1}f_2(\tau,x_1(\tau))d\tau\Big)\Big| \le L_{f_2}, \end{aligned}$$

so, for any $(x_1, x_2)\in\Omega_1\times\Omega_2$,

$$|T_1(x_1,x_2)(t)| \le \frac{3(\lambda_1 M_1 + L_{f_1})}{2\Gamma(\alpha_1+1)},$$
$$|T_2(x_1,x_2)(t)| \le \frac{3(\lambda_2 M_2 + L_{f_2})}{2\Gamma(\alpha_2+1)}.$$

Thus,

$$\begin{aligned} ||T(x_1,x_2)|| &= ||T_1(x_1,x_2)|| + ||T_2(x_1,x_2)|| \\ &\le \frac{3(\lambda_1 M_1 + L_{f_1})}{2\Gamma(\alpha_1+1)} + \frac{3(\lambda_2 M_2 + L_{f_2})}{2\Gamma(\alpha_2+1)}. \end{aligned}$$

Thus the operator T is uniformly bounded. Next, T is proven to be equi-continuous. For any $0 \le t_1 \le t_2 \le 1$,

$$
\begin{aligned}
&|T_1(x_1,x_2)(t_2) - T_1(x_1,x_2)(t_1)| \\
&\le \Big|\frac{\lambda_1}{\Gamma(\alpha_1)}\int_0^{t_2}(t_2-s)^{\alpha_1-1}x_1(s)ds \\
&\quad -\frac{\lambda_1}{\Gamma(\alpha_1)}\int_0^{t_1}(t_1-s)^{\alpha_1-1}x_1(s)ds\Big| \\
&\quad +\Big|\frac{1}{\Gamma(\alpha_1)}\int_0^{t_2}(t_2-s)^{\alpha_1-1}\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^{s}(s-\tau)^{\beta_1-1} \\
&\quad \times f_1(\tau,x_2(\tau))d\tau - \frac{1}{2\Gamma(\beta_1)}\int_0^1(1-\tau)^{\beta_1-1} \\
&\quad \times f_1(\tau,x_2(\tau))d\tau\Big)ds - \frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}(t_1-s)^{\alpha_1-1} \\
&\quad \times\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^{s}(s-\tau)^{\beta_1-1}f_1(\tau,x_2(\tau))d\tau \\
&\quad -\frac{1}{2\Gamma(\beta_1)}\int_0^1(1-\tau)^{\beta_1-1}f_1(\tau,x_2(\tau))d\tau\Big)ds\Big| \\
&\le \frac{\lambda_1}{\Gamma(\alpha_1)}\Big|\int_0^{t_1}((t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1})x_1(s)ds \\
&\quad +\int_{t_1}^{t_2}(t_1-s)^{\alpha_1-1}x_1(s)ds\Big| \\
&\quad +\frac{1}{\Gamma(\alpha_1)}\Big|\int_0^{t_1}((t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}) \\
&\quad \times\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^{s}(s-\tau)^{\beta_1-1}f_1(\tau,x_2(\tau))d\tau \\
&\quad -\frac{1}{2\Gamma(\beta_1)}\int_0^1(1-\tau)^{\beta_1-1}f_1(\tau,x_2(\tau))d\tau\Big)ds \\
&\quad +\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^{s}(s-\tau)^{\beta_1-1} \\
&\quad \times f_1(\tau,x_2(\tau))d\tau - \frac{1}{2\Gamma(\beta_1)}\int_0^1(1-\tau)^{\beta_1-1} \\
&\quad \times f_1(\tau,x_2(\tau))d\tau\Big)ds\Big| \\
&\le \frac{\lambda_1 M_1 + L_{f_1}}{\Gamma(\alpha_1+1)}(t_1^{\alpha_1} - t_2^{\alpha_1} + 2(t_2-t_1)^{\alpha_1}).
\end{aligned}
$$

Analogously, it can be obtained that

$$
\begin{aligned}
&|T_2(x_1,x_2)(t_2) - T_2(x_1,x_2)(t_1)| \\
&\le \frac{\lambda_2 M_2 + L_{f_2}}{\Gamma(\alpha_2+1)}(t_1^{\alpha_2} - t_2^{\alpha_2} + 2(t_2-t_1)^{\alpha_2}).
\end{aligned}
$$

As $t_2 \to t_1$, the operator T is equi-continuous according to the Arzelá-Ascoli theorem. The operator T is observed to be completely continuous.

In the second step, we consider the following set.

$$
\Omega = \{(x_1,x_2) \in X | (x_1,x_2) = \mu T(x_1,x_2), 0 < \mu < 1\}.
$$

We now show that it is bounded. Let $(x_1,x_2) \in \Omega$, then $(x_1,x_2) = \mu T(x_1,x_2), \mu \in (0,1)$, we have

$$
x_1(t) = \mu T_1(x_1,x_2)(t), x_2(t) = \mu T_2(x_1,x_2)(t).
$$

According to (H_1), it is found that

$$
\begin{aligned}
&|x_1(t)| = \mu|T_1(x_1,x_2)(t)| \\
&\le \frac{3\lambda_1||x_1||_\infty}{2\Gamma(\alpha_1+1)} + \frac{3}{2\Gamma(\alpha_1+1)} \\
&\quad \times\phi_q\left(\frac{3}{2\Gamma(\beta_1+1)}\phi_p(||a_1||_\infty + ||b_1||_\infty||x_2||_\infty)\right) \\
&\le \frac{3\lambda_1||x_1||_\infty}{2\Gamma(\alpha_1+1)} + \frac{3}{2\Gamma(\alpha_1+1)}\left(\frac{3}{2\Gamma(\beta_1+1)}\right)^{q-1} \\
&\quad \times(||a_1||_\infty + ||b_1||_\infty||x_2||_\infty) \\
&= \frac{3\lambda_1||x_1||_\infty}{2\Gamma(\alpha_1+1)} + \frac{3^q(||a_1||_\infty + ||b_1||_\infty||x_2||_\infty)}{2^q\Gamma(\alpha_1+1)(\Gamma(\beta_1+1))^{q-1}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|x_2(t)| = \mu|T_2(x_1,x_2)(t)| \\
&\le \frac{3\lambda_2||x_2||_\infty}{2\Gamma(\alpha_2+1)} + \frac{3^q(||a_2||_\infty + ||b_2||_\infty||x_1||_\infty)}{2^q\Gamma(\alpha_2+1)(\Gamma(\beta_2+1))^{q-1}}.
\end{aligned}
$$

Consequently, it yields

$$
\begin{aligned}
&||x_1|| + ||x_2|| \\
&\le \Big[\frac{3\lambda_1}{2\Gamma(\alpha_1+1)} + \frac{3^q||b_2||}{2^q\Gamma(\alpha_2+1)(\Gamma(\beta_2+1))^{q-1}}\Big]||x_1|| \\
&\quad +\Big[\frac{3\lambda_2}{2\Gamma(\alpha_2+1)} + \frac{3^q||b_1||}{2^q\Gamma(\alpha_1+1)(\Gamma(\beta_1+1))^{q-1}}\Big]||x_2|| \\
&\quad +\frac{3^q||a_1||}{2^q\Gamma(\alpha_1+1)(\Gamma(\beta_1+1))^{q-1}} + \frac{3^q||a_2||}{2^q\Gamma(\alpha_2+1)(\Gamma(\beta_2+1))^{q-1}}.
\end{aligned}
$$

Using the condition described in Eq. (8), the following relation is obtained.

$$
\begin{aligned}
&||(x_1,x_2)|| \\
&\le \frac{\frac{3^q||a_1||}{2^q\Gamma(\alpha_1+1)(\Gamma(\beta_1+1))^{q-1}} + \frac{3^q||a_2||}{2^q\Gamma(\alpha_2+1)(\Gamma(\beta_2+1))^{q-1}}}{\Delta},
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta = \min\Big\{&1 - \Big(\frac{3\lambda_1}{2\Gamma(\alpha_1+1)} + \frac{3^q||b_2||}{2^q\Gamma(\alpha_2+1)(\Gamma(\beta_2+1))^{q-1}}\Big), \\
&1 - \Big(\frac{3\lambda_2}{2\Gamma(\alpha_2+1)} + \frac{3^q||b_1||}{2^q\Gamma(\alpha_1+1)(\Gamma(\beta_1+1))^{q-1}}\Big)\Big\}.
\end{aligned}
$$

It shows that $||(x_1,x_2)||$ is bounded. As a consequence of Schaefer's fixed point theorem, it is thus concluded that system (1) has at least one solution. It completes the proof.

Theorem 3.2 If $1 < p \le 2$, $f_1, f_2 : X \times X \to X \times X$ are continuous functions satisfying (H_2) and (H_3), then the system (1) has a unique solution as per the following relation

$$
\begin{aligned}
d = \max\Big\{&\frac{3\lambda_1}{2\Gamma(\alpha_1+1)} + \frac{9(q-1)\triangle_2^{q-2}L_2}{4\Gamma(\alpha_2+1)\Gamma(\beta_2+1)}, \\
&\frac{3\lambda_2}{2\Gamma(\alpha_2+1)} + \frac{9(q-1)\triangle_1^{q-2}L_1}{4\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}\Big\} < 1, \qquad (9)
\end{aligned}
$$

where $\Delta_1 = \frac{3M_{f_1}}{2\Gamma(\beta_1)}$, $\Delta_2 = \frac{3M_{f_2}}{2\Gamma(\beta_2)}$.

Proof According to (H_3), for all $t \in [0,1]$ and $x_1, x_2 \in X$, it is estimated that

$$\begin{aligned}&\Big|\frac{1}{\Gamma(\beta_1)}\int_0^t (t-s)^{\beta_1-1} f_1(s,x_2(s))ds\\&\quad-\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1} f_1(s,x_2(s))ds\Big|\\&\leq \Big|\frac{1}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1} g_1(s)ds\Big|\\&\quad+\Big|\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1} g_1(s)ds\Big|\leq\triangle_1.\end{aligned}$$

Analogously,

$$\begin{aligned}&\Big|\frac{1}{\Gamma(\beta_2)}\int_0^t (t-s)^{\beta_2-1} f_2(s,x_1(s))ds\\&\quad-\frac{1}{2\Gamma(\beta_2)}\int_0^1 (1-s)^{\beta_2-1} f_2(s,x_1(s))ds\Big|\leq\triangle_2.\end{aligned}$$

If $1 < p \leq 2$, then $q \geq 2$, for $x_i, \bar{x}_i \in X (i = 1, 2)$ in the light of (ii) in Lemma 2.2. Assuming that (H_2) holds, we obtain the following expression.

$$\begin{aligned}&\Big|\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^t (t-s)^{\beta_1-1} f_1(s,x_2(s))ds\\&\quad+\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1} f_1(s,x_2(s))ds\Big)\\&\quad-\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^t (t-s)^{\beta_1-1} f_1(s,\bar{x}_2(s))ds\\&\quad+\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1} f_1(s,\bar{x}_2(s))ds\Big)\Big|\\&\leq \frac{3(q-1)\triangle_1^{q-2}}{2\Gamma(\beta_1+1)}\Big|f_1(s,x_2(s))-f_1(s,\bar{x}_2(s))\Big|\\&\leq \frac{3(q-1)\triangle_1^{q-2}L_1}{2\Gamma(\beta_1+1)}||x_2-\bar{x}_2||.\end{aligned}$$

Similarly, the following equations are derived as well.

$$\begin{aligned}&\Big|\phi_q\Big(\frac{1}{\Gamma(\beta_2)}\int_0^t (t-s)^{\beta_2-1} f_2(s,x_1(s))ds\\&\quad+\frac{1}{2\Gamma(\beta_2)}\int_0^1 (1-s)^{\beta_2-1} f_2(s,x_1(s))ds\Big)\\&\quad-\phi_q\Big(\frac{1}{\Gamma(\beta_2)}\int_0^t (t-s)^{\beta_2-1} f_2(s,\bar{x}_1(s))ds\\&\quad+\frac{1}{2\Gamma(\beta_2)}\int_0^1 (1-s)^{\beta_2-1} f_2(s,\bar{x}_1(s))ds\Big)\Big|\\&\leq \frac{3(q-1)\triangle_2^{q-2}L_2}{2\Gamma(\beta_2+1)}||x_1-\bar{x}_1||.\end{aligned}$$

For $(x_1, x_2), (\bar{x}_1, \bar{x}_2) \in X \times X$, the following inequality is obtained

$$\begin{aligned}&|T_1(x_1,x_2)(t)-T_1(\bar{x}_1,\bar{x}_2)(t)|\\&\leq\frac{\lambda_1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}|x_1(s)-\bar{x}_1(s)|ds\\&\quad+\frac{\lambda_1}{2\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}|x_1(s)-\bar{x}_1(s)|ds\\&\quad+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\Big|\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^s (s-\tau)^{\beta_1-1}\\&\quad\times f_1(\tau,x_2(\tau))d\tau+\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-\tau)^{\beta_1-1}\\&\quad\times f_1(\tau,x_2(\tau))d\tau\Big)\\&\quad-\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^s (s-\tau)^{\beta_1-1} f_1(\tau,\bar{x}_2(\tau))d\tau\\&\quad+\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-\tau)^{\beta_1-1} f_1(\tau,\bar{x}_2(\tau))d\tau\Big)\Big|ds\\&\quad+\frac{1}{2\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}\Big|\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^s (s-\tau)^{\beta_1-1}\\&\quad\times f_1(\tau,x_2(\tau))d\tau+\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-\tau)^{\beta_1-1}\\&\quad\times f_1(\tau,x_2(\tau))d\tau\Big)\\&\quad-\phi_q\Big(\frac{1}{\Gamma(\beta_1)}\int_0^s (s-\tau)^{\beta_1-1} f_1(\tau,\bar{x}_2(\tau))d\tau\\&\quad+\frac{1}{2\Gamma(\beta_1)}\int_0^1 (1-\tau)^{\beta_1-1} f_1(\tau,\bar{x}_2(\tau))d\tau\Big)\Big|ds\\&\leq\frac{3\lambda_1}{2\Gamma(\alpha_1+1)}||x_1-\bar{x}_1||+\frac{9(q-1)\triangle_1^{q-2}L_1}{4\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}||x_2-\bar{x}_2||.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&|T_2(x_1,x_2)(t)-T_2(\bar{x}_1,\bar{x}_2)(t)|\\&\leq\frac{3\lambda_2}{2\Gamma(\alpha_2+1)}||x_2-\bar{x}_2||\\&\quad+\frac{9(q-1)\triangle_2^{q-2}L_2}{4\Gamma(\alpha_2+1)\Gamma(\beta_2+1)}||x_1-\bar{x}_1||.\end{aligned}$$

Hence,

$$\begin{aligned}&|T(x_1,x_2)(t)-T(\bar{x}_1,\bar{x}_2)(t)|\\&=|T_1(x_1,x_2)(t)-T_1(\bar{x}_1,\bar{x}_2)(t)|\\&\quad+|T_2(x_1,x_2)(t)-T_2(\bar{x}_1,\bar{x}_2)(t)|\\&\leq\Big(\frac{3\lambda_1}{2\Gamma(\alpha_1+1)}+\frac{9(q-1)\triangle_2^{q-2}L_2}{4\Gamma(\alpha_2+1)\Gamma(\beta_2+1)}\Big)||x_1-\bar{x}_1||\\&\quad+\Big(\frac{3\lambda_2}{2\Gamma(\alpha_2+1)}+\frac{9(q-1)\triangle_1^{q-2}L_1}{4\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}\Big)||x_2-\bar{x}_2||\\&\leq d||(x_1,x_2)-(\bar{x}_1,\bar{x}_2)||.\end{aligned}$$

Therefore, T has unique solutions because condition (9) is satisfied.

Theorem 3.3 If assumptions (H_2) and (H_3) hold, the matrix U converges to zero, then the system of equations (1) is stable in the sense of Ulam-Hyers stability.

Proof. According to Theorem 3.2, we have

$$\begin{aligned}&||T_1(x_1,x_2)-T_1(\bar{x}_1,\bar{x}_2)||\\&\leq l_1||x_1-\bar{x}_1||+l_2||x_2-\bar{x}_2||,\end{aligned}\tag{10}$$

where

$$l_1=\frac{3\lambda_1}{2\Gamma(\alpha_1+1)},\ l_2=\frac{9(q-1)\triangle_1^{q-2}L_1}{4\Gamma(\alpha_1+1)\Gamma(\beta_1+1)},$$

l_1 and l_2 are non-negative real numbers. Moreover, by using a similar approach, the following equation is obtained.

$$\begin{aligned}&||T_2(x_1,x_2)-T_2(\bar{x}_1,\bar{x}_2)||\\&\leq l_3||x_1-\bar{x}_1||+l_4||x_2-\bar{x}_2||,\end{aligned}\tag{11}$$

where

$$l_3=\frac{3\lambda_2}{2\Gamma(\alpha_2+1)},l_4=\frac{9(q-1)\triangle_2^{q-2}L_2}{4\Gamma(\alpha_2+1)\Gamma(\beta_2+1)},$$

l_3 and l_4 are non-negative real numbers. Combining (10) and (11) yields the following relation.

$$U = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}.$$

Since U converges to zero when combined with Theorem 2.1, the system of equations (1) is UH stable.

IV. Examples

Example 4.1 Consider the following coupled system

$$\begin{cases} {}^C D_{0+}^{4/5}\phi_{3/2}[({}^C D_{0+}^{1/2}+(1/10))x_1(t)]=f_1(t,x_2(t)), \\ {}^C D_{0+}^{4/5}\phi_{3/2}[({}^C D_{0+}^{1/2}+(1/5))x_1(t)]=f_2(t,x_1(t)), \\ x_1(0)=-x_1(1), {}^C D_{0+}^{1/2}x_1(0)=-{}^C D_{0+}^{1/2}x_1(1), \\ x_2(0)=-x_2(1), {}^C D_{0+}^{1/2}x_2(0)=-{}^C D_{0+}^{1/2}x_2(1). \end{cases} \tag{12}$$

Here,

$$\alpha_1=\alpha_2=\frac{1}{2}, \beta_1=\beta_2=\frac{4}{5}, p=\frac{3}{2},$$
$$q=3, \lambda_1=\frac{1}{10}, \lambda_2=\frac{1}{5}.$$

For demonstrating the application of Theorem 3.1, it is assumed that,

$$f_1(t,x_2(t))=\frac{1}{5}t+\frac{1}{10}x_2(t),$$
$$f_2(t,x_1(t))=\frac{1}{8}t+\frac{1}{20}x_1(t).$$

Then $||a_1||=\frac{1}{5}, ||a_2||=\frac{1}{8}, ||b_1||=\frac{1}{10}, ||b_2||=\frac{1}{20}$. By routine calculation we can get assumption (H_1) holds. Next, the following calculations are obtained.

$$\frac{3\times 1/10}{2\Gamma(3/2)}+\frac{3^3\times 1/20}{2^3\Gamma(3/2)(\Gamma(9/5))^2}\approx 0.389<1,$$
$$\frac{3\times 1/5}{2\Gamma(3/2)}+\frac{3^3\times 1/10}{2^3\Gamma(3/2)(\Gamma(9/5))^2}\approx 0.778<1.$$

According to Theorem 3.1, the coupled system (12) has at least one solution.

For illustrating Theorem 3.2, consider the following situation

$$f_1(t,x_2(t))=\frac{(1-t)\sin(x_2(t))}{30},$$
$$f_2(t,x_1(t))=\frac{(1-t)\sin(x_1(t))}{20}.$$

There exist $g_1(t)=\frac{1-t}{30}, g_2(t)=\frac{1-t}{20}, L_1=\frac{1}{30}, L_2=\frac{1}{20}$, then

$$M_{f_1}=\int_0^1(1-s)^{\beta_1-1}g_1(s)ds=\frac{1}{54},$$
$$M_{f_2}=\int_0^1(1-s)^{\beta_2-1}g_2(s)ds=\frac{1}{36},$$

such that

$$|f_1(t,x_2(t))|\le\frac{1-t}{30}=g_1(t),$$
$$|f_2(t,x_1(t))|\le\frac{1-t}{20}=g_2(t),$$

for arbitrary $t\in[0,1]$ and $x_1,\bar{x}_1,x_2,\bar{x}_2\in X$, we have

$$\begin{aligned}&|f_1(t,x_2(t))-f_1(t,\bar{x}_2(t))|\\&=\frac{1-t}{30}|\sin x_2(t)-\sin\bar{x}_2(t)|\\&\le\frac{1}{30}||x_2-\bar{x}_2||,\end{aligned}$$

and

$$\begin{aligned}&|f_2(t,x_1(t))-f_2(t,\bar{x}_1(t))|\\&=\frac{1-t}{20}|\sin x_1(t)-\sin\bar{x}_1(t)|\\&\le\frac{1}{20}||x_1-\bar{x}_1||.\end{aligned}$$

Thus, the assumption (H_2) is valid. By using the given data, we find that,

$$\Delta_1=\frac{3M_{f_1}}{2\Gamma(\beta_1)}=0.0239,\ \Delta_2=\frac{3M_{f_2}}{2\Gamma(\beta_2)}=0.0358,$$

so

$$\begin{aligned}d=\max&\left\{\frac{3/10}{2\Gamma(3/2)}+\frac{(9\times 2\times 0.0358)/20}{4\times\Gamma(3/2)\Gamma(9/5)},\right.\\&\left.\frac{3/5}{2\times\Gamma(3/2)}+\frac{(9\times 2\times 0.0239)/30}{4\times\Gamma(3/2)\Gamma(9/5)}\right\}\\&=\max\{0.1791,0.3429\}=0.3429<1.\end{aligned}$$

Thus, all the conditions of Theorem 3.2 hold, and there is a unique solution for the system of equations (12).

Considering the data given in Theorem 3.2, the values are calculated as follows:

$$l_1=\frac{3/10}{2\Gamma(3/2)}\approx 0.1693, l_2=\frac{(9\times 2\times 0.0239)/30}{4\times\Gamma(3/2)\Gamma(9/5)}\approx 0.0043,$$
$$l_3=\frac{3/5}{2\Gamma(3/2)}\approx 0.3386, l_4=\frac{(9\times 2\times 0.0358)/20}{4\times\Gamma(3/2)\Gamma(9/5)}\approx 0.0098.$$

The matrix U takes the following values

$$U=\begin{pmatrix}0.1693 & 0.0043\\ 0.3386 & 0.0098\end{pmatrix}.$$

The matrix U has eigenvalues equal to 0.178 and 0.0011. Clearly $\Upsilon(U)<1$, hence, by Definition 2.3, matrix U converges to zero. Therefore, the system of equations (12) is stable with respect to Ulam-Hyers.

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