Tiling Systems and Domino Systems for Partial Array Languages

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Abstract—**In this paper we study the closure properties of partial array tiling systems and define the new class of partial array languages termed as $hv-$local partial array languages. Further we develop a special kind of partial array tiling systems called partial array domino systems with the relevance of family of $hv-$local partial array languages. We prove that the class of partial array languages generated by partial array tiling systems is equal to the class of partial array languages generated by partial array domino systems. Finally we study the automata characterization for $hv-$local partial array languages.**

Keywords: partial languages, local partial languages, tiling system, domino system.

1 Introduction

In DNA computation, DNA molecules are viewed as finite strings that encode information. During the encoding process, some parts of the information may be unseen, and these parts can be visible by using partial words. Research has been extensively conducted on how to efficiently learn a concept through its representations. Efficient methods for learning concepts through the representation of the concepts have been presented in the literature [1, 9, 10, 12, 14, 15, 16]. In 1974, Fischer and Paterson introduced partial words as strings with do not care symbols [4]. In 1999, Berstel and Boasson started combinatorics on partial words [2] and have been studied by Blanchet-Sadri [3] in detail. In 2007, Sasikala et.al developed an automaton that can recognize partial languages [17].

To recognize or generate a two-dimensional word, various formal models are employed. These strategies were derived from the problems associated with image processing and pattern recognition [5, 13]. Giammarresi and Restivo proposed the concept of recognizable languages (or families of recognizable languages) [6] in 1991. Thomas studied various logic tiling and automata ideas [19]. Giammarresi et.al explained how rectangular pictures were recognizable by tiling [7, 8]. In 1997, Latteux and Simplot studied several characterizations of recognizable picture languages and domino tiling [11]. To study differences in gene expression between wild type and signaling mutants, Sweety et al. [18] extended tiling systems to finite partial array languages. The do not care symbol $\Diamond$ alone cannot be used as a tile to distinguish a local partial array language from a local array language.

The paper is structured as follows: The basic definitions are given in section 2. In section 3 we study the closure properties of partial array tiling systems. In section 4 we define $hv-$local partial array languages and prove that every $hv-$local partial array language is a local partial array language but the converse is not true. Further we discuss partial array domino system and prove that the equivalence relation between the class of partial array language is recognizable by partial array tiling system and partial array domino system. Finally we study the automata characterization for $hv-$local partial array languages in section 5.

2 Preliminaries

Let Σ denotes a finite alphabet and Σ^* denotes the collection of all finite words including the empty word λ over Σ. A two-dimensional word over Σ is a two-dimensional rectangular array of elements from Σ of size (h, v). Σ^{**} denotes the collection of all arrays over Σ such that the array language $L \subseteq \Sigma^{**}$. For every array X of size (h, v), $\widehat{X}$ is the array of size $(h+2, v+2)$ constructed by enclosing X with a special symbol $\# \notin \Sigma$. Let L_1 and L_2 be two string languages over Σ then the row-column concatenation of L_1 and L_2 denoted by $L_1 \oplus L_2$ is an array language $L = L_1 \oplus L_2 \subseteq \Sigma^{**}$ such that $X \in L$ if and only if the strings corresponding to the rows belong to L_1 and to the columns of X belong to L_2 respectively.

A partial word over Σ is a sequence of elements in Σ_p where $\Sigma_p = \Sigma \cup \{\Diamond\}$ and $\Diamond \notin \Sigma$. Formally a partial word w over Σ of length n is a partial function such that $w : \{1, 2, 3, \cdots, n\} \to \Sigma$. For $1 \le i \le n$, if $w = w(1)w(2)\cdots w(i)\cdots w(n)$ is defined then i belongs to the domain set of w, otherwise i belongs to the hole set of w. If w is a partial word of length n over Σ, then the

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companion of w represented by w_p is the total function, $w_p : \{1, 2, \cdots n\} \to \Sigma_p$ defined by

$$w_p(i) = \begin{cases} w(i) & \text{if } i \in \text{domain}(w), \\ \Diamond & \text{if } i \in \text{hole}(w). \end{cases}$$

Σ_p^* denotes the collection of all finite partial words including the empty word λ over Σ. A partial language L over Σ_p is said to be local if there exists a tuple (I, C, H, J) where $I, J \subseteq \Sigma$, $C \subseteq \Sigma^2$, $H \subseteq \Sigma_p^2$ such that $L = \{x \in \Sigma_p^* : P_1(x) \in I,\ F_2(x) \in C,\ PF_2(x) \in H,\ S_1(x) \in J\}$ where

- $P_1(x)$- Prefix of x of length 1
- $S_1(x)$- Suffix of x of length 1
- $F_2(x)$- Set of subwords of x of length 2
- $PF_2(x)$- Set of subpartial words of x of length 2.

The class of all local partial language is denoted by $\mathfrak{L}_p$. A partial array X of size (h, v) over Σ is a partial function $X : \{(1,1), \cdots\cdots (h,v)\} \to \Sigma$. For $1 \leq i \leq h$, $1 \leq j \leq v$ if $X(i,j)$ is defined then (i,j) belongs to the set of domain of X, otherwise (i,j) belongs to the set of hole of X. If X is a partial array of size (h,v) over Σ, then the companion of X represented by X_p is the total function, $X_p : \{(1,1), \cdots\cdots (h,v)\} \to \Sigma_p$ defined by

$$X_p(i,j) = \begin{cases} X(i,j) & \text{if } (i,j) \in \text{domain}(X), \\ \Diamond & \text{if } (i,j) \in \text{hole}(X). \end{cases}$$

Σ_p^{**} denotes the collection of all finite partial arrays over Σ. For any partial array $X \in \Sigma_p^{**}$, $B_{r,c}(X)$ is the set of all partial subarrays of X of size (r,c). A tile is a partial subarray of size $(2,2)$. A domino is a partial subarray of size $(1,c)$ or $(r,1)$. A domino of size $(1,c)$ is called a horizontal domino and a domino of size $(r,1)$ is called a vertical domino. The row-column catenation of any two local partial languages $L_1, L_2 \subseteq \Sigma_p^*$ is called a partial array language $L = L_1 \oplus L_2 \subseteq \Sigma_p^{**}$ such that the strings corresponding to rows belong to L_1 and the strings corresponding to columns belong to L_2. A nondeterministic (deterministic) finite partial automata is a quintuple $\mathcal{M} = (S_p, \Sigma_p, \delta, S_0, S_f)$ where

- $S_p = S \cup S_h$ and $S \cap S_h = \varnothing$ in which S be the finite collection of states and S_h be the finite collection of hole states,
- Σ_p be the input alphabet,
- $S_0 \subset S_p$ be the collection of initial states,
- $S_f \subseteq S_p$ be the collection of final states,
- $\delta : S_p \times \Sigma_p \to 2^{S_p}$ $(\delta : S_p \times \Sigma_p \to S_p)$ be the transition function.

For all $a \in \Sigma$ and $s, s_h \in S_p$,

1. $\delta(s, a) = A$ for some $A \subset S$
2. $\delta(s, \Diamond) = B$ for some $B \subset S_h$
3. $\delta(s_h, \Diamond) = C$ for some $C \subset S_h$.

A finite partial automata $\mathcal{M} = (S_p, \Sigma_p, \delta, S_0, S_f)$ is said to be finite partial local automata if for every

- $a \in \Sigma$, the set $\{\delta(s,a) : s \in S_p\}$ contains atmost one element.
- $\Diamond \in \Sigma_p$, the set $\{\delta(s,\Diamond) : s \in S_p\}$ contains atmost one element from S_h.

3 Partial array Tiling Systems

In this section we examine some of the characteristics of the class of partial array tiling system in terms of closure under various operations.

Definition 1. *[18] Let $\Gamma_p = \Gamma_p \cup \{\Diamond\}$ be a finite alphabet. A partial array language $L \subseteq \Gamma_p^{**}$ is said to be a* local partial array language *if there exists a set of tiles $\theta \subseteq \Gamma_p^{2\times 2}$ such that*

$$L = \left\{X \in \Gamma_p^{**} : B_{22}(\widehat{X}) \subseteq \theta\right\}.$$

In this case, we write $L = L(\theta)$.

Definition 2. *[18] A partial array tiling system is a four tuple $P_T = (\Sigma_p, \Gamma_p, \theta, \psi)$ where*

- *Σ_p, Γ_p are two finite alphabets*
- *θ is finite set of tiles over the alphabet $\Sigma_p \cup \{\#\}$*
- *$\psi : \Gamma_p \to \Sigma_p$ is a projection.*

A partial array tiling system recognizes the partial array language over Σ_p as $L = \psi(L')$ where $L' = L(\theta)$ is the local partial array language over Γ_p with the set of tiles θ.

Theorem 1. *The class of partial array language recognized by partial array tiling system is closed under projection.*

Proof. Consider the partial array language $L_1 \subseteq \Sigma_{p_1}^{**}$ and $L_2 \subseteq \Sigma_{p_2}^{**}$ such that $L_2 = \delta(L_1)$ where $\delta : \Sigma_{p_1} \to \Sigma_{p_2}$. We have to show that, if partial array tiling system recognizes L_1 then partial array tiling system recognizes L_2. If L_1 is generated by $P_{T_1} = (\Sigma_{p_1}, \Gamma_p, \theta, \psi_1)$ then there exists a local partial array language, $L' \subseteq \Gamma_p^{**}$ and a projection $\psi : \Gamma_p \to \Sigma_{p_1}$ such that $L_1 = \psi_1(L')$. Now

$$\delta(L_1) = \delta(\psi_1(L'))$$

$$= (\delta \circ \psi_1)(L')$$
$$= \psi_2(L'))$$

where $\psi_2 = (\delta \circ \psi_1) : \Gamma_p \to \Sigma_{p_2}$. Hence L_2 is generated by $P_{T_2} = (\Sigma_{p_2}, \Gamma_p, \theta, \psi_2)$. □

Theorem 2. *The class of partial array language recognized by partial array tiling system is closed under row and column concatenation operations.*

Proof. Consider the partial array language $L_1, L_2 \subseteq \Sigma_p^{**}$. Let the column concatenation of L_1 and L_2 be denoted as $L = L_1 \oplus L_2$. Let $P_{T_1} = (\Sigma_p, \Gamma_{p_1}, \theta_1, \psi_1)$ and $P_{T_2} = (\Sigma_p, \Gamma_{p_2}, \theta_2, \psi_2)$ be the partial array tiling system such that $L_1 = L(P_{T_1})$ and $L_2 = L(P_{T_2})$. Consider $P_T = (\Sigma_p, \Gamma_p, \theta, \psi)$ where $\Gamma_p = \Gamma_{p_1} \cup \Gamma_{p_2}$, $\theta = \theta_1 \cup \theta_2$. Without loss of assumption let Γ_{p_1} and Γ_{p_2} are disconnected. Let $\Gamma_p = \Gamma_{p_1} \cup \Gamma_{p_2}$. Now we define the set of tiles

$$\theta_1' = \left\{ \begin{array}{|c|c|}\hline c_1 & c_2 \\ \hline d_1 & d_2 \\ \hline\end{array} : \begin{array}{|c|c|}\hline c_1 & c_2 \\ \hline d_1 & d_2 \\ \hline\end{array} \in \theta_1 \textit{ and } c_2,\ d_2 \notin \# \right\}$$

$$\theta_2' = \left\{ \begin{array}{|c|c|}\hline e_1 & e_2 \\ \hline f_1 & f_2 \\ \hline\end{array} : \begin{array}{|c|c|}\hline e_1 & e_2 \\ \hline f_1 & f_2 \\ \hline\end{array} \in \theta_2 \textit{ and } e_1,\ f_1 \notin \# \right\}$$

$$\theta_3' = \left\{ \begin{array}{|c|c|}\hline c_1 & e_1 \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline c_2 & e_2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline d_1 & f_1 \\ \hline d_2 & f_2 \\ \hline\end{array} : \right.$$
$$\begin{array}{|c|c|}\hline c_1 & \# \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline c_2 & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline d_1 & \# \\ \hline d_2 & \# \\ \hline\end{array} \in \theta_1,$$
$$\left. \begin{array}{|c|c|}\hline \# & e_1 \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & e_2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & f_1 \\ \hline \# & f_2 \\ \hline\end{array} \in \theta_2. \right\}$$

Then $\theta = \theta_1' \cup \theta_2' \cup \theta_3'$ and the projection $\psi : \Gamma_p \to \Sigma_p$ is defined as

$$\psi(a) = \begin{cases} \psi_1(a) & \text{if } a \in \Gamma_{p_1} \\ \psi_2(a) & \text{if } a \in \Gamma_{p_2} \end{cases}$$

then $L = L_1 \oplus L_2$. Similarly we can obtain a tiling system for the row concatenation $L = L_1 \ominus L_2$. □

Theorem 3. *The class of partial array language recognized by partial array tiling system is closed under union and intersection.*

Proof. Consider the partial array language $L_1, L_2 \subseteq \Sigma_p^{**}$. Let $P_{T_1} = (\Sigma_p, \Gamma_{p_1}, \theta_1, \psi_1)$ and $P_{T_2} = (\Sigma_p, \Gamma_{p_2}, \theta_2, \psi_2)$ be the partial array tiling system such that $L_1 = L(P_{T_1})$ and $L_2 = L(P_{T_2})$. Consider $P_T = (\Sigma_p, \Gamma_p, \theta, \psi)$ where $\Gamma_p = \Gamma_{p_1} \cup \Gamma_{p_2}$, $\theta = \theta_1 \cup \theta_2$ and the projection $\psi : \Gamma_p \to \Sigma_p$ is defined as

$$\psi(a) = \begin{cases} \psi_1(a) & \text{if } a \in \Gamma_{p_1} \\ \psi_2(a) & \text{if } a \in \Gamma_{p_2} \end{cases}$$

then $L = L(P_{T_1}) \cup L(P_{T_2})$. Therefore $L = L_1 \cup L_2$ is generated by $P_T = (\Sigma_p, \Gamma_p, \theta, \psi)$.

To prove that the class of partial array language recognized by partial array tiling system is closed under intersection we have to construct partial array tiling system for $L = L_1 \cap L_2$. Consider $\Gamma_p \subseteq \Gamma_{p_1} \times \Gamma_{p_2}$ such that $(c_1, d_1) \in \Gamma_p \Leftrightarrow \psi_1(c_1) = \psi_2(d_1)$. Then $\theta = \theta_1 \cap \theta_2$ is defined as

$$\theta = \left\{ \begin{array}{|c|c|}\hline (c_1,\ c_2) & (e_1,\ e_2) \\ \hline (d_1,\ d_2) & (f_1,\ f_2) \\ \hline\end{array} : \begin{array}{|c|c|}\hline c_1 & e_1 \\ \hline d_1 & f_1 \\ \hline\end{array} \in \theta_1, \right.$$
$$\left. \begin{array}{|c|c|}\hline c_2 & e_2 \\ \hline d_2 & f_2 \\ \hline\end{array} \in \theta_2 \right\}$$

and the projection $\psi : \Gamma_p \to \Sigma_p$ is defined as $\psi((c_1, d_1)) = \psi(c_1) = \psi(d_1)$ for all $(c_1, d_1) \in \Gamma_{p_1} \times \Gamma_{p_2}$ then $L = L(P_{T_1}) \cap L(P_{T_2})$. Therefore $L = L_1 \cap L_2$ is recognizable by partial array tiling system $P_T = (\Sigma_p, \Gamma_p, \theta, \psi)$. □

Definition 3. *Let L be a local partial array language over Γ_p and $L = L(\theta)$ for some finite $\theta \subseteq \Gamma_p^{2\times 2}$. θ is said to be* minimal *if $L = L(\theta')$ for some finite $\theta' \subseteq \Gamma_p^{2\times 2}$ such that $\theta \subseteq \theta'$.*

Theorem 4. *Let L be a local partial array language then there is a minimal set θ such that $L = L(\theta)$.*

Proof. Let X be a partial array language over Γ_p. For $i = 1, 2, \cdots, n$, θ_i be a subsets of $\theta \subseteq \Gamma_p^{2\times 2}$ and $L = L(\theta_i)$. If $\theta = \theta_1 \cap \theta_2 \cdots \cap \theta_n$ then for $i = 1, 2, \cdots, n$, $B_{22}(\widehat{X}) \subseteq \theta_i$ such that $B_{22}(\widehat{X}) \subseteq \theta$. Thus $L = L(\theta)$ and $\theta \subseteq \theta_i$. □

4 Partial array domino Systems

In this section, we define new class of partial array languages called $hv-$local partial array languages.

Definition 4. *A partial array language $L \subseteq \Gamma_p^{**}$ is said to be $hv-$local partial array language if there exists a system $S = (\Delta, \Delta_\Diamond)$ where $\Delta, \Delta_\Diamond$ are the finite sets of dominoes over the alphabets $\Gamma \cup \{\#\}$ and $\Gamma_p \cup \{\#\}$ respectively such that*

$$L = \left\{ X \in \Gamma_p^{**} : B_{1,2}(\widehat{X}) \cup B_{2,1}(\widehat{X}) \subseteq S \right\}.$$

In this case, we write $L = L(S)$.

Example 1. *Consider the system $S = (\Delta, \Delta_\Diamond)$ over $\Gamma_p = \{a, b\} \cup \{\Diamond\}$ with*

$$\Delta = \left\{ \begin{array}{|c|c|}\hline \# & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline b & b \\ \hline\end{array}, \right.$$
$$\left. \begin{array}{|c|c|}\hline b & \# \\ \hline\end{array}, \begin{array}{|c|}\hline a \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline a \\ \hline a \\ \hline\end{array}, \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}, \begin{array}{|c|}\hline b \\ \hline b \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline b \\ \hline\end{array} \right\},$$

$$\Delta_\Diamond = \left\{ \begin{array}{|c|c|}\hline \# & \Diamond \\ \hline\end{array}, \begin{array}{|c|c|}\hline \Diamond & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline a & \Diamond \\ \hline\end{array}, \begin{array}{|c|c|}\hline \Diamond & b \\ \hline\end{array}, \right.$$
$$\left. \begin{array}{|c|}\hline \Diamond \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline \Diamond \\ \hline a \\ \hline\end{array}, \begin{array}{|c|}\hline b \\ \hline \Diamond \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline \Diamond \\ \hline\end{array} \right\}.$$

Then

$$L = L(S) = \left\{ \begin{matrix} \Diamond & b \\ a & \Diamond \end{matrix}, \begin{matrix} \Diamond & b & b \\ a & \Diamond & b \\ a & a & \Diamond \end{matrix}, \begin{matrix} \Diamond & b & b & b \\ a & \Diamond & b & b \\ a & a & \Diamond & b \\ a & a & a & \Diamond \end{matrix}, \ldots\ldots \right\}.$$

Hence L is $hv-$local partial array language.

Theorem 5. *Every $hv-$local partial array language is a local partial array language.*

Proof. Let $L \subseteq \Gamma_p^{**}$ be a $hv-$local partial array language. Then there exists a system $S = (\Delta, \Delta_\Diamond)$ where $\Delta, \Delta_\Diamond$ are the finite sets of dominoes such that $L = \{X \in \Gamma_p^{**} : B_{1,2}(\widehat{X}) \cup B_{2,1}(\widehat{X}) \subseteq S\}$. We construct a set of squares θ such that $\theta = \{p \in (\Gamma_p \cup \{\#\})^{2,2} : B_{1,2}(p) \cup B_{2,1}(p) \subseteq S\}$. Let $L' = \{X \in \Gamma_p^{**} : B_{2,2}(\widehat{X}) \subseteq \theta\}$. Therefore L' is local partial array language. Let the partial array $X \in L'$ such that $B_{1,2}(\widehat{X}) \subseteq B_{1,2}(B_{2,2}(\widehat{X})) \subseteq B_{1,2}(\theta) \subseteq S$ and $B_{2,1}(\widehat{X}) \subseteq B_{2,1}(B_{2,2}(\widehat{X})) \subseteq B_{2,1}(\theta) \subseteq S$ where $B_{2,2}(\widehat{X}) \subseteq \theta$. Hence $X \in L$. Conversely, $X \in L$ and $x \in B_{2,2}(\widehat{X})$. Then $B_{1,2}(x) \subseteq B_{1,2}(\widehat{X}) \subseteq S$ and $B_{2,1}(x) \subseteq B_{2,1}(\widehat{X}) \subseteq S$. Therefore $x \in \theta$ and $X \in L'$. Therefore L is local partial array language. □

Example 2. *Consider the $hv-$local partial array language from Example 1*

$$L = \left\{ \begin{matrix} \Diamond & b \\ a & \Diamond \end{matrix}, \begin{matrix} \Diamond & b & b \\ a & \Diamond & b \\ a & a & \Diamond \end{matrix}, \begin{matrix} \Diamond & b & b & b \\ a & \Diamond & b & b \\ a & a & \Diamond & b \\ a & a & a & \Diamond \end{matrix}, \ldots\ldots \right\}.$$

Then L is local partial array language with

$$\theta = \left\{ \begin{matrix} \# & a & \# & \Diamond & \# & a & \# & \# & \# & \# & \# & \# \\ \# & \#, & \# & a, & \# & a, & \# & \Diamond, & \Diamond & b, & b & b, \end{matrix} \right.$$

$$\left. \begin{matrix} \# & \# & b & \# & b & \# & a & a & a & \Diamond & \Diamond & b & b & b & b & b \\ b & \#, & b & \#, & \Diamond & \#, & a & a, & a & a, & a & \Diamond, & \Diamond & b, & b & b \end{matrix} \right\}.$$

Remark 1. *If L is local partial array language then L need not be $hv-$local partial array language.*

Example 3. *Consider the local partial array language over $\Gamma_p = \{a\} \cup \{\Diamond\}$*

$$L = \left\{ \begin{matrix} \Diamond & a \\ a & \Diamond \end{matrix}, \begin{matrix} \Diamond & a & a \\ a & \Diamond & a \\ a & a & \Diamond \end{matrix}, \begin{matrix} \Diamond & a & a & a \\ a & \Diamond & a & a \\ a & a & \Diamond & a \\ a & a & a & \Diamond \end{matrix}, \ldots\ldots \right\}$$

with

$$\theta = \left\{ \begin{matrix} \# & a & \# & \Diamond & \# & a & \# & \# & a & a & a & \Diamond & a & \Diamond \\ \# & \#, & \# & a, & \# & a, & \# & \Diamond, & a & a, & \# & \#, & a & a, \end{matrix} \right.$$

$$\left. \begin{matrix} \Diamond & a & a & \# & a & \# & \# & \# & \# & \# & \# & \# & a & a \\ a & \Diamond, & \Diamond & \#, & a & \#, & a & \#, & a & a, & \Diamond & a, & \# & \# \end{matrix} \right\}.$$

Then L is not $hv-$local partial array language corresponding to the system $S = (\Delta, \Delta_\Diamond)$ with

$$\Delta = \left\{ \begin{array}{|c|c|}\hline \# & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline a & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}, \right.$$

$$\left. \begin{array}{|c|}\hline a \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline a \\ \hline a \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline a \\ \hline\end{array} \right\},$$

$$\Delta_\Diamond = \left\{ \begin{array}{|c|c|}\hline \# & \Diamond \\ \hline\end{array}, \begin{array}{|c|c|}\hline \Diamond & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline a & \Diamond \\ \hline\end{array}, \begin{array}{|c|c|}\hline \Diamond & a \\ \hline\end{array}, \right.$$

$$\left. \begin{array}{|c|}\hline \Diamond \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline \Diamond \\ \hline a \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline \Diamond \\ \hline\end{array}, \begin{array}{|c|}\hline a \\ \hline \Diamond \\ \hline\end{array} \right\}$$

because if it is $hv-$local partial array language then an array in which all the entries are a will be in L.

Theorem 6. *Every $hv-$local partial array language is equal to the row-column concatenation of any two local partial languages.*

Proof. Consider $hv-$local partial array language L so that $L = L(S)$ where $S = (\Delta, \Delta_\Diamond)$. Consider the local systems (I_1, C_1, H_1, J_1) and (I_2, C_2, H_2, J_2) recognizes the local partial languages L_1 and L_2 respectively such that $L = L_1 \oplus L_2 \in \mathfrak{L}_p \oplus \mathfrak{L}_p$ where

$$I_1 = \left\{a : \begin{array}{|c|c|}\hline \# & a \\ \hline\end{array} \in \Delta,\ a \in \Gamma_p\right\}$$

$$C_1 = \left\{ab : \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \in \Delta,\ a, b \in \Gamma\right\}$$

$$H_1 = \left\{a\Diamond : \begin{array}{|c|c|}\hline a & \Diamond \\ \hline\end{array} \in \Delta_\Diamond,\ \ \Diamond a : \begin{array}{|c|c|}\hline \Diamond & a \\ \hline\end{array} \in \Delta_\Diamond\right\}$$

$$J_1 = \left\{a : \begin{array}{|c|c|}\hline a & \# \\ \hline\end{array} \in \Delta,\ a \in \Gamma_p\right\}$$

and

$$I_2 = \left\{a : \begin{array}{|c|}\hline a \\ \hline \# \\ \hline\end{array} \in \Delta,\ a \in \Gamma_p\right\}$$

$$C_2 = \left\{ab : \begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} \in \Delta,\ a, b \in \Gamma\right\}$$

$$H_2 = \left\{a\Diamond : \begin{array}{|c|}\hline \Diamond \\ \hline a \\ \hline\end{array} \in \Delta_\Diamond,\ \ \Diamond a : \begin{array}{|c|}\hline a \\ \hline \Diamond \\ \hline\end{array} \in \Delta_\Diamond\right\}$$

$$J_2 = \left\{a : \begin{array}{|c|}\hline \# \\ \hline a \\ \hline\end{array} \in \Delta,\ a \in \Gamma_p\right\}.$$

Conversely, consider $L \in \mathfrak{L}_p \oplus \mathfrak{L}_p$ then $L = L_1 \oplus L_2$. Let

$$\Delta_1 = \left\{\begin{array}{|c|c|}\hline \# & a \\ \hline\end{array} :\ a :\in I_1\right\}$$

$$\Delta_2 = \left\{\begin{array}{|c|c|}\hline a & b \\ \hline\end{array} :\ ab \in C_1\right\}$$

$$\Delta_3 = \left\{\begin{array}{|c|c|}\hline a & \# \\ \hline\end{array} :\ a \in J_1\right\}$$

$$\Delta_4 = \left\{\begin{array}{|c|}\hline a \\ \hline \# \\ \hline\end{array} :\ a \in I_2\right\}$$

$$\Delta_5 = \left\{\begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} :\ ab \in C_2\right\}$$

$$\Delta_6 = \left\{ \begin{array}{|c|}\hline \# \\ \hline a \\ \hline\end{array} : \ a \in J_2 \right\}$$

$$\Delta_{\Diamond_1} = \left\{ \begin{array}{|c|c|}\hline a & \Diamond \\ \hline\end{array} : \ a\Diamond \in H_1, \ \begin{array}{|c|c|}\hline \Diamond & a \\ \hline\end{array} : \ \Diamond a \in H_1 \right\}$$

$$\Delta_{\Diamond_2} = \left\{ \begin{array}{|c|}\hline \Diamond \\ \hline a \\ \hline\end{array} : \ a\Diamond \in H_2, \ \begin{array}{|c|}\hline a \\ \hline \Diamond \\ \hline\end{array} : \ \Diamond a \in H_2 \right\}$$

such that $\Delta = (\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5 \cup \Delta_6)$ and $\Delta_\Diamond = (\Delta_{\Diamond_1} \cup \Delta_{\Diamond_2})$. Therefore $L = L(S)$ where $S = (\Delta, \Delta_\Diamond)$. Hence L is $hv-$local partial array language. □

Example 4. *Consider the hv−local partial array language*

$$L = \left\{ \begin{array}{cc} a & \Diamond \\ \Diamond & b \end{array}, \ \begin{array}{ccc} a & a & \Diamond \\ a & \Diamond & b \\ \Diamond & b & b \end{array}, \ \begin{array}{cccc} a & a & a & \Diamond \\ a & a & \Diamond & b \\ a & \Diamond & b & b \\ \Diamond & b & b & b \end{array}, \ \ldots\ldots \right\}$$

recognizable by the system $S = (\Delta, \Delta_\Diamond)$ *with*

$$\Delta = \left\{ \begin{array}{|c|c|}\hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline b & b \\ \hline\end{array}, \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline b & \# \\ \hline\end{array}, \begin{array}{|c|}\hline b \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline a \\ \hline\end{array}, \begin{array}{|c|}\hline a \\ \hline a \\ \hline\end{array}, \begin{array}{|c|}\hline b \\ \hline b \\ \hline\end{array} \right\}$$

$$\Delta_\Diamond = \left\{ \begin{array}{|c|c|}\hline \# & \Diamond \\ \hline\end{array}, \begin{array}{|c|c|}\hline \Diamond & b \\ \hline\end{array}, \begin{array}{|c|c|}\hline a & \Diamond \\ \hline\end{array}, \begin{array}{|c|c|}\hline \Diamond & \# \\ \hline\end{array}, \begin{array}{|c|}\hline \Diamond \\ \hline \# \\ \hline\end{array}, \begin{array}{|c|}\hline \# \\ \hline \Diamond \\ \hline\end{array}, \begin{array}{|c|}\hline a \\ \hline \Diamond \\ \hline\end{array}, \begin{array}{|c|}\hline \Diamond \\ \hline b \\ \hline\end{array} \right\}.$$

Let $I_1 = \{\Diamond, a\}$, $C_1 = \{aa, bb\}$, $H_1 = \{a\Diamond, \Diamond b\}$, $J_1 = \{\Diamond, b\}$ *and* $I_2 = \{\Diamond, b\}$, $C_2 = \{aa, bb\}$, $H_2 = \{\Diamond a, b\Diamond\}$, $J_1 = \{\Diamond, a\}$. *Let the local partial languages* L_1 *and* L_2 *be recognizable by* (I_1, C_1, H_1, J_1) *and* (I_2, C_2, H_2, J_2) *respectively such that* $L_1 = \{a^* \Diamond b^*\}$ *and* $L_2 = \{b^* \Diamond a^*\}$. *Then* $L = L_1 \oplus L_2$.

Definition 5. *A partial array domino system is a quintuple* $P_D = (\Sigma_p, \Gamma_p, \Delta, \Delta_\Diamond, \psi)$ *where*

- Σ_p, Γ_p *are two finite alphabets*
- $\Delta, \Delta_\Diamond$ *are two finite sets of dominoes over* $\Sigma_p \cup \{\#\}$
- $\psi : \Gamma_p \to \Sigma_p$ *is a projection.*

A partial array domino system recognizes a partial array language over Σ_p *as* $L = \psi(L')$ *where* $L' = L(S)$ *is the hv−local partial array language over* Γ_p *with the system* $S = (\Delta, \Delta_\Diamond)$.

Theorem 7. *Let* L *be a local partial array language over* Σ_p. *Then there exists hv−local partial array language* L' *over* $\Gamma_p = \Gamma \cup \{\Diamond\}$ *and a projection* $\psi : \Gamma_p \to \Sigma_p$ *such that* $L = \psi(L')$.

Proof. Consider the local partial array language L which is generated by a finite set of tiles over $\Sigma_p \cup \{\#\}$. Therefore $L = L(\theta)$. Let $\Gamma_p = \theta \subseteq (\Sigma_p \cup \{\#\})^{2\times 2}$ and

$$\Delta_1 = \left\{ \begin{array}{|cc|cc|}\hline \# & c_1 & c_1 & c_2 \\ \# & \# & \# & \# \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & c_1 \\ \# & \# \\ \hline\end{array}, \begin{array}{|cc|}\hline c_1 & c_2 \\ \# & \# \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_2 = \left\{ \begin{array}{|cc|cc|}\hline \# & c_2 & c_2 & c_4 \\ \# & c_1 & c_1 & c_3 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & c_2 \\ \# & c_1 \\ \hline\end{array}, \begin{array}{|cc|}\hline c_2 & c_4 \\ c_1 & c_3 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_3 = \left\{ \begin{array}{|cc|cc|}\hline \# & \# & \# & \# \\ \# & c_2 & c_2 & c_3 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ \# & c_2 \\ \hline\end{array}, \begin{array}{|cc|}\hline \# & \# \\ c_2 & c_3 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_4 = \left\{ \begin{array}{|cc|cc|}\hline c_1 & c_2 & c_2 & c_5 \\ c_3 & c_4 & c_4 & c_6 \\ \hline\end{array} : \begin{array}{|cc|}\hline c_1 & c_2 \\ c_3 & c_4 \\ \hline\end{array}, \begin{array}{|cc|}\hline c_2 & c_5 \\ c_4 & c_6 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_5 = \left\{ \begin{array}{|cc|}\hline \# & \# \\ \# & c_1 \\ \hline \# & c_1 \\ \# & c_2 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ \# & c_1 \\ \hline\end{array}, \begin{array}{|cc|}\hline \# & c_1 \\ \# & c_2 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_6 = \left\{ \begin{array}{|cc|}\hline \# & \# \\ c_1 & c_2 \\ \hline c_1 & c_2 \\ c_3 & c_4 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ c_1 & c_2 \\ \hline\end{array}, \begin{array}{|cc|}\hline c_1 & c_2 \\ c_3 & c_4 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_7 = \left\{ \begin{array}{|cc|}\hline \# & \# \\ c_1 & \# \\ \hline c_1 & \# \\ c_2 & \# \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ c_1 & \# \\ \hline\end{array}, \begin{array}{|cc|}\hline c_1 & \# \\ c_2 & \# \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_8 = \left\{ \begin{array}{|cc|}\hline c_1 & c_2 \\ c_3 & c_4 \\ \hline c_3 & c_4 \\ c_5 & c_6 \\ \hline\end{array} : \begin{array}{|cc|}\hline c_1 & c_2 \\ c_3 & c_4 \\ \hline\end{array}, \begin{array}{|cc|}\hline c_3 & c_4 \\ c_5 & c_6 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_{\Diamond_1} = \left\{ \begin{array}{|cc|cc|}\hline \# & \Diamond & \Diamond & c_2 \\ \# & \# & \# & \# \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \Diamond \\ \# & \# \\ \hline\end{array}, \begin{array}{|cc|}\hline \Diamond & c_2 \\ \# & \# \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_{\Diamond_2} = \left\{ \begin{array}{|cc|cc|}\hline \# & c_2 & c_2 & c_4 \\ \# & \Diamond & \Diamond & c_3 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & c_2 \\ \# & \Diamond \\ \hline\end{array}, \begin{array}{|cc|}\hline c_2 & c_4 \\ \Diamond & c_3 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_{\Diamond_3} = \left\{ \begin{array}{|cc|cc|}\hline \# & \# & \# & \# \\ \# & \Diamond & \Diamond & c_3 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ \# & \Diamond \\ \hline\end{array}, \begin{array}{|cc|}\hline \# & \# \\ \Diamond & c_3 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_{\Diamond_4} = \left\{ \begin{array}{|cc|}\hline \# & \# \\ \# & \Diamond \\ \hline \# & \Diamond \\ \# & c_2 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ \# & \Diamond \\ \hline\end{array}, \begin{array}{|cc|}\hline \# & \Diamond \\ \# & c_2 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_{\Diamond_5} = \left\{ \begin{array}{|cc|}\hline \# & \# \\ c_1 & \Diamond \\ \hline c_1 & \Diamond \\ c_3 & c_4 \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ c_1 & \Diamond \\ \hline\end{array}, \begin{array}{|cc|}\hline c_1 & \Diamond \\ c_3 & c_4 \\ \hline\end{array} \in \Gamma_p \right\}$$

$$\Delta_{\Diamond_6} = \left\{ \begin{array}{|cc|}\hline \# & \# \\ \Diamond & \# \\ \hline \Diamond & \# \\ c_2 & \# \\ \hline\end{array} : \begin{array}{|cc|}\hline \# & \# \\ \Diamond & \# \\ \hline\end{array}, \begin{array}{|cc|}\hline \Diamond & \# \\ c_2 & \# \\ \hline\end{array} \in \Gamma_p \right\}.$$

Here $\Delta = (\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5 \cup \Delta_6 \cup \Delta_7 \cup \Delta_8)$ and $\Delta_\Diamond = (\Delta_{\Diamond_1} \cup \Delta_{\Diamond_2} \cup \Delta_{\Diamond_3} \cup \Delta_{\Diamond_4} \cup \Delta_{\Diamond_5} \cup \Delta_{\Diamond_6})$. Let $L' =$

$L(S)$ with $S = (\Delta, \Delta_\Diamond)$ is $hv-$local partial array language over Γ_p. Define a mapping $\psi : \Gamma_p \to \Sigma_p$ by $\psi\left(\boxed{\begin{array}{cc} c_1 & c_2 \\ c_3 & c_4 \end{array}}\right) = c_1$. Therefore $L = \psi(L')$. □

Theorem 8. *The class of partial array language recognizable by partial array domino system is equal to the class of partial array language recognizable by partial array tiling system.*

Proof. This follows from Theorem 5 and Theorem 7. □

Definition 6. *Let L be an $hv-$local partial array language over Γ_p and $L = L(\Delta)$ for some finite set $\Delta \subseteq \Gamma_p \cup \#$. Δ is said to be* minimal *if $L = L(\Delta')$ for some finite set $\Delta \subseteq \Gamma_p \cup \#$ then $\Delta \subseteq \Delta'$.*

Theorem 9. *Let L be an $hv-$local partial array language then there is a minimal set Δ such that $L = L(\Delta)$.*

Proof. Let X be a partial array language over Γ_p. For $i = 1, 2, \cdots, n$, Δ_i be a subsets of $\Delta \subseteq \Gamma_p^{2\times 2}$ and $L = L(\Delta_i)$. If $\Delta = \Delta_1 \cap \Delta_2 \cdots \cap \Delta_n$ then for $i = 1, 2, \cdots, n$, $B_{1,2}(\widehat{X}) \cup B_{2,1}(\widehat{X}) \subseteq \Delta_i$ such that $B_{1,2}(\widehat{X}) \cup B_{2,1}(\widehat{X}) \subseteq \Delta$. Thus $L = L(\Delta)$ and $\Delta \subseteq \Delta_i$. □

5 Automata Characterization of $hv-$local partial array languages

Definition 7. *[18] A non-deterministic (deterministic) finite online tesselation h-automaton is a quintuple $\mathcal{M} = (S_p, \Sigma_p, \delta, S_0, S_f)$ where*

- *$S_p = S \cup S_h$ and $S \cap S_h = \emptyset$ in which S be the finite collection of states and S_h be the finite collection of hole states,*
- *Σ_p be the input alphabet,*
- *$S_0 \subset S_p$ be the collection of initial states,*
- *$S_f \subseteq S_p$ be the collection of final states,*
- *$\delta : S_p \times S_p \times \Sigma_p \to 2^{S_p}$ ($\delta : S_p \times S_p \times \Sigma_p \to S_p$) be the transition function.*

For all $a \in \Sigma$ and $s, s_h \in S_p$,

1. *$\delta(s, s, a) = A$ for some $A \subset S$*
2. *$\delta(s, s, \Diamond) = B$ for some $B \subset S_h$*
3. *$\delta(s_h, s_h, a) = C$ for some $C \subset S$*
4. *$\delta(s_h, s_h, \Diamond) = D$ for some $D \subset S_h$*

A computation by $\mathcal{M}$ on an $\omega\omega-$partial array X where

$$\widehat{X} = \begin{array}{cccccc} \# & \# & \# & \cdot & \# & \# \\ \# & a_{m1} & a_{m2} & \cdot & a_{mn} & \# \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \# & a_{21} & a_{22} & \cdot & a_{2n} & \# \\ \# & a_{11} & a_{12} & \cdot & a_{1n} & \# \\ \# & \# & \# & \cdot & \# & \# \end{array}$$

with $a_{ij} \in \Sigma_p$ and $\# \notin \Sigma_p$ is consummate as follows:

When $t = 0$, $s_0 \in S_0$ be the initial state corresponding to all positions of $\widehat{X}$ having $\#$. The state corresponding to each position (i, j) by δ subject to the states corresponding to the positions $(i-1, j)$, $(i, j-1)$ and the symbol a_{ij}. If $a_{ij} \in \Sigma$ then g_{ij} be the state corresponding to the position (i, j) and if $a_{ij} = \Diamond$ then h_{ij} be the state corresponding to the position (i, j). When $t = 1$, a state from $\delta(s_0, s_0, a_{11})$ corresponding to the position $(1, 1)$ holding a_{11}. If $a_{11} \in \Sigma$ then the state g_{11} corresponding to the position $(1, 1)$ and if $a_{11} = \Diamond$ then the state h_{11} corresponding to the position $(1, 1)$. When $t = 2$, states corresponding concurrently to the positions $(2, 1)$ and $(1, 2)$ possessing a_{21} and a_{12}.

Case (i):
If the state g_{11} corresponds to the position $(1, 1)$ then the state corresponding to the position $(2, 1)$ belongs to $\delta(s_0, g_{11}, a_{21})$ and to the position $(1, 2)$ belongs to $\delta(g_{11}, s_0, a_{12})$. There are given as follows:

- *$\delta(s_0, g_{11}, a_{21}) = g_{21}$ if $a_{21} \in \Sigma$*
- *$\delta(g_{11}, s_0, a_{12}) = g_{12}$ if $a_{12} \in \Sigma$*
- *$\delta(s_0, g_{11}, \Diamond) = h_{21}$*
- *$\delta(g_{11}, s_0, \Diamond) = h_{12}$.*

Case (ii):
If the state h_{11} corresponds to the position $(1, 1)$ then the state corresponding to the position $(2, 1)$ belongs to $\delta(s_0, h_{11}, a_{21})$ and to the position $(1, 2)$ belongs to $\delta(h_{11}, s_0, a_{12})$. There are given as follows:

- *$\delta(s_0, h_{11}, a_{21}) = g_{21}$ if $a_{21} \in \Sigma$*
- *$\delta(h_{11}, s_0, a_{12}) = g_{12}$ if $a_{12} \in \Sigma$*
- *$\delta(s_0, h_{11}, \Diamond) = h_{21}$*
- *$\delta(h_{11}, s_0, \Diamond) = h_{12}$.*

If $X \in \Sigma_p^{\omega\omega}$, the sequence $\Sigma = s_{11}s_{12}s_{21}s_{31}s_{22}s_{13}\cdots s_{mn}$ of states from S_p is said to be a run of $\mathcal{M}$ and it is an element of S_p^{ω}. The collection of runs is represented as $r(\Sigma)$. The language of partial arrays recognized by $\mathcal{M}$ is denoted by $L(\mathcal{M})$.

Now we introduce online tesselation h-local automaton.

Definition 8. *A online tesselation h-automaton $\mathcal{M} = (S_p, \Sigma_p, \delta, S_0, S_f)$ is said to be online tesselation h-local automaton if for every*

- $a \in \Sigma$, *the set* $\{\delta(s_1, s_2, a) : s_1, s_2 \in S_p\}$ *contains atmost one element.*
- $\Diamond \in \Sigma_p$, *the set* $\{\delta(s_1, s_2, \Diamond) : s_1, s_2 \in S_p\}$ *contains atmost one element from* S_h.

Theorem 10. *If L is an hv-local partial array language then L is recognized by online tesselation h-local automaton.*

Proof. Let L be an hv-local partial array language. Then $L = L_1 \oplus L_2$ where L_1 and L_2 are local partial languages. Let L_1 and L_2 be recognized by local partial finite automata $\mathcal{M}_1 = (S_{p_1}, \Sigma_p, \delta_1, S_0', S_{f_1})$ and $\mathcal{M}_2 = (S_{p_2}, \Sigma_p, \delta_2, S_0', S_{f_2})$ respectively. Consider online tesselation h-local automata $\mathcal{M} = (S_p, \Sigma_p, \delta, S_0, S_f)$ where

- $S_p = S_{p_1} \times S_{p_2}$
- $S_0 = (S_0', S_0'')$
- $S_f = S_{f_1} \times S_{f_2}$

and the transition function δ is defined as

- $\delta((p_1, q_1), (r_1, s_1), a) = (\delta_1(p_1, a), \delta_2(s_1, a))$
- $\delta((p_1, q_1), (r_1, s_1), \Diamond) = (\delta_1(p_1, \Diamond), \delta_2(s_1, \Diamond))$.

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are local partial automata then $\{\delta_1(p_1, a) : p_1 \in S_{p_1}\}$, $\{\delta_1(p_1, \Diamond) : p_1 \in S_{p_1}\}$ and $\{\delta_1(s_1, a) : s_1 \in S_{p_2}\}$, $\{\delta_1(s_1, \Diamond) : s_1 \in S_{p_2}\}$ has atmost one element such that $\{\delta((p_1, q_1), (r_1, s_1), a) : (p_1, q_1), (r_1, s_1) \in S_p\}$ and $\{\delta((p_1, q_1), (r_1, s_1), \Diamond) : (p_1, q_1), (r_1, s_1) \in S_p\}$ has atmost one element. Therefore L is recognized by online tesselation h-local automata. □

Theorem 11. *If L is recognized by online tesselation h-local automata then L is hv-local partial array language.*

Proof. Let L is recognized by online tesselation h-local automata $\mathcal{M} = (S_p, \Sigma_p, \delta, S_0, S_f)$. Consider the partial finite automata $\mathcal{M}_1 = (S_{p_1}, \Sigma_p, \delta_1, S_0, S_f)$ and $\mathcal{M}_2 = (S_{p_2}, \Sigma_p, \delta_2, S_0, S_f)$ where $\delta_1(p, a) = \delta(p, q, a)$, $\delta_1(p, \Diamond) = \delta(p, q, \Diamond)$ for some $q \in S_p$ and $\delta_2(q, a) = \delta(p, q, a)$, $\delta_2(q, \Diamond) = \delta(p, q, \Diamond)$ for some $p \in S_p$. Since $\mathcal{M}$ is online tesselation h-local automata then $\mathcal{M}_1$ and $\mathcal{M}_2$ are also local partial finite automata. If the languages L_1 and L_2 recognized by $\mathcal{M}_1$ and $\mathcal{M}_2$ then L_1 and L_2 are local partial languages and $L = L_1 \oplus L_2$. Therefore by Theorem 6, L is hv-local partial array language. □

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