# On the Hilfer Fractional Volterra-Fredholm Integro Differential Equations

Karim Ivaz, Ismael Alasadi and Ahmed Hamoud

*Abstract*—**In this work, we investigate a class of nonlocal Volterra-Fredholm integro-differential equations involving Hilfer fractional derivatives and almost sectorial operators. We prove our results by applying Arzela-Ascoli theorem and Schauder's fixed point technique. Moreover, we show the fundamental properties of the representation of the solution by discussing a case related to the associated semigroup. For that, we consider compactness properties.**

*Index Terms*—**Hilfer fractional derivatives, mild solutions, almost sectorial operators, measure of compactness.**

## I. Introduction

FRACTIONAL differential equations have attracted much attention and have been the focus of many studies due mainly to their varied applications in many fields of engineering, biology, physics, biophysics, chemistry, control theory, economy and so on; see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

We also refer to the work in [2], where Debbouche and Antonov studied the question of the existence of nonlinear Hilfer fractional differential equations with controls. Sufficient conditions are also established, where the time fractional derivative is the Hilfer derivative. In [11], Zhang and Zhou studied fractional Cauchy problems with almost sectorial operators of the form

$$\begin{cases} {}^{L}D^{\alpha}_{0^+}\varpi(\rho) = A\varpi(\rho) + g(\rho, \varpi(\rho)) \\ I^{(1-\alpha)}_{0^+}\varpi(0) = \varpi_0, \ \rho \in [0, a], \end{cases}$$

where ${}^{L}D^{\alpha}_{0^+}$ is the Riemann-Liouville derivative of order $\alpha$, $I^{(1-\alpha)}_{0^+}$ is the Riemann-Liouville integral of order $1-\alpha$, $0 < \alpha < 1$, $A$ is an almost sectorial operator on a complex Banach space, and $g$ is a given function.

Recently, in many published works, Hilfer fractional differential equations have received attention [2], [6], [12].

Furati et al. in [13], considered nonlinear fractional differential equation involving Hilfer fractional derivative

$$\begin{aligned} &D^{\alpha,\beta}_{a^+}\varpi(\rho) = g(\rho, \varpi(\rho)), \ 0 < \alpha < 1, \ 0 \le \beta \le 1, \\ &I^{(1-\psi)}_{a^+}\varpi(a^+) = \varpi_a, \rho > a, \ \psi = \alpha + \beta - \alpha\beta, \end{aligned} \quad (1)$$

where $D^{\alpha,\beta}_{a^+}(.), I^{(1-\psi)}_{a^+}(.)$ are Hilfer fractional derivative and Riemann-Liouville fractional integral, respectively, $\varpi_a \in \mathbb{R}$. The author used the Banach fixed point theorem to investigate the existence and uniqueness and stability of global solutions in the weighted space. Dheigude and Bhairat in [14], disccused the existence, uniqueness and continuous dependence of solution for the problem (1) by using successive approximations and generalized Gronwall inequality. Oliveira and de Oliveira in [15], proposed a new fractional derivative the Hilfer-Katugampola fractional derivatives ${}^{\rho}D^{\alpha,\beta}_{a^+}(.)$ and generalized fractional integral ${}^{\rho}I^{(1-\psi)}_{a^+}(.)$. The authors used Banach fixed point theorem to obtain the existence and uniqueness of solution for a weighted Cauchy-type problem (1).

In [16], Jaiswal and Bahuguna studied equations of Hilfer fractional derivatives with almost sectorial operators in the abstract sense:

$$\begin{cases} D^{\alpha,\psi}_{0^+}\varpi(\rho) + A\varpi(\rho) = g(\rho, \varpi(\rho)) \\ I^{(1-\alpha)(1-\psi)}_{0^+}\varpi(0) = \varpi_0, \ \rho \in (0, T], \end{cases}$$

Motivated by these results, here, we extend the previous available results of the literature to a class of Hilfer fractional integro-differential equations in which the closed operator is almost sectorial. Moreover, we also consider compactness case of the semigroup operator.

We consider nonlocal Volterra-Fredholm integro-differential equations involving Hilfer fractional derivatives and almost sectorial operators:

$$D^{\alpha,\psi}_{0^+}\varpi(\rho) + A\varpi(\rho) = g\Big(\rho, \varpi(\rho), B\varpi(\rho), Z\varpi(\rho)\Big), \quad (2)$$

$$I^{(1-\alpha)(1-\psi)}_{0^+}[\varpi(\rho)]|_{\rho=0} + h(\varpi(\rho)) = \varpi_0, \quad (3)$$

where $D^{\alpha,\psi}_{0^+}$ is the Hilfer fractional derivative of order $\alpha \in (0, 1)$ and type $\psi \in [0, 1]$. We assume that $A$ is an almost sectorial operator on a Banach space $Y$ with norm $\|.\|$. Let $f_i : J \times Y \longrightarrow Y, \ i = 1, 2, \ \rho \in J := (0, T], \ g : J \times Y \times Y \times Y \longrightarrow Y$ and $h : C(J : Y) \longrightarrow Y$ to be given abstract functions to be specified later. For brevity, we take

$$B\varpi(\rho) = \int_0^{\rho} K(\rho, s) f_1(s, \varpi(s))ds,$$

$$Z\varpi(\rho) = \int_0^{T} H(\rho, s) f_2(s, \varpi(s))ds.$$

The paper is structured as follows. In Section II, we present necessary information about the Hilfer derivative, almost sectorial operators, measures of compactness, mild solutions of Eqs. (2) and (3) and some useful definitions, results and lemmas. We discuss fundamental results for mild solutions to Eqs. (2) and (3) and we prove the solvability question, when associated semigroups is compact in Section III.

## II. Preliminaries

In this section, we recall some basic definitions and properties of the fractional calculus theory and auxiliary lemmas which will be used throughout this paper, see ([4],

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Karim Ivaz is a professor at the Department of Applied Mathematics, University of Tabriz, Tabriz, Iran. e-mail: (ivaz@tabrizu.ac.ir).

Ismael Alasadi is a researcher at the Department of Mathematics, Thi Qar Directorates of Education, Ministry of Education, Iraq. Department of Applied Mathematics, University of Tabriz, Tabriz, Iran. e-mail: (7207183617ismael@gmail.com).

Ahmed Hamoud is an assistant professor at the Department of Mathematics, Taiz University, Taiz, Yemen. e-mail: (ahmed.hamoud@taiz.edu.ye)

[6], [10], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28]).

**Definition 1.** *[4] The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ with a lower limit $a$ for a function $x : [a, +\infty) \longrightarrow \mathbb{R}$ is defined as*

$$I^{\alpha}_{a^+}x(\rho) = \frac{1}{\Gamma(\alpha)}\int_a^{\rho}(\rho - s)^{\alpha-1}x(s)ds,$$

*provided the right hand side is defined almost everywhere (a.e.) on $[a, +\infty)$.*

**Remark 2.** *If $a = 0$, then we write $I^{\alpha}_{a^+}f(\rho) = (g_\alpha * f)(\rho)$, where*

$$g_\alpha(\rho) = \begin{cases} \frac{1}{\Gamma(\alpha)}\rho^{\alpha-1}, & \rho > 0,\\ 0, & \rho \le 0,\end{cases}$$

*and, as usual, $*$ denotes the convolution of functions. Note that $\lim_{\alpha\longrightarrow 0^+} g_\alpha(\rho) = \delta(\rho)$ with $\delta$ the delta Dirac function.*

**Definition 3.** *[4] The left-sided Riemann-Liouville fractional derivative of order $\alpha > 0,\ n-1 \le \alpha < n,\ n \in \mathbb{N}$, for a function $x : [a, +\infty) \longrightarrow \mathbb{R}$, is defined by*

$$^LD^{\alpha}_{a^+}x(\rho) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{d\rho^n}\int_a^{\rho}\frac{1}{(\rho - s)^{\alpha-n+1}}x(s)ds,$$

*provided the right hand side is defined a.e. on $[a, +\infty)$.*

**Definition 4.** *[4] The left-sided Caputo's fractional derivative of order $\alpha > 0,\ n-1 < \alpha < n,\ n \in \mathbb{N}$, for a function $x : [a, +\infty) \longrightarrow \mathbb{R}$, is defined by*

$$\begin{aligned} ^CD^{\alpha}_{a^+}x(\rho) &= \frac{1}{\Gamma(n-\alpha)}\int_a^{\rho}\frac{1}{(\rho - s)^{\alpha-n+1}}x^{(n)}(s)ds\\ &= I^{n-\alpha}_{a^+}x^{(n)}(\rho),\ \rho > a,\end{aligned}$$

*provided the right hand side is defined a.e. on $[a, +\infty)$.*

**Definition 5.** *[16] The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and type $\psi \in [0, 1]$, of a function $x : [a, +\infty) \longrightarrow \mathbb{R}$, is defined as*

$$D^{\alpha,\psi}_{a^+}x(\rho) = \Big[I^{(1-\alpha)\psi}_{0^+}D\Big(I^{(1-\alpha)(1-\psi)}_{a^+}x\Big)\Big](\rho).$$

**Remark 6.** *(i) If $\psi = 0,\ 0 < \alpha < 1$ and $a = 0$, then the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative:*

$$D^{\alpha,0}_{a^+}x(\rho) = \frac{d}{d\rho}I^{(1-\alpha)}_{a^+}x(\rho) = {}^LD^{\alpha}_{a^+}x(\rho).$$

*(ii) If $\psi = 1,\ 0 < \alpha < 1$ and $a = 0$, then the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative:*

$$D^{\alpha,1}_{a^+}x(\rho) = I^{(1-\alpha)}_{a^+}\frac{d}{d\rho}x(\rho) = {}^CD^{\alpha}_{a^+}x(\rho).$$

The motivation to consider our problem can be found in [11]. Here, we generalize the results in [11]. Let $X \subset Y$ be bounded. The Hausdorff measure of non-compactness is considered as

$$Q(X) = \inf\Big\{\theta > 0 : X \subset \bigcup_{j=1}^{m} B_\theta(x_j),\ \ x_j \in Y,\ m \in \mathbb{N}\Big\},$$

while the Kurtawoski measure of noncompactness $F$ on a bounded set $B \subset Y$ is given by

$$F(X) = \inf\Big\{\epsilon > 0 : X \subset \bigcup_{j=1}^{m} M_j,\ \ \text{and}\ \ \text{diam}(M_j) \le \epsilon\Big\},$$

with the following properties:
1. $X_1 \subset X_2 \Longrightarrow Q(X_1) \le Q(LX_2)$, where $X_1, X_2 \subset Y$ and bounded;
2. $Q(X) = 0$ if and only if $X$ is relatively compact in $Y$;
3. $Q(\{z\} \cup X) = Q(X)$ for all $z \in YX \subseteq Y$;
4. $Q(X_1 \cup X_2) \le \max\{Q(X_1), Q(X_2)\}$;
5. $Q(X_1 + X_2) \le Q(X_1) + Q(X_2)$;
6. $Q(rX) \le |r|Q(X)$ for $r \in \mathbb{R}$.

Let $M \subset C(J, Y)$ and $M(r) = \{\varpi(r) \in Y : \varpi \in M\}$. One defines

$$\int_0^{\rho} M(r)dr := \Big\{\int_0^{\rho}\varpi(r)dr : \varpi \in M\Big\},\ \rho \in J.$$

**Proposition 7.** *[16] If $M \subset C(J, Y)$ is equicontinuous and bounded, then $\rho \longrightarrow Q(M(\rho))$ is continuous on $J$. Furthermore,*

$$\begin{aligned} Q(M) &= \max\Big\{Q(M(\rho)), Q\Big(\int_0^{\rho}\varpi(r)dr\Big)\Big\}\\ &\le \int_0^{\rho} Q(\varpi(r))dr.\end{aligned}$$

**Proposition 8.** *[2] Let $\{\varpi_n : J \longrightarrow Y,\ n \in \mathbb{N}\}$ be Bochner integrable functions. This implies that $\|\varpi_n\| \le m(\rho)$ a.e. for $n \in \mathbb{N}$ and $m \in L^1(J, \mathbb{R}^+)$. Then, $\zeta(\rho) = Q(\{\varpi_n(\rho)\}_{n=1}^{\infty}) \in L^1(J, \mathbb{R}^+)$ and satisfies*

$$Q(\{\int_0^{\rho}\varpi_n(r)dr :\ n \in \mathbb{N}\}) \le 2\int_0^{\rho}\zeta(r)dr.$$

**Proposition 9.** *[2] Let $M$ be a bounded set. Then, for any $q > 0$, there exists a sequence $\{\varpi_n\}_{n=1}^{\infty} \subset M$ such that*

$$Q(M) \le 2Q\{\varpi_n\}_{n=1}^{\infty} + q.$$

**Almost Sectorial Operators**

Let $0 < \beta < \pi$ and $-1 < \beta < 0$. We define $S^0_\beta := \{\varpi \in C \setminus \{0\} :\ |\arg\ \varpi| < \beta\}$ and its closure by $S_\beta$, such that $S_\beta = \{\varpi \in C \setminus \{0\} :\ |\arg\ \varpi| \le \beta\} \cup \{0\}$.

**Definition 10.** *[4] For $-1 < \beta < 0$ and $0 < w < \frac{\pi}{2}$, we define $\{Q^{\beta}_w\}$ as the family of all closed and linear operators $A : D(A) \subset Y \longrightarrow Y$ such that*
*1. $\sigma(A)$ is contained in $S_w$;*
*2. for all $\beta \in (w, \pi)$ there exists $M_\beta$ such that*

$$\|R(z, A)\|_{L(Y)} \le M_\beta|z|^{\beta},$$

*where $R(z, A) = (zI - A)^{-1}$ is the resolvent operator and $A \in Q^{\beta}_w$ is said to be an almost sectorial operator on $Y$.*

**Proposition 11.** *[4] Let $A \in Q^{\beta}_w$ for $-1 < \beta < 0$ and $0 < w < \frac{\pi}{2}$. Then the following properties hold:*
*1. $Q(\rho)$ is analytic and $\frac{d^n}{d\rho^n}Q(\rho) = (-A^nQ(\rho)(\rho \in S^0_{\frac{\pi}{2}})$;*
*2. $Q(\rho + s) = Q(\rho)Q(s),\ \forall\rho, s \in^0_{\frac{\pi}{2}}$;*
*3. $\|Q(\rho)\|_{L(Y)} \le C_0\rho^{-\beta-1}(\rho > 0)$, where $C_0 = C_0(\beta) > 0$ is a constant;*
*4. if $\sum_Q = \{x \in Y : \lim_{\rho\longrightarrow 0_+} Q(\rho)x = x\}$, then $D(A^q) \subset \sum_Q$ if $q > 1 + \beta$;*
*5. $R(r, -A) = \int_0^{\infty} e^{-rs}Q(s)ds,\ r \in \mathbb{C}$ with $Re(r) > 0$.*

We use the following Wright-type function [8]:

$$M_\alpha(q) = \sum_{n\in\mathbb{N}} \frac{(-q)^{n-1}}{\Gamma(1-\alpha n)(n-1)!},\ q \in \mathbb{C}.$$

For $-1 < \sigma < \infty,\ r > 0$, the following properties hold:
(A1) $M_\alpha(q) \geq 0,\ \rho > 0$;
(A2) $\int_0^\infty q^\sigma M_\alpha dq = \frac{\Gamma(1+\sigma)}{\Gamma(1+\alpha\sigma)}$;
(A3) $\int_0^\infty \frac{\alpha}{q^{1+\alpha}} e^{-rq} M_\alpha(\frac{1}{q^\alpha}) dq = e^{-r^\alpha}$.

The characteristic operators $\{S_\alpha(\rho)\}_{\rho\in S^0_{\frac{\pi}{2}-w}}$ and $\{T_\alpha(\rho)\}|_{\rho\in S^0_{\frac{\pi}{2}-w}}$ are defined by

$$S_\alpha(\rho) := \int_0^\infty M_\alpha(q) Q(\rho^\alpha q) dq$$

and

$$T_\alpha(\rho) := \int_0^\infty \alpha q M_\alpha(q) Q(\rho^\alpha q) dq.$$

**Theorem 12.** *For each fixed* $\rho \in S^0_{\frac{\pi}{2}-w}$, $S_\alpha(\rho)$ *and* $T_\alpha(\rho)$ *are bounded linear operators on* $Y$. *Moreover,*

$$\|S_\alpha(\rho)\| \leq C_1 \rho^{-\alpha(1+\beta)},\ \|T_\alpha(\rho)\| \leq C_2 \rho^{-\alpha(1+\beta)},\ \rho > 0,$$

*where* $C_1$ *and* $C_2$ *are constants dependent on* $\alpha$ *and* $\beta$.

**Theorem 13.** *The operators* $S_\alpha(\rho)$ *and* $T_\alpha(\rho)$ *are continuous in the uniform operator topology for* $\rho > 0$. *For* $s > 0$, *the continuity is uniform on* $[s, \infty)$.

Define $W_r(J) := \{\varpi \in C(J,Y) : \|\varpi\| \leq r\}$. Our main results are proved in relation to the following hypotheses:
**(H1).** For $\rho \in J,\ g(\rho, ., ., .) : J \times Y \times Y \times Y \longrightarrow Y$ and $f(\rho, .) : Y \longrightarrow Y$ are continuous functions and, for each $\varpi \in C(J,Y),\ g(., \varpi, B\varpi, Z\varpi) : J \longrightarrow Y$ and $f(., \varpi) : J \longrightarrow Y$ are strongly measurable.
**(H2).** There exist functions $k_1, k_2 \in L^1(J, \mathbb{R}^+)$ satisfying

$$\|g(\rho, ., ., .)\| \leq k_1(\rho) + k_2(\rho)e^{-\delta\rho},\ \forall \varpi \in W_r(J)$$

and almost all $\rho$ on $J$ and

$$I^{-\alpha\beta}_{0^+}\left[k_1(\rho) + k_2(\rho)e^{-\delta\rho}\right] \in C(J, \mathbb{R}),$$

$$\lim_{\rho\longrightarrow 0^+} \rho^{(1+\alpha\beta)(1-\psi)} I^{(-\alpha\beta)}_{0^+}\left[k_1(\rho) + k_2(\rho)e^{-\delta\rho}\right] = 0.$$

**(H3).** Function $h : C(J,Y) \longrightarrow Y$ is completely continuous and there exists a positive constant $k$ such that

$$\|h(\varpi)\| \leq k.$$

**(H4).** We assume that

$$\sup_{[0,T]} \rho^{(1+\alpha\beta)(1-\psi)} \|S_{\alpha,\psi}(\rho)[\varpi_0 + k]k + \rho^{(1+\alpha\beta)(1-\psi)} \times \int_0^\rho (\rho - r)^{-\alpha\beta-1}[k_1(r) + k_2(r)e^{-\delta r}]dr \leq r,$$

for $r > 0,\ \varpi_0 \in D(A^q)$, and $q > 1 + \beta$, where $S_{\alpha,\psi}(\rho) = I^{\psi(1-\alpha)}_{0+} \rho^{\alpha-1} T_\alpha(\rho)$.

For the next two lemmas, we refer to [2].

**Lemma 14.** *The fractional Cauchy problem* (2)-(3) *is equivalent to the integral equation*

$$\begin{aligned}\varpi(\rho) &= \frac{[\varpi_0 - h(\varpi(\rho))]}{\Gamma(\psi(1-\alpha)+\alpha)} \rho^{(1-\alpha)(\psi-1)} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^\rho (\rho - r)^{\alpha-1}[-A\varpi(r) \\ &\quad + g(r, \varpi(r), (B\varpi)r, (Z\varpi)r)]dr,\ \rho \in J. \quad (4)\end{aligned}$$

**Lemma 15.** *If* $\varpi$ *is a solution to the integral equation* (4), *then it satisfies*

$$\begin{aligned}\varpi(\rho) &= S_{\alpha,\psi}(\rho)[\varpi_0 - h(\varpi(\rho))] \\ &\quad + \int_0^\rho R_\alpha(\rho - r) g(r, \varpi(r), (B\varpi)r, (Z\varpi)r)dr,\end{aligned}$$

*where* $S_{\alpha,\psi}(\rho) = I^{\psi(1-\alpha)}_{0^+} R_\alpha(\rho)$ *with* $R_\alpha = \rho^{\alpha-1} T_\alpha(\rho)$.

**Definition 16.** *[4] By a mild solution of the Cauchy problem in Eqs.* (2) *and* (3), *we mean a function* $\varpi \in C(J,Y)$ *that satisfies*

$$\begin{aligned}&\varpi(\rho) \\ &= S_{\alpha,\psi}(\rho)[\varpi_0 - h(\varpi(\rho))] \\ &\quad + \int_0^\rho R_\alpha(\rho - r) g(r, \varpi(r), (B\varpi)r, (Z\varpi)r)dr, \rho \in J.\end{aligned}$$

*We define operator* $\Psi : W_r(J) \longrightarrow W_r(J)$ *as*

$$\begin{aligned}&(\Psi\varpi)(\rho) \\ :=\ & S_{\alpha,\psi}(\rho)[\varpi_0 - h(\varpi(\rho))] \\ +\ & \int_0^\rho (\rho - r)^{\alpha-1} T_\alpha(\rho - r) g(r, \varpi(r), (B\varpi)r, (Z\varpi)r)dr\end{aligned}$$

**Lemma 17.** *[4] The operators* $R_\alpha(\rho)$ *and* $S_{\alpha,\psi}(\rho)$ *are bounded linear operators on* $Y$ *for every fixed* $\rho \in S^0_{\frac{\pi}{2}-w}$. *Also, for* $\rho > 0$, *we have*

$$\begin{aligned}\|R_\alpha(\rho)\varpi\| &\leq C_2 \rho^{-1-\alpha\beta}\|\varpi\|,\ \|S_{\alpha,\psi}(\rho)\varpi\| \\ &\leq \frac{\Gamma(-\alpha\beta)}{\Gamma(\psi(1-\alpha) - \alpha\beta)} C_2 \rho^{\psi(1-\alpha)-\alpha\beta-1}\|\varpi\|.\end{aligned}$$

**Proposition 18.** *[4] The operators* $R_\alpha(\rho)$ *and* $S_{\alpha,\psi}(\rho)$ *are strongly continuous for* $\rho > 0$.

## III. Main Results

In this section, we state and prove our main results.

**Theorem 19.** *Let* $A \in Q^\beta_w$ *for* $-1 < \beta < 0$ *and* $0 < w < \frac{\pi}{2}$. *Assuming that (H1)-(H4) are satisfied, then the operator* $\{\Psi\varpi : \varpi \in W_r(J)\}$ *is equicontinuous, provided* $\varpi_0 \in D(A^q)$ *with* $q > 1 + \beta$.

*Proof:* For $\varpi \in W_r(J)$ and $\rho_1 = 0 < \rho_2 \leq T$, we have

$$\begin{aligned}&\Big\|(\Psi\varpi)(\rho_2) - (\Psi\varpi)(0)\Big\| \\ =\ & \Big\|\rho_2^{(1+\alpha\beta)(1-\psi)}\Big(S_{\alpha,\psi}(\rho_2)[\varpi_0 - h(\varpi(\rho_2))] \\ +\ & \int_0^{\rho_2} (\rho_2 - r)^{\alpha-1} T_\alpha(\rho_2 - r) \\ \times\ & g(r, \varpi(r), (B\varpi)r, (Z\varpi)r)dr\Big)\Big\| \\ \leq\ & \Big\|\rho_2^{(1+\alpha\beta)(1-\psi)} S_{\alpha,\psi}(\rho_2)\Big\|(\varpi_0 + k) \\ +\ & \Big\|\rho_2^{(1+\alpha\beta)(1-\psi)} \int_0^{\rho_2} (\rho_2 - r)^{\alpha-1} T_\alpha(\rho_2 - r) \\ \times\ & g(r, \varpi(r), (B\varpi)r, (Z\varpi)r)dr\Big)\Big\| \\ & \longrightarrow 0 \quad \text{as} \quad \rho_2 \longrightarrow 0.\end{aligned}$$

Now, let $0 < \rho_1 < \rho_2 \leq T$. One has

$$\begin{aligned}
&\Big\|(\Psi\varpi)(\rho_2)-(\Psi\varpi)(\rho_1)\Big\| \\
\leq\ & \Big\|\rho_2^{(1+\alpha\beta)(1-\psi)}S_{\alpha,\psi}(\rho_2)[\varpi_0-h(\varpi(\rho_2))] \\
&-\rho_1^{(1+\alpha\beta)(1-\psi)}S_{\alpha,\psi}(\rho_1)[\varpi_0-h(\varpi(\rho_1))]\Big\| \\
&+\Big\|\rho_2^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho_2}(\rho_2-r)^{\alpha-1} \\
&\times T_\alpha(\rho_2-r)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr \\
&-\rho_1^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho_1}(\rho_1-r)^{\alpha-1} \\
&\times T_\alpha(\rho_1-r)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr\Big\| \\
\leq\ & \Big\|\rho_2^{(1+\alpha\beta)(1-\psi)}S_{\alpha,\psi}(\rho_2)[\varpi_0-h(\varpi(\rho_2))] \\
&-\rho_1^{(1+\alpha\beta)(1-\psi)}S_{\alpha,\psi}(\rho_1)[\varpi_0-h(\varpi(\rho_1))]\Big\| \\
&+\Big\|\rho_2^{(1+\alpha\beta)(1-\psi)}\int_{\rho_1}^{\rho_2}(\rho_2-r)^{\alpha-1} \\
&\times T_\alpha(\rho_2-r)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr\Big\| \\
&+\Big\|\rho_2^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho_1}(\rho_2-r)^{\alpha-1} \\
&\times T_\alpha(\rho_2-r)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr \\
&-\rho_1^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho_1}(\rho_1-r)^{\alpha-1} \\
&\times T_\alpha(\rho_2-r)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr\Big\| \\
&+\Big\|\rho_1^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho_1}(\rho_1-r)^{\alpha-1} \\
&\times T_\alpha(\rho_2-r)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr \\
&-\rho_1^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho_1}(\rho_1-r)^{\alpha-1} \\
&\times T_\alpha(\rho_1-r)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr\Big\| \\
=\ & J_1+J_2+J_3+J_4.
\end{aligned}$$

From the strong continuity of $S_{\alpha,\psi}(\rho)$, we have $J_1\longrightarrow 0$ as $\rho_2\longrightarrow\rho_1$. Also,

$$\begin{aligned}
J_2\ \leq\ & C_2\rho_2^{(1+\alpha\beta)(1-\psi)}\int_{\rho_1}^{\rho_2}(\rho_2-r)^{-\alpha\beta-1} \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr \\
\leq\ & C_2\Big|\rho_2^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho_2}(\rho_2-r)^{-\alpha\beta-1} \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr \\
&-\rho_2^{(1+\beta)(1-\psi)}\int_0^{\rho_1}(\rho_1-r)^{-\alpha\beta-1} \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr\Big| \\
\leq\ & C_2\int_0^{\rho_1}\Big|\rho_1^{(1+\alpha\beta)(1-\psi)}(\rho_1-r)^{-\alpha\beta-1} \\
&-\rho_2^{(1+\alpha\beta)(1-\psi)}(\rho_2-r)^{-\alpha\beta-1}\Big| \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr.
\end{aligned}$$

Then, by using (H2) and the dominated convergence theorem, $J_2\longrightarrow 0$ as $\rho_2\longrightarrow\rho_1$.

Similarly, we can see that $J_3, J_4\longrightarrow 0$ as $\rho_2\longrightarrow\rho_1$.

Hence, $\|(\Psi\varpi)(\rho_2)-(\Psi\varpi)(\rho_1)\|\longrightarrow 0$, independently of $\varpi\in W_r(J)$ as $\rho_2\longrightarrow\rho_1$. Therefore, $\{\Psi\varpi:\varpi\in W_r(J)\}$ is equicontinuous. ■

**Theorem 20.** *Let $-1<\beta<0,\ 0<w<\frac{\pi}{2}$, and $A\in Q_w^\beta$. Then, under hypotheses (H1)-(H4), the operator $\{\Psi\varpi:\varpi\in W_r(J)\}$ is continuous and bounded, provided $\varpi_0\in D(A^q)$ with $q>1+\beta$.*

*Proof:* We verify that $\Psi$ maps $W_r(J)$ into itself. Taking $\varpi\in W_r(J)$ and defining

$$v(\rho):=\rho^{(1+\alpha\beta)(1-\psi)}\varpi(\rho),$$

we have $v\in W_r(J)$. Let $\rho\in[0,T]$,

$$\begin{aligned}
&\Big\|\Psi\varpi(\rho)\Big\| \\
\leq\ & \Big\|\rho^{(1+\alpha\beta)(1-\psi)}S_{\alpha,\psi}(\rho)[\varpi_0-h(\varpi(\rho))]\Big\| \\
&+\rho^{(1+\alpha\beta)(1-\psi)}\Big\|\int_0^\rho(\rho-r)^{\alpha-1}T_\alpha(\rho-r) \\
&\times g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr\Big\| \\
\leq\ & \rho^{(1+\alpha\beta)(1-\psi)}\Big\|S_{\alpha,\psi}(\rho)[\varpi_0-h(\varpi(\rho))]\Big\| \\
&+\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho(\rho-r)^{-\alpha\beta-1} \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr \\
\leq\ & \sup_{[0,T]}\Big(\rho^{(1+\alpha\beta)(1-\psi)}\Big\|S_{\alpha,\psi}(\rho)\|[\|\varpi_0\|+k]\Big\| \\
&+\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho(\rho-r)^{-\alpha\beta-1} \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr\Big) \\
\leq\ & r.
\end{aligned}$$

Thus, $\|\Psi\varpi\|\leq r$ for any $\varpi\in W_r(J)$. Now, to verify $\Psi$ is continuous in $W_r(J)$, let $\varpi_n,\varpi\in W_r(J), n=1,2,...$ with $\lim_{n\longrightarrow\infty}\varpi_n=\varpi$; that is, $\lim_{n\longrightarrow\infty}\varpi_n(\rho)=\varpi(\rho)$; $\lim_{n\longrightarrow\infty}\rho^{-(1+\alpha\beta)(1-\psi)}\varpi_n(\rho)=\rho^{-(1+\alpha\beta)(1-\psi)}\varpi(\rho)$ on $J$. Then, (H1) implies that

$$\begin{aligned}
& g(\rho,\varpi_n(\rho),(B\varpi_n)(\rho),(Z\varpi_n)(\rho)) \\
=\ & g(\rho,\rho^{-(1+\alpha\beta)(1-\psi)}\varpi_n(\rho),\rho^{-(1+\alpha\beta)(1-\psi)}(B\varpi_n)(\rho), \\
& \rho^{-(1+\alpha\beta)(1-\psi)}(Z\varpi_n)(\rho)) \\
\longrightarrow\ & g(\rho,\rho^{-(1+\alpha\beta)(1-\psi)}\varpi(\rho),\rho^{-(1+\alpha\beta)(1-\psi)}(B\varpi)(\rho), \\
& \rho^{-(1+\alpha\beta)(1-\psi)}(Z\varpi)(\rho)),
\end{aligned}$$

as $n\longrightarrow\infty$. From (H2), we obtain the inequality

$$\begin{aligned}
& (\rho-r)^{-\alpha\beta-1}|g(\rho,\varpi_n(\rho),B(\varpi_n(\rho)),Z(\varpi_n(\rho)))| \\
\leq\ & 2(\rho-r)^{-(\alpha\beta)(1-\psi)}[k_1(r)+k_2(r)e^{-\delta r}],
\end{aligned}$$

that is,

$$\int_0^\rho(\rho-r)^{-\alpha\beta-1}\|g(\rho,\varpi_n(\rho),B(\varpi_n(\rho)),Z(\varpi_n(\rho)))\|dr$$
$$\longrightarrow 0\ \text{ as }\ n\longrightarrow\infty.$$

Let $\rho\in[0,T]$. Now,

$$\begin{aligned}
&\Big\|\Psi\varpi_n(\rho)-\Psi\varpi(\rho)\Big\| \\
\leq\ & \rho^{(1+\alpha\beta)(1-\psi)}\Big\|\int_0^\rho(\rho-r)^{\alpha-1} \\
&\times T_\alpha(\rho-r)[g(\rho,\varpi_n(\rho),B(\varpi_n(\rho)),Z(\varpi_n(\rho))) \\
&-g(\rho,\varpi(\rho),B(\varpi(\rho)),Z(\varpi(\rho)))]dr\Big\|.
\end{aligned}$$

Applying Theorem 12, we have

$$\Big\|\Psi\varpi_n(\rho)-\Psi\varpi(\rho)\Big\| \le C_2\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho \times(\rho-r)^{-\alpha\beta-1}\Big\|g(\rho,\varpi_n(\rho),B(\varpi_n(\rho)),Z(\varpi_n(\rho))) -g(\rho,\varpi(\rho),B(\varpi(\rho)),Z(\varpi(\rho)))\Big\|dr,$$

which tends to 0 as $n\longrightarrow\infty$, i.e., $\Psi\varpi_n\longrightarrow\Psi\varpi$ pointwise on $J$. Moreover, Theorem 19 implies that $\Psi\varpi_n\longrightarrow\Psi\varpi$ uniformly on $J$ as $n\longrightarrow\infty$, that is, $\Psi$ is continuous. ■

We prove the existence of a mild solution to the problem in Eqs. (2)-(3). Here, we assume $Q(\rho)$ to be compact.

**Theorem 21.** *Let* $-1<\beta<0$, $0<w<\frac{\pi}{2}$ *and* $A\in Q_w^\beta$. *If* $Q(\rho)(\rho>0)$ *is compact and (H1)-(H4) hold, then there exists a mild solution of the problem* (2)-(3) *in* $W_r(J)$ *for every* $\varpi_0\in D(A^q)$ *with* $q>1+\beta$.

*Proof:* Because we assume $Q(\rho)$ to be compact, then the equicontinuity of $Q(\rho)(\rho>0)$ is ensured. Moreover, by Theorems 19 and 20, $\Psi: W_r(J)\longrightarrow W_r(J)$ is continuous and bounded and $\varepsilon: W_r(J)\longrightarrow W_r(J)$ is bounded and continuous, and $\{\varepsilon\varpi:\varpi\in W_r(J)\}$ is equicontinuous. We can write $\Delta: W_r(J)\longrightarrow W_r(J)$ by $(\Delta\varpi)(\rho)=(\Delta^1\varpi)(\rho)+(\Delta^2\varpi)(\rho)$, where

$$\begin{aligned}
&(\Delta^1\varpi)(\rho)\\
=\;&\rho^{(1+\alpha\beta)(1-\psi)}S_{\alpha,\psi}(\rho)[\varpi_0-h(\varpi(\rho))]\\
=\;&\frac{\rho^{(1+\alpha\beta)(1-\psi)}}{\Gamma(\psi(1-\alpha))}\int_0^\rho(\rho-r)^{\psi(1-\alpha)-1}r^{\alpha-1}\\
&\int_0^\infty \alpha qM_\alpha(q)Q(r^\alpha q)(\varpi_0-h(\varpi))dqdr\\
=\;&\frac{\alpha\rho^{(1+\alpha\beta)(1-\psi)}}{\Gamma(\psi(1-\alpha))}\int_0^\rho\int_0^\infty(\rho-r)^{\psi(1-\alpha)-1}r^{\alpha-1}\\
&\times qM_\alpha(q)Q(r^\alpha q)(\varpi_0-h(\varpi))dqdr,
\end{aligned}$$

and

$$(\Delta^2\varpi)(\rho) = \rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho(\rho-r)^{\alpha-1}T_\alpha(\rho-r) \times g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dr.$$

For $\sigma>0$ and $q\in(0,\rho)$, we define an operator $\Delta^1_{q,\sigma}$ on $W_r(J)$ by

$$\begin{aligned}
&(\Delta^1_{q,\sigma}\varpi)(\rho)\\
=\;&\frac{\rho^{(1+\alpha\beta)(1-\psi)}}{\Gamma(\psi(1-\alpha))}\int_q^\rho\int_\sigma^\infty(\rho-r)^{(1-\alpha)\psi-1}r^{\alpha-1}\\
&\times qM_\alpha(q)Q(r^\alpha q)(\varpi_0-h(\varpi))dqdr\\
=\;&\frac{\alpha\rho^{(1+\alpha\beta)(1-\psi)}}{\Gamma(\psi(1-\alpha))}\Upsilon(\Delta^\alpha\sigma)\int_q^\rho\int_\sigma^\infty(\rho-r)^{\psi(1-\alpha)-1}\\
&\times r^{\alpha-1}qM_\alpha(q)\\
&\times Q(r^\alpha q-q^\alpha\sigma)(\varpi_0-h(\varpi))dqdr.
\end{aligned}$$

Since $\Upsilon(\Delta^\alpha\sigma)$ is compact, $V^1_{q,\sigma}(\rho)=\{(\Delta^1_{q,\sigma}\varpi)(\rho),\varpi\in W_r(J)\}$ is precompact in $\varpi$ for all $q\in(0,\rho)$ and $\delta>0$. Moreover, for any $\varpi\in W_r(J)$, one has

$$\begin{aligned}
&\|(\Delta^1\varpi)(\rho)-(\Delta^1_{q,\sigma}\varpi)(\rho)\|\\
\le\;&\Lambda(\alpha,\psi)\Big\|\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho\int_0^\sigma(\rho-r)^{(1-\alpha)\psi-1}\\
&\times r^{\alpha-1}qM_\alpha(q)Q(r^\alpha q)(\varpi_0-h(\varpi))dqdr\Big\|\\
&+\Lambda(\alpha,\psi)\Big\|\rho^{(1+\alpha\beta)(1-\psi)}\int_0^q\int_\sigma^\infty(\rho-r)^{(1-\alpha)\psi-1}\\
&\times r^{\alpha-1}qM_\alpha(q)Q(r^\alpha q)(\varpi_0-h(\varpi))dqdr\Big\|\\
\le\;&\Lambda(\alpha,\psi)\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho\int_0^\sigma(\rho-r)^{(1-\alpha)\psi-1}\\
&\times r^{\alpha-1}qM_\alpha(q)r^{-\alpha\psi-\alpha}\|\varpi_0-h(\varpi)\|q^{-\beta-1}dqdr\\
&+\Lambda(\alpha,\psi)\rho^{(1+\alpha\beta)(1-\psi)}\int_0^q\int_\sigma^\infty(\rho-r)^{(1-\alpha)\psi-1}\\
&\times r^{\alpha-1}qM_\alpha(q)r^{-\alpha\beta-\alpha}\|\varpi_0-h(\varpi)\|q^{-\beta-1}dqdr\\
=\;&\Lambda(\alpha,\psi)\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho(\rho-r)^{(1-\alpha)\psi-1}\\
&\times r^{-\alpha\beta-1}\|\varpi_0-h(\varpi)\|dr\int_0^\sigma q^{-\beta}M_\alpha(q)dq\\
&+\Lambda(\alpha,\psi)\rho^{(1+\alpha\beta)(1-\psi)}\int_0^q(\rho-r)^{(1-\alpha)\psi-1}\\
&\times r^{-\alpha\beta-1}\|\varpi_0-h(\varpi)\|dr\int_\sigma^\infty q^{-\beta}M_\alpha(q)dq\\
\le\;&\Lambda\rho^{-\alpha\psi(1+\beta)}\|\varpi_0-h(\varpi)\|\int_0^\sigma q^{-\beta}M_\alpha(q)dq\\
&+\Lambda\rho^{-\alpha\psi(1+\beta)}[\|\varpi_0\|+k]\int_0^q(\rho-r)^{(1-\alpha)\psi-1}\\
&\times r^{-\alpha\beta-1}dr\int_\sigma^\infty q^{-\beta}M_\alpha(q)dq\\
&\longrightarrow 0\ \text{ as }\ q\longrightarrow 0\ \text{ and }\ \sigma\longrightarrow 0,
\end{aligned}$$

where $\Lambda(\alpha,\psi)=\frac{\alpha}{\Gamma(\psi(1-\alpha))}$. Therefore, $V^1_{q,\sigma}(\rho)=\{(\Delta^1_{q,\sigma}\varpi)(\rho),\ \varpi\in W_r(J)\}$ are arbitrarily close to $V^1(\rho)=\{(\Delta^1\varpi)(\rho),\ \varpi\in W_r(J)\}$ for $\rho>0$. Hence, $V^1(\rho)$, for $\rho>0$, is precompact in $Y$. For $q\in(0,\rho)$ and $\sigma>0$, we can present an operator $\Delta^2_{q,\sigma}$ on $W_r(J)$ by

$$\begin{aligned}
&(\Delta^2_{q,\sigma}\varpi)(\rho)\\
=\;&\alpha\rho^{(1+\alpha\beta)(1-\psi)}\int_0^{\rho-q}\int_\sigma^\infty qM_\alpha(q)(\rho-r)^{\alpha-1}\\
&\times Q((\rho-r)^\alpha q)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dqdr\\
=\;&\alpha\rho^{(1+\alpha\beta)(1-\psi)}\Upsilon(q^\alpha\sigma)\int_0^{\rho-q}\int_\sigma^\infty\\
&\times qM_\alpha(q)(\rho-r)^{\alpha-1}Q((\rho-r)^\alpha q-q^\alpha\sigma)\\
&\times g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dqdr.
\end{aligned}$$

Thus, due to the compactness of $Q(q^\alpha\sigma)$, $V^2_{q,\sigma}(\rho)=\{\Delta^2_{q,\sigma}(\rho),\ \varpi\in W_r(J)\}$ is precompact in $Y$ for all $q\in(0,\rho)$ and $\sigma>0$. For every $\varpi\in W_r(J)$, we get

$$\begin{aligned}
&\|(\Delta^2\varpi)(\rho) - (\Delta^2_{q,\sigma}\varpi)(\rho)\| \\
\leq\ & \Big\|\alpha\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho\int_0^\sigma qM_\alpha(q)(\rho-r)^{\alpha-1} \\
&\times Q((\rho-r)^\alpha q)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dqdr\Big\| \\
&+\Big\|\alpha\rho^{(1+\alpha\beta)(1-\psi)}\int_{\rho-q}^\rho\int_\sigma^\infty qM_\alpha(q)(\rho-r)^{\alpha-1} \\
&\times Q((\rho-r)^\alpha q)g(r,\varpi(r),(B\varpi)r,(Z\varpi)r)dqdr\Big\| \\
\leq\ & \alpha C_0\rho^{(1+\alpha\beta)(1-\psi)}\int_0^\rho(\rho-r)^{-\alpha\beta-1} \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr\int_0^\sigma q^{-\beta}M_\alpha(q)dq \\
&+\alpha C_0\rho^{(1+\alpha\beta)(1-\psi)}\int_{\rho-q}^\rho(\rho-r)^{-\alpha\beta-1} \\
&\times[k_1(r)+k_2(r)e^{-\delta r}]dr\int_0^\infty q^{-\beta}M_\alpha(q)dq \\
&\longrightarrow 0 \ \text{ as } \ \sigma\longrightarrow 0.
\end{aligned}$$

Therefore, $V^2_{q,\sigma}(\rho) = \{\Delta^2_{q,\sigma}(\rho),\ \varpi\in W_r(J)\}$ are arbitrarily close to $V^2(\rho) = \{\Delta^2(\rho),\ \varpi\in W_r(J)\},\ \ \rho>0$. This implies the relative compactness of $V^2(\rho),\ \rho>0$ in $Y$. Also, $V(\rho) = \{\Delta(\rho),\ \varpi\in W_r(J)\}$ is relatively compact in $Y\ \forall\rho\in[0,T]$. It follows, from the Arzela-Ascoli theorem, that $\{\Delta(\rho),\ \varpi\in W_r(J)\}$ is relatively compact for $\Delta$, it is continuous, and $\{\Delta(\rho),\ \varpi\in W_r(J)\}$ is relatively compact. This implies, by the Schauder's fixed point theorem, the existence of a fixed point $v^*\in W_r(J)$ of $\Delta$. Let $\varpi^*(\rho) := \rho^{(1+\alpha\beta)(\psi-1)}v^*(\rho)$. Then, $\varpi^*$ is a mild solution of the problem (2)-(3). ■

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